APPRAOCHES TO RESOLUTION OF WEYL MODULES

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1. Introduction

In the nine years of our collaboration, we published only two joint articles ([5],[6]), yet it was nevertheless true that we did quite a bit more mathematics together in that period than the written record shows. In this article, we will try to indicate the areas that we were exploring, the results that we obtained, and some of the philosophy that guided our explorations.

The main objective with which we started out was to find a characteristic-free resolution of the Weyl modules of certain shapes. For one of us, this was of interest because it would provide a description of the higher syzygies, and hence more invariants, of $GL_n$. For the other, it was a way of dealing with the problem of determining intertwining numbers in modular representation theory, and also of discovering $\mathbb{Z}$-forms of rational representations (problems that have been discussed in [1], [2]). We were both well aware of the work of Lascoux, [14], but over the course of many years, one of us had become acutely aware of the fact that it would take more than minor adjustments on his resolutions in characteristic zero to find what we were looking for. The search for a solution to this problem has by now quite a long history.

In the early 1980’s, Akin and Buchsbaum (referred to henceforth as A&B) started working on the problem of resolving Schur modules in terms of direct sums of tensor products of exterior powers (or fundamental representations). For two-rowed Schur modules, using the ‘fundamental exact sequence’ (this will be expanded on in a later section)

\[
0 \rightarrow \begin{array}{c} p + t + 1 \\ q - t - 1 \end{array} \rightarrow \begin{array}{c} t + 1 \\ q \end{array} \rightarrow \begin{array}{c} p \\ q \end{array} \rightarrow 0,
\]

and a Koszul-like complex (which we called an ‘arithmetic Koszul Complex’, and which will be explained in section 4.1), an induction argument on the number of overlaps between the two rows led to a description of the desired type of resolution ([1]).

The three-rowed case presented quite another aspect. For one thing, the analogue of the exact sequence above led to a type of representation that had never been studied in the classical situation. To be precise, the analogue of the above sequence is the following:

\[
0 \rightarrow \begin{array}{c} t_1 \\ t_2 + 1 \\ r - t_2 - 1 \end{array} \rightarrow \begin{array}{c} t_2 + 1 \\ r \end{array} \rightarrow \begin{array}{c} t_1 \\ p \end{array} \rightarrow 0.
\]

Thus one sees that the kernel of the epimorphism on the right is not any of the classical shapes, but what M. Klucznik calls an ‘almost skew-shape’ (see [12]). If one starts with an epimorphism of the type above between two
almost skew-shapes:

\[
\begin{array}{c}
p \\
q \rightarrow \\
s \\
s - 1 \rightarrow \\
\end{array}
\quad \begin{array}{c}
\rightarrow
\\
\rightarrow
\\
\rightarrow
\\
\end{array}
\begin{array}{c}
t \\
q \rightarrow 0 (t \geq s),
\end{array}
\]

the kernel is again an almost skew-shape, and the fundamental exact sequence in this case is:

\[
\begin{array}{c}
0 \rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
s - 1 \\
p + t - s + 1 \\
q \\
r - (t - s + 1)
\end{array}
\begin{array}{c}
t \\
p \\
q \rightarrow 0.
\end{array}
\]

Of course, for three-rowed shapes, there is essentially only one type of almost skew-shape; in the case of \(n\)-rowed shapes, the number of distinct types of almost skew-shapes is \(n - 2\).

Using the arithmetic Koszul complexes and induction on the number of triple overlaps, A&B had, in the summer of 1982, succeeded in constructing the resolutions of Schur modules of skew-shapes having no more than two triple overlaps. The computations were so \textit{ad hoc}, and the way ahead so unclear, that it seemed imprudent at that time to go ahead with the project of resolving skew-shapes in general; it was almost impossible to think of resolving almost skew-shapes in general and, in fact, because the resolutions that we were looking for weren’t projective resolutions, an existence proof for these types of resolutions had to be found. In [3], resolutions of Weyl modules whose terms are direct sums of tensor products of divided powers were then considered. Terms of this kind are projective over the appropriate Schur algebra, so that using the same fundamental exact sequences for Weyl modules that we had been using for Schur modules, and a similar induction argument on the number of overlaps, an existence proof was obtained (see [2]). In fact, the existence of resolutions of the desired type was proved simultaneously for all the shapes that were being considered: partitions, skew-shapes and almost skew-shapes. By means of a duality between Schur and Weyl modules which carries exterior powers to divided powers and vice versa, the resolutions of Schur modules in terms of fundamental representations were also shown to exist. But, although some interesting applications of this fact were found and studied, there seemed little hope of finding an explicit description of the resolutions of Weyl modules associated to skew shapes, and even less of ever obtaining resolutions of almost skew-shapes.

In the summer of 1990, we met in Rome, and became aware that our work over the past several years had a great deal in common. As Boston was a considerably more convenient place for the two of us to meet than Rome, we decided to get together regularly once we returned to the Boston area and see if we could learn each other’s languages and methods. So we began
to look at the resolutions of Weyl modules together. A description of the contents of the various sections will follow, but first let us summarize the overall principles that have come into play.

One of the first observations we made, was that the standard kind of map that had been constantly used by A&B to present Weyl and Schur modules and their resolutions was ‘some sort of’ polarization map. This map is one that takes a tensor product of two or more elements, diagonalizes one of them in suitable degrees, and then multiplies one of the resulting factors by another term of the tensor product (see the first paragraph of section 2 for a more careful description). The next observation that was made was that if one used the letter-place algebra to describe the tensor products that we were dealing with, these ‘sort of’ polarizations could be viewed precisely as place polarizations (defined and discussed in section 2). The advantage to adopting this different perspective is that compositions of polarizations satisfy certain standard (Capelli-like) identities, and we soon realized that many of the baroque computations that had previously been done could now proceed in a straightforward, almost predictable manner.

Once the letter-place approach had been adopted, it soon became evident that the Bar Complex was a more convenient homological tool than the arithmetic Koszul Complex mentioned above (all of these complexes are discussed in section 4). A short while after that it became evident that a generalization of the classical Bar Complex was demanding to be defined, and we developed the generalized Bar Complex and Bar Module to handle the slightly more complicated homological facets of our problem ([6]). The essential idea behind the generalization is this: if one considers the ‘bar’ in the classical situation as a skew-symmetric variable, and the terms of the Bar Complex as elements of the free product of the given algebra with the exterior algebra on this variable, then the boundary map of the complex is simply given by polarizing the bar (or variable) to 1. Thus, one might consider an algebra, A, together with an exterior algebra, Λ, on any number of variables \( \{x, y, \ldots\} \), and take the free product of these algebras. A boundary map automatically exists: the sum of the polarizations of all (or any subset of) the ‘separator’ variables \( \{x, y, \ldots\} \). In section 4 we develop all these notions, and more, in greater detail. The attempt to use the Bar Constructions as our main homological tool has influenced the way many of our subsequent proofs are shaped.

As the algebra of polarizations is closely associated with the Heisenberg-Weyl algebra, we devote a section of the article to study the relations between our polarization algebra and this algebra. It is an area that we have just recently begun to explore from the point of view of resolutions, and we hope that it will prove useful not only as a tool in developing interesting identities among polarizations, but also as a way of analyzing the ‘virtual operators’ that play an important role in defining our complexes.

Because the methods we use in this paper are a mixture of components that have not often been blended, we have included some proofs that are
fairly detailed. Since a plethora of such proofs can be both soporific and overwhelming, we have withheld a number of them when it was clear that the procedure was not too different from ones that had already been used. We have used freely, and without definition in this article, the terms ‘Weyl module’ (see [4]), ‘Schur algebra’ (see [3] and [10]), as well as most of the usual constructions in multilinear algebra such as exterior, divided, and symmetric powers (all of these are quickly summarized in [4]).

In brief, then, what we have is the following:

In our first non-introductory section, we interpret many of the maps that came up in the earlier study of resolutions as ‘place polarizations’, and see that there are very fundamental identities that they must satisfy. These natural ‘Capelli identities’ explain a great deal of the farrago that A&B encountered in their earlier work. We will give a detailed example, after the section on Bar Complexes, to illustrate this. Of course, to talk about ‘place polarizations’ we have to use the notion of ‘place’ described in ([8],[11]), so that in the same section we also give a brief hands-on description of the use of the letter-place algebra. For those to whom the notion of letter-place seems too basis-dependent to be of much intrinsic value, we give an illustration of its use in defining the Pieri filtration on certain types of skew-shapes.

Following that, we review the ‘fundamental exact sequences’ that play a central role in setting the strategy of our approach. We offer, for what it may be worth, an heuristic ‘proof’ of the exactness. (The first strict proof of this result is found in [3]; it involves a spectral sequence argument which is far from heuristic.) This review is followed by a section on Bar Complexes. The classical complexes were generalized in [6], and they play an important role in organizing the terms that appear in the resolutions. Since Bar Complexes are usually associated to an action of an algebra on a module, we have to set up and study the actions that arise in our situation. We therefore develop a section on the action of our polarization operators on certain complexes, which we call ‘stems’. We also see that some of the actions aren’t transitive, but homotopy-transitive; this leads us to study not only the actions, but also certain homotopies associated with them. Following this we have the section on Heisenberg-Weyl algebras mentioned above; then we devote the final section to a description of the terms of the resolution, and a proof of the fact that this description is indeed valid.

2. Letter Place Algebra and Capelli Identities

In addition to the maps whose images define the Schur and Weyl modules, the so-called Schur and Weyl maps of [4], another type of map was used repeatedly, namely one of the sort $D_{p+k} \otimes D_{q-k} \rightarrow D_p \otimes D_q$, which sends an element $x \otimes y$ of $D_{p+k} \otimes D_{q-k}$ to $\sum x_p \otimes x'_k y$, where $\sum x_p \otimes x'_k$ is the component of the diagonal of $x$ in $D_p \otimes D_k$. Here $D_l$ stands for the $l^{th}$ divided power of a free $R$-module, where $R$ is a commutative ring. In fact, generalizations of this map to ones where there were more factors, were components of what was called in [4] the ‘box map’. This box map is the
map whose cokernel is the Weyl module corresponding to an appropriate shape.

2.1. Letter Place Algebra. The first step taken in our collaboration was to look at tensor products such as $D_p \otimes D_q$ from the viewpoint of letter-place algebras. For the sake of those who are not expert in this area, we will provide a very brief description of this notion. The main idea is that in addition to the positive alphabet, or basis of the underlying free module, that we use in describing elements of the divided power, we also have a ‘place alphabet’ of positive places. For example, an element $w \otimes w' \in D_p \otimes D_q$ would be written, in letter-place algebra, as $(w|1)^{(p)}(w'|2)^{(q)}$ to indicate that it is the tensor product of a basis element of degree $p$ in the first factor, and one of degree $q$ in the second. This is then collected in double tableau form as

$$
\begin{array}{c|c}
  w & 1^{(p)} \\
  w' & 2^{(q)}
\end{array}
$$

If we further agree that the symbol $(v|1)^{(r)}2^{(s)}$ means $\sum v(r) \otimes v(s)$, where this stands for the diagonalization of the element $v \in D_r \otimes D_s$ into its image under the diagonalization map in $D_r \otimes D_s$, then we can also talk about the double tableau

$$
\begin{array}{c|c}
  w & 1^{(p)}2^{(k)} \\
  w' & 2^{(q-k)}
\end{array}
$$

which means the element $\sum w(p) \otimes w(k)w' \in D_p \otimes D_q$. Ordering the basis elements of the underlying free module and the place alphabet as well, we can now talk about ‘standard’ and ‘double standard’ tableaux. A major result of [8] or [11] on letter-place algebra is that the set of double standard tableaux form a basis for $D_p \otimes D_q$. In general, of course, one talks about $D_{p_1} \otimes D_{p_2} \otimes \ldots \otimes D_{p_n}$ in letter-place terms, where the ‘places’ run from 1 to $n$. In a similar way, one can talk about mixed tensor products of divided powers and exterior powers, and negative as well as positive places. In our case, however, we will generally be using a positive letter alphabet, and most usually a positive place alphabet. In the section on fundamental exact sequences, we will give a very brief description of the use of negative places with positive letters, but that will be the only section where that notion will be used.

To illustrate the basis theorem, let us suppose that $p < q$, and that we have the element $a^{(p)} \otimes b^{(q)} \in D_p \otimes D_q$. Then, although \( \left( \begin{array}{c|c} a^{(p)} & 1^{(p)} \\ b^{(q)} & 2^{(q)} \end{array} \right) \) is a basis element of $D_p \otimes D_q$, it isn’t a double standard tableau (even assuming $a < b$ and $1 < 2$) since $p < q$. It fails to be standard because the top row is shorter than the second row. But

$$
\left( \begin{array}{c|c} a^{(p)} & 1^{(p)} \\ b^{(q)} & 2^{(q)} \end{array} \right) = a^{(p)} \otimes b^{(q)},
$$
with $a < b$, $1 < 2$, and $p < q$ can be straightened to
\[
\sum_{k \geq 0} \binom{p - q}{k} \left( \frac{a^{(p)}_b b^{(q-p+k)}}{b^{(p-k)}} \right) \left( \frac{1^{(p)}_2 (q-p+k)}{2^{(p-k)}} \right).
\]

One way to see this is to write
\[
\left( \frac{a^{(p)}}{b^{(q)}} \right) \left( \frac{1^{(p)}}{2^{(q)}} \right) = \sum_{l=0}^{p} c_l \left( \frac{a^{(p)}_b b^{(q-p+l)}}{b^{(p-l)}} \right) \left( \frac{1^{(p)}_2 (q-p+l)}{2^{(p-l)}} \right),
\]
and then to determine the coefficients $c_l$. Rewriting the above, we get
\[
a^{(p)}_l b^{(q)} = \sum_{l=0}^{p} c_l \sum_{k=0}^{p} \binom{q-k}{p-l} a^{(p-k)} b^{(k)} \otimes a^{(k)} b^{(q-k)},
\]
and we want the $c_l$ to be such that
\[
\sum_{l=0}^{p} c_l \binom{q-k}{p-l} = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}.
\]
Clearly, if we set $c_l = \binom{p-q}{l}$, then for $k = 0$, the sum above is
\[
\sum_{l=0}^{p} \binom{p-q}{l} \binom{q}{p-l} = \binom{p}{p} = 1,
\]
while for $k > 0$, we get
\[
\sum_{l=0}^{p} \binom{p-q}{l} \binom{q-k}{p-l} = \binom{p-k}{p} = 0
\]
as we wanted.

For the reader who wants a more general definition of the letter-place set-up, we will borrow the brief summary given in [7]. We assume that we are given two free modules, $L$ and $P^+$, and we construct (see [11]) a bilinear pairing, $(\cdot)$ of the divided power algebras $D(L)$ and $D(P^+)$ into $D(L \otimes P^+)$. (We have decorated $P^+$ with a sign following the definition in [11] which really applies to a direct sum of free modules $P = P^+ \oplus P^-$.) Elements of $P^+$ are said to be \textit{positively signed} and elements of $P^-$ are \textit{negatively signed}; for the moment we are specializing to the case that $P^- = 0$.

Suppose that $L$ and $P^+$ have bases $\mathcal{L}$ and $\mathcal{P}$. We identify the basis $\{l \otimes p | l \in \mathcal{L}, p \in \mathcal{P}^+\}$ of $L \otimes P^+$ with the set $\{(l|p) | l \in \mathcal{L}, p \in \mathcal{P}^+\}$ of “letterplaces”. The algebra $D(L \otimes P^+)$ can now be identified with the commutative associative algebra $D([\mathcal{L}|\mathcal{P}^+])$ generated by all $(l|p)$ and satisfying the relations
\[
b_i^{(j)} b_i^{(k)} = \binom{j+k}{j} b_i^{(j+k)} \quad \text{and} \quad b_i^{(0)} = 1.
\]
for all $b = (l|p)$. For $l_1, \ldots, l_k \in \mathcal{L}$ and $p_1, \ldots, p_{k'} \in \mathcal{P}^+$, we have
\[
(\beta) \quad (l_1 l_2 \cdots l_k | p_1 p_2 \cdots p_{k'}) = \begin{cases} \sum_{\sigma \in S_k} (l_{\sigma(1)}|p_1) \cdots (l_{\sigma(k)}|p_k) & \text{when } k = k'; \\ 0 & \text{otherwise} \end{cases}
\]
If we consider the case that we have a trivial $P^+$ but a non-trivial $P^-$, we have the bilinear pairing $(\langle \rangle : D(L) \otimes \Lambda(P^-) \to \Lambda(L \otimes P^-)$ (as in [11]). If we are now given a basis $P^-$ of $P^-$, then identifying $\Lambda(L \otimes P^-)$ with the free exterior algebra $\Lambda([L|P^-])$ on generators $\{(l|q) | l \in L, q \in P^-\}$, the bilinear pairing $(\langle \rangle$ still satisfies equation $(\beta)$.

For both $P = P^-$ and $P = P^+$, the relations $(\beta)$ determine $(\langle \rangle$ over the integers, $\mathbb{Z}$.

As an example, consider the following:

Suppose we have bases $L = \{a, b, c\}, P^+ = \{1, 2\}, P^- = \{1, 2\}$. We have

$$ (aabc|1222) = 12 (a|1) (a|2) (b|2) (c|2) + 6 (b|1) (a|2) (a|2) (c|2) + 6 (c|1) (a|2) (a|2) (b|2), $$

so (over $\mathbb{Z}$) bilinearity implies that

$$ (a^{(2)}bc|12^{(3)}) = (a|1) (a|2) (b|2) (c|2) + (c|1) (a|2) (a|2) (b|2). $$

However, using the negative places 1 and 2, and the fact that $22 = 0$, we have

$$ (aabc|1222) = 0. $$

2.2. Capelli Identities. Let’s now look at the map $D_{p+k} \otimes D_{q-k} \to D_0 \otimes D_q$ from this other point of view. We may take an element $\left(\begin{array}{c|c}
    w & 1^{(p+k)}2^{(l)}\\
    \hline
    w' & 2^{(q-k-l)}
\end{array}\right) \in D_{p+k} \otimes D_{q-k}$, and consider the place polarization, $\partial_{21}$, from place 1 to place 2, raised to the $k^{th}$ divided power. This sends the tableau above to

$$ \left(\begin{array}{c|c}
    w & 1^{(p+k)}2^{(l)}\\
    \hline
    w' & 2^{(q-k-l)}
\end{array}\right) = \left(\begin{array}{c|c}
    w & 1^{(p+k+l)}\\
    \hline
    w' & 2^{(q-k-l)}
\end{array}\right). $$

In the more traditional notation, this means that we have taken the element $\sum w(p+k) \otimes w(l)w'$ to $(k+l) \sum w(p) \otimes w(k+l)w'$. But this is exactly the same thing as taking the element $\sum w(p+k) \otimes w(l)w'$, diagonalizing the first factor into its degree $p$ and degree $k$ parts, and then multiplying with the second factor (using the coassociativity and cocommutativity of the divided power algebra).

The examples above lead us then to examine place polarizations and the identities that obtain among them in the case that there are more than just two places (although, and this will be discussed in the penultimate section, one can also ask for the identities one has if one takes $\partial_{21}^{(k)} \partial_{12}^{(l)}$). Again as an example, let us take three places, and consider the easily verified equation:

$$ \partial_{32}\partial_{21} - \partial_{21}\partial_{32} = \partial_{31}. $$

This is a typical Capelli identity, and we actually have more:

$$ \partial_{32}^{(k)} \partial_{21}^{(l)} = \sum_{\alpha \geq 0} \partial_{21}^{(l-\alpha)} \partial_{32}^{(k-\alpha)} \partial_{31}^{(\alpha)}. $$

Of course, from this one easily obtains the corresponding identity:

$$ \partial_{21}^{(l)} \partial_{32}^{(k)} = \sum_{\alpha \geq 0} (-1)^{\alpha} \partial_{32}^{(k-\alpha)} \partial_{21}^{(l-\alpha)} \partial_{31}^{(\alpha)}. $$
More generally, if one has places 1, . . . , n, one may consider, for any \(i\) and \(j\), the place polarization \(\partial_{ij}\) and its powers, divided or otherwise. We’ll assume, unless specified otherwise, that \(i \neq j\). One then proves immediately that

**Proposition 2.1.** The divided powers of the place polarizations satisfy the following identities:

1. If \(k \neq i\), then
   \[
   \partial^{(r)}_{ij} \partial^{(s)}_{jk} = \sum_{\alpha \geq 0} \partial^{(s-\alpha)}_{jk} \partial^{(r-\alpha)}_{ij} \partial^{(\alpha)}_{ik};
   \]
   \[
   (2.1)
   \]

2. If \(i \neq k\) and \(j \neq l\), then
   \[
   \partial^{(s)}_{jk} \partial^{(r)}_{il} = \partial^{(r)}_{il} \partial^{(s)}_{jk}.
   \]

**Proof.** That \(\partial^{(s)}_{jk} \partial^{(r)}_{il} = \partial^{(r)}_{il} \partial^{(s)}_{jk}\) if \(i \neq k\) and \(j \neq l\), is easy to see. We first assume that \(l \neq k\), and take a tableau \(\left( \begin{array}{cc|c} w & w' & j(q) \\ w'' & j(t) \end{array} \right)\). Applying \(\partial^{(s)}_{jk} \partial^{(r)}_{il}\) we get:

   \[
   \partial^{(s)}_{jk} \partial^{(r)}_{il} \left( \begin{array}{cc|c} w & w' & j(q) \\ w'' & j(t) \end{array} \right) = \partial^{(s)}_{jk} \left( \begin{array}{cc|c} w & w' & j(q) \\ w'' & j(t) \end{array} \right) = \left( \begin{array}{cc|c} w & w' & j(q) \\ w'' & j(t) \end{array} \right).
   \]

It’s clear that if we apply \(\partial^{(r)}_{il} \partial^{(s)}_{jk}\), we get the same result. If we assume now that \(l = k\), then we need only look at a tableau of the form \(\left( \begin{array}{cc|c} w & j(q) \\ w' \end{array} \right)\), and we get

   \[
   \partial^{(s)}_{jl} \partial^{(r)}_{il} \left( \begin{array}{cc|c} w & j(q) \\ w' \end{array} \right) = \partial^{(s)}_{jl} \left( \begin{array}{cc|c} w & j(q) \\ w' \end{array} \right) = \left( \begin{array}{cc|c} w & j(q) \\ w' \end{array} \right).
   \]

Applying \(\partial^{(r)}_{il} \partial^{(s)}_{jk}\), we get the same result, since \(j^{(s)}(r) = j^{(r)}(s)\). However, if we apply \(\partial^{(r)}_{ij} \partial^{(s)}_{jk}\) to the tableau \(\left( \begin{array}{cc|c} w & j(q) \\ k(t) \end{array} \right)\), we get

   \[
   \partial^{(r)}_{ij} \partial^{(s)}_{jk} \left( \begin{array}{cc|c} w & j(q) \\ k(t) \end{array} \right) = \partial^{(r)}_{ij} \left( \begin{array}{cc|c} w & j(q) \\ k(t) \end{array} \right) = \sum_{\alpha} \left( \begin{array}{cc|c} w & j^{(q-r+\alpha)}(r-\alpha) \\ k(t-s)j^{(s)}(\alpha) \end{array} \right).
   \]

But

   \[
   \partial^{(s-\alpha)} \partial^{(r-\alpha)} \partial^{(\alpha)} \left( \begin{array}{cc|c} w & j(q) \\ k(t) \end{array} \right) = \partial^{(s-\alpha)} \partial^{(r-\alpha)} \left( \begin{array}{cc|c} w & j(q) \\ k(t) \end{array} \right) = \partial^{(s-\alpha)} \partial^{(r-\alpha)} \left( \begin{array}{cc|c} w & j(q) \\ k(t) \end{array} \right).
   \]
Thus, summing, we get the first identity of the Proposition. The second identity, involving alternating sign, may be proved in the same way or, more efficiently, by induction. For \( r = s = 1 \), we have from the first identity that
\[
\partial_{ij} \partial_{jk} = \partial_{jk} \partial_{ij} + \partial_{ik}
\]
or,
\[
\partial_{jk} \partial_{ij} = \partial_{ij} \partial_{jk} - \partial_{ik}.
\]
Then by induction on \( r \) we get
\[
r \partial_{jk} \partial_{ij}^{(r)} = \partial_{jk} \partial_{ij}^{(r-1)} = \{ \partial_{ij} \partial_{jk} - \partial_{ik} \} \partial_{ij}^{(r-1)} = \partial_{ij} \partial_{jk}^{(r-1)} \partial_{ij} - \partial_{ij} \partial_{jk} \partial_{ij}^{(r-2)} \partial_{ik} - \partial_{ij} \partial_{jk} \partial_{ij}^{(r-1)} \partial_{ik}
\]
\[
= r \partial_{ij}^{(r)} \partial_{jk} - (r-1) \partial_{ij}^{(r-1)} \partial_{ik} - \partial_{ij}^{(r-1)} \partial_{ik}
\]
\[
= r \left( \partial_{ij}^{(r)} \partial_{jk} - \partial_{ij}^{(r-1)} \partial_{ik} \right).
\]
Canceling the \( r \) (which we may do generically), we get the result for any \( r \). Proceeding similarly by induction on \( s \), we obtain the general result. Of course, a straightforward calculation on tableaux gives a simple and direct proof of the identity.

2.3. Pieri Filtrations. As promised in the introduction, we'll briefly indicate here how one can do 'intrinsic' module theory using letter-place methods. Namely, one can use the letter-place basis to define the equivariant filtration on, say, a two-rowed skew-shape that gives rise to the Pieri decomposition of the corresponding Weyl module. For suppose we have the two-rowed skew-shape:

\[
(A) \begin{array}{c}
t \\
p & q
\end{array}
\]

As indicated earlier, this is the image of \( D_p \otimes D_q \) under the Weyl map ([4]), and the letter-place basis for \( D_p \otimes D_q \) is the set of all double standard tableaux \( \left\{ \left( \begin{array}{cc} w & \frac{1(p)2(l)}{2(q-l)} \\
\frac{1}{2} & \frac{1}{2}
\end{array} \right) \right\} \) with \( q \leq p + l \), and where \( w \) and \( w' \) are words in the letter alphabet (the place alphabet consisting in this case merely of 1 and 2 in their usual order). Now the Pieri decomposition theorem for the above shape tells us that the Weyl module has an equivariant filtration such that the associated graded is the direct sum of the Weyl modules corresponding to the partitions \( (p + l, q - l) \) with \( q - l \leq p \) and \( l \leq t \). (The condition that \( q - l \leq p \) arises from the fact that when you take \( t - l \) boxes away from the \( p + t \) boxes in the filled-in top row, they can’t overlap any boxes in the second row.) Except for the condition that \( l \leq t \), these conditions are remarkably close to the standard basis for \( D_p \otimes D_q \). However, if we use the fact that the image of any basis element \( \left( \begin{array}{cc} w & \frac{1(p)2(l)}{2(q-l)} \\
\frac{1}{2} & \frac{1}{2}
\end{array} \right) \) with \( l > t \) is zero under the Weyl map (this will be demonstrated in a later section), and that the straightening of a double tableau increases the shape...
of the tableau in the dominance order ([11]), we see that the equivariant Pieri filtration on our Weyl module is that obtained by taking, for each \( l \), the image of the submodule of \( D_p \otimes D_q \) generated by all standard tableaux \( \left\{ \begin{array}{c} w \\ w' \end{array} \right| \begin{array}{c} 1^{(p)} 2^{(k)} \\ 2^{(q-k)} \end{array} \} \) with \( k \geq l \). For skew-shapes of Pieri type (i.e., skew-shapes which are a partition modulo a one-rowed partition), this argument works for any number of rows; how to extend this to arbitrary (almost) skew-shapes in the form of a Littlewood-Richardson decomposition is still being explored.

3. Fundamental Exact Sequences

After attempting a number of other routes to obtain resolutions of Weyl modules, we finally returned to the procedure that had been begun in [3], namely to use certain exact sequences of these modules, and construct the resolutions as mapping cones. The main strategy is summed up in the following elementary lemma:

**Lemma 3.1.** Let
\[
0 \rightarrow A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C \rightarrow 0
\]
be an exact sequence of modules, let \( X \overset{\varepsilon_1}{\rightarrow} A \) and \( Y \overset{\varepsilon_2}{\rightarrow} B \) be exact left complexes over \( A \) and \( B \), and let \( X \overset{f}{\rightarrow} Y \) be a map over \( \alpha \) (i.e., the map \( f \) is a map of complexes, and the diagram
\[
\begin{array}{ccc}
X & \overset{f}{\rightarrow} & Y \\
\downarrow{\varepsilon_1} & & \downarrow{\varepsilon_2} \\
A & \overset{\alpha}{\rightarrow} & B
\end{array}
\]
is commutative). Then the mapping cone of \( f \), \( M(f) \), is an exact complex over \( C \), with augmentation map \( \varepsilon_3 : M(f) \rightarrow C \) given by \( \beta \varepsilon_2 \).

**Proof.** Recall that \( X \overset{\varepsilon_1}{\rightarrow} A \) is an exact left complex over \( A \) means that \( X \) is a negative complex (written \( \cdots \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0 \)), \( \varepsilon_1 \) is a map of complexes (i.e., \( \varepsilon_1 : X_0 \rightarrow A \) is a map such that \( X_1 \rightarrow X_0 \overset{\varepsilon_1}{\rightarrow} A \) is zero), and \( H_*(X) \overset{(\varepsilon_1)_*}{\rightarrow} A \) is an isomorphism. This boils down to the simple description:
\[
\cdots \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \overset{\varepsilon_1}{\rightarrow} A \rightarrow 0
\]
is an exact sequence.

Recall too that the mapping cone of \( f \) looks like:
\[
\cdots \rightarrow X_n \oplus Y_{n+1} \rightarrow \cdots \rightarrow X_0 \oplus Y_1 \rightarrow Y_0 \rightarrow 0,
\]
with the map \( X_n \oplus Y_{n+1} \rightarrow X_{n-1} \oplus Y_n \) given by \( (x, y) \mapsto (\partial x, (-1)^n f_n(x) + \partial(y)) \).

We use \( \partial \) to denote the boundary maps of both complexes, \( X \) and \( Y \), and \( f_n \) is the component of \( f \) in dimension \( n \) : \( f_n \) maps \( X_n \) to \( Y_n \). The meaning of the statement that the augmentation map \( \varepsilon_3 : M(f) \rightarrow C \) is given by \( \beta \varepsilon_2 : Y_0 \rightarrow B \) is clear.
With the terminology and notation settled, the proof is straightforward. If we let \( X' \) stand for what is usually written as \( X(-1) \), we have the short exact sequence of complexes:

\[
0 \to Y \to M(f) \to X' \to 0
\]

and the corresponding long exact sequence of homology:

\[
\cdots \to H_n(X) \to H_n(Y) \to H_n(M(f)) \to H_{n-1}(X) \to \cdots \to \\
H_1(M(f)) \to H_0(X) \xrightarrow{\alpha} H_0(Y) \xrightarrow{\beta} H_0(M(f)) \to 0.
\]

The vanishing of the positive-dimensional homology of \( X \) and \( Y \) and the fact that \( \alpha \) is injective, give us the vanishing of \( H_k(M(f)) \) for positive \( k \), and the fact that \( C \) is the cokernel of \( \beta \) tells us that \( H_0(M(f)) \approx C \).

The idea now is to apply this to certain fundamental exact sequences of Weyl modules, where \( C \) is the Weyl module we want to resolve, and \( A \) and \( B \) are ‘simpler’ Weyl modules whose resolutions we already know. Because the notation for these modules can get a little complicated, we will first develop standard notation for the three-rowed case, and then give an heuristic argument for the exactness of a certain class of sequences of Weyl modules. After that, we will proceed to the general \( n \)-rowed case. We will not give a rigorous proof of the exactness of these sequences; a complete proof may be found in [4], [12], [15] or [16].

3.1. Three-rowed Case. Assume that we have the three-rowed shape:

\[
\begin{array}{c}
  & t_1 & \\
  t_2 &  & \sigma \\
  & p_1 & \\
  & & p_2 \\
  & & p_3
\end{array}
\]

As it stands, it looks as though \( t_1 \) and \( t_2 \) are both non-negative. We’ll assume that \( t_1 \) is indeed non-negative, but let’s allow \( t_2 \) to be positive, negative or zero; set \( \tau = \min(0, -t_2) \). (This convention about \( t_2 \) is different from the one we used in the Introduction, where we used the letter \( s \) as well as \( t_2 + 1 \) in first introducing almost skew-shapes. It is, however, the one that we’ll adopt from now on.) We’ll assume, also, that \( p_3 - t_2 \leq p_2 \).

The Weyl map, whose image is the Weyl module associated to the shape, is a map from \( D_{p_1} \otimes D_{p_2} \otimes D_{p_3} \) into a tensor product of exterior powers. It is at this point that we want to talk about positive letters and negative places. Rather than do this in complete generality, we will first look at an example, and leave it to the reader to consult [8] and [11] for more detail.

Suppose we look at the almost skew-shape given by \((4, 3, 2)/(1, 0, 1)\). In this case, \( p_1 = p_2 = 3 \), and \( p_3 = 1 \). We also have \( t_1 = 1 \) and \( t_2 = -1 \). In pictures, this looks like:
The Weyl map, then, is a map from $D_3 \otimes D_3 \otimes D_1$ into $\Lambda^1 \otimes \Lambda^3 \otimes \Lambda^2 \otimes \Lambda^1$. To describe the tensor product, $\Lambda^1 \otimes \Lambda^3 \otimes \Lambda^2 \otimes \Lambda^1$, we will use four places, but they will be negative (i.e., the algebra of these places is the exterior algebra on the basis consisting of these negative places). Let us denote them by $1', 2', 3'$ and $4'$. To make what we are going to say about this example clear, we will look at what we mean by a tableau involving positive letters and negative places as an element in the tensor product of exterior powers.

If we take a word, $w$, in $D_3$, say, we will let the symbol $[w|1'2'3']$ stand for the element in $\Lambda^1 \otimes \Lambda^3 \otimes \Lambda^2 \otimes \Lambda^1$ obtained by diagonalizing $w$ three-fold into (necessarily) linear components. Then a double tableau of the type

$$\begin{array}{c|c}
  w_1 & 1'2'3' \\
  w_2 & 1'3' \\
\end{array}$$

is an element in $\Lambda^2 \otimes \Lambda^1 \otimes \Lambda^2$, where $w_1$ and $w_2$ have been diagonalized and then multiplied in the exterior algebra. For example, we take $w_1$ to be $a^{(3)}$ and $w_2$ to be $b^{(2)}$. Then

$$\begin{array}{c|c c}
  a^{(3)} & 1'2'3' \\
  b^{(2)} & 1'3' \\
\end{array} = a \wedge b \otimes a \otimes a \wedge b.$$
of the form
\[
\begin{bmatrix}
w_1 & t_1 - \tau + 1, t_1 - \tau + 2, \ldots, t_1 - \tau + p_1 \\
w_2 & 1 - \tau, 2 - \tau, \ldots, p_2 - \tau \\
w_3 & 1 - t_2 - \tau, 2 - t_2 - \tau, \ldots, p_3 - t_2 - \tau
\end{bmatrix}.
\]
where the \(w's\) are words in a positive letter alphabet and the numerals are negative places. In general, we'll denote this Weyl module by \([p_1, p_2, p_3; t_1, t_2]\). (In most instances it will be the case that \(t_1 + t_2 \geq 0\), but for technical reasons we won't insist on that. If we happen on a shape where \(t_1 + t_2 < 0\), it can be easily shown that the corresponding Weyl module is isomorphic to the module \([p_1, p_2, p_3; t_1, -t_1]\). Notice that if \(t_2 \geq 0\), we have a skew shape, and that \(\tau = -t_2\). In that case, the places in the first row start at \(t_1 + t_2 + 1\) and go up to \(t_1 + t_2 + p_1\); the places in the second row start at \(1 + t_2\) and go up to \(p_2 + t_2\), while in the third row the places go from 1 to \(p_3\). If \(t_2 < 0\), then \(\tau = 0\) and the first row places start at \(t_1 + 1\) ending at \(t_1 + p_1\), the second row places go from 1 to \(p_2\), while the third row places go from \(1 - t_2\) to \(p_3 - t_2\). It's convenient to identify the module just described with the module one obtains by adding a fixed constant to all the negative places.

3.2. Heuristic Proof of Exactness (Three Rows). Given the Weyl module \([p_1, p_2, p_3; t_1, t_2]\), with \(t_1 + t_2 \geq 0\), we always have the map
\[
[p_1, p_2, p_3; t_1, t_2 + 1] \xrightarrow{\partial} [p_1, p_2, p_3; t_1, t_2]
\]
given by polarizing the place \(1 - (t_2 + 1) - \tau' (= -t_2 - \tau')\) to \(p_3 + 1 - t_2 - \tau\), where
\[
\tau' = \min(0, -(t_2 + 1)).
\]
We should notice that when \(t_2 \geq 0\), the result of this polarization makes all the negative places \(\geq 2\). By subtracting 1 from all the negative places, we end up in \([p_1, p_2, p_3; t_1, t_2]\).

We're going to show the following:

**Proposition 3.2.** We have a short exact sequence
\[
0 \to X \to [p_1, p_2, p_3; t_1, t_2 + 1] \xrightarrow{\partial} [p_1, p_2, p_3; t_1, t_2] \to 0,
\]
where
\[
X = [p_1, p_2 + t_2 + 1, p_3 - t_2 - 1; t_1 + t_2 + 1, -(t_2 + 1)] \text{ when } t_2 \geq 0;
\]
\[
X = [p_1 + t_1 + t_2 + 1, p_2, p_3 - t_1 - t_2 - 1; -t_2 - 1, -t_1] \text{ when } t_2 < 0.
\]

**Proof.** (heuristic) From the remarks made above, we see that the module \(X\) is isomorphic to \([p_1 + t_1 + t_2 + 1, p_2, p_3 - t_1 - t_2 - 1; -t_2 - 1, t_2 + 1]\). The strategy behind showing the exactness of this sequence is this: we assume that we are working with a three-letter positive alphabet, so that our words \(w_1, w_2, \text{ and } w_3\) are \(a^{(p_1)}\), \(b^{(p_2)}\), \(c^{(p_3)}\) respectively. We also assume that the letters in the tableaux we work with are row-increasing. We don't insist that the columns be increasing, but the entries in a given column must be distinct, or else the tableau is zero.

Now suppose that \(t_2 \geq 0\) and consider a tableau in \([p_1, p_2, p_3; t_1, t_2 + 1]\). What must it be like in order to be in the kernel of the map \(\partial\)? We must
have

\[
\begin{bmatrix}
a^{(p_1)} & t_1 + t_2 + 2, \ldots, t_1 + t_2 + 1 + p_1 \\
b^{(p_2)} & t_2 + 2, \ldots, t_2 + 1 + p_2 \\
b^{(t_2+1)c(p_3-t_2-1)} & 1, \ldots, p_3
\end{bmatrix}.
\]

When we say “we must have”, we’re following the philosophy of ‘parsimony’: what is the least change necessary to produce an element of the kernel? Since the \(a\)'s are not necessarily engaged by the bottom row when we push it one step to the right, we must engage the \(b\)'s. We could precede a single \(b\) in the bottom row by \(a\)’s, but that isn’t as parsimonious as we like to be. Clearly, this is not a proof; it is the ‘heuristic’ argument that we refer to in the heading. Assuming that it is valid, it is clear that the tableaux of the above form are equal to the tableaux that describe the module \(X\) of our theorem when \(t_2 \geq 0\).

We now take a look at the case when \(t_2 < 0\). To be in the kernel of our map \(\partial\), our tableau in \([p_1, p_2, p_3; t_1, t_2 + 1]\) must be of the form

\[
\begin{bmatrix}
a^{(p_1)} & t_1 + 1, t_1 + 2, \ldots, t_1 + p_1 \\
b^{(p_2)} & 1, 2, \ldots, p_2 \\
a^{(t_1+t_2+1)c(p_3-t_2-1)} & -t_2, 1 - t_2, \ldots, p_3 - 1 - t_2
\end{bmatrix}.
\]

Here again, “must be” has to be read with caution. In any event, we can’t have any \(b\)'s in the bottom row, since they will get killed by those in the second row. So we have to have \(a\)'s, and that does the trick when the bottom row is slid over one box to the right. These tableaux are seen to be equal to the ones that we use to describe the module \(X\) in the case \(t_2 < 0\). 

3.3. The n-rowed Case. We now go to the general case of an \(n\)-rowed shape whose Weyl module, corresponding to the shape

\[
\begin{array}{cccc}
t_1 & & & p_1 \\
\vdots & & \ddots & \\
t_{n-2} & & & p_{n-2} \\
t_{n-1} & & & p_{n-1} \\
\end{array}
\]

will be denoted by \([p_1, p_2, \ldots, p_n; t_1, \ldots, t_{n-1}]\). As before, we assume that \(t_i \geq 0\) for \(i = 1, \ldots, n-2\), and that \(p_n - t_{n-1} \leq p_{n-1}\). As in the three-rowed case, it will usually be assumed that \(t_1 + \cdots + t_{n-1} \geq 0\), but we won’t insist that this always be true. Also as above, the module that we’re talking about
is the one generated by all tableaux of the form
\[
\begin{pmatrix}
  w_1 & 1 + \sigma_1 - \tau, \cdots, p_1 + \sigma_1 - \tau \\
  \vdots & \vdots \\
  w_{n-i} & 1 + \sigma_{n-i} - \tau, \cdots, p_{n-i} + \sigma_{n-i} - \tau \\
  \vdots & \vdots \\
  w_{n-1} & 1 - \tau, \cdots, p_{n-1} - \tau \\
  w_n & 1 - t_{n-1} - \tau, \cdots, p_n - t_{n-1} - \tau \\
\end{pmatrix},
\]
where \(\sigma_{n-i} = t_{n-2} + \cdots + t_{n-i}\). For future use, we’ll set \(\sigma_n = \sigma_{n-1} = 0\).

**Definition 3.1.** If \(t_1 + \cdots + t_{n-1} \geq 0\), we say that the module is of type \(i\) for \(i = 0, 1, \ldots, n-2\) if \(i\) is the smallest integer such that \(t_{n-1-i} + \cdots + t_{n-1} \geq 0\). If \(t_1 + \cdots + t_{n-1} < 0\), we will say the module is of type \(n-1\).

If the module is of type 0, we have a skew shape. If the module is of type 1, then this means that the penultimate row “completely covers” the bottom row, while the row above that one does not. By “completely covers” we mean that this row extends properly to the left beyond the row being covered (i.e., it can’t be flush). In general, if the module is of type \(i\), this means that row \(n-i\) completely covers the bottom row, but row \(n-i-1\) does not. The condition that \(t_1 + \cdots + t_{n-1} \geq 0\) ensures that the top row doesn’t completely cover the bottom one. If that condition is absent, the top row does completely cover the bottom one but, as in the preceding section, such a module is seen to be isomorphic to the one obtained by moving the bottom row flush with the top one (on the left). To make it a little clearer, when \(t_{n-1} < 0\), the shape (of type \(i\)) that we’re dealing with is

\[
\begin{array}{cccccccc}
  & t_1 & \cdots & \cdots & \cdots & \cdots & p_n \\
  & p_1 & & & & & \\
  & p_2 & & & & & \\
  & \vdots & & & & & \\
  & t_{n-i} & \cdots & \cdots & \cdots & & \\
  & \vdots & & & & & \\
  & p_{n-i} & \cdots & \cdots & \cdots & & \\
  & \vdots & & & & & \\
  & t_{n-2} & \cdots & \cdots & \cdots & & \\
  & \vdots & & & & & \\
  & p_{n-2} & \cdots & \cdots & \cdots & & \\
  & p_{n-1} & \cdots & \cdots & \cdots & & \\
  & p_n & \cdots & \cdots & \cdots & & \\
  & -t_{n-1} & \cdots & \cdots & \cdots & & \\
\end{array}
\]

In this picture, we also have \(t_1 + \cdots + t_{n-1} \geq 0\). If the inequality were reversed, we could consider the shape in which we set \(t'_{n-1} = -(t_1 + \cdots + t_{n-2})\). This would then have the left side of the bottom row ‘flush’ with the left end of the top row. The corresponding Weyl modules would be isomorphic.

**Theorem 3.3.** Let \([p_1, p_2, \ldots, p_n; t_1, \ldots, t_{n-1}]\) be an \(n\)-rowed shape, with \(t_1 + \cdots + t_{n-1} \geq 0\), and suppose it is of type \(i\). Then we have a canonical surjection

\[
[p_1, p_2, \ldots, p_n; t_1, \ldots, t_{n-1} + 1] \xrightarrow{\partial} [p_1, p_2, \ldots, p_n; t_1, \ldots, t_{n-1}]
\]
obtained by polarizing the place \( 1 - (t_{n-1} + 1) - \tau' \) to \( p + 1 - t_{n-1} - \tau \), where \( \tau' = \min(0, -(t_{n-1} + 1)) \). The kernel of this surjection is the module \( \mathbf{X} \), where

\[
\mathbf{X} = \begin{pmatrix}
[p_1, \ldots, p_{n-i-2}, p_{n-i-1} + \sigma_{n-i-1} + t_{n-i-1} + 1, \\
p_{n-i}, \ldots, p_{n-1}, p_{n-1} - (\sigma_{n-i} + t_{n-i}) + 1, \\
t_1, \ldots, t_{n-i} + \sigma_{n-i-1} + t_{n-i-1} + 1, \\
-(\sigma_{n-i} + t_{n-i-1} + 1), t_{n-i}, \ldots, t_{n-2} - (\sigma_{n-i-1})
\end{pmatrix}.
\]

**Proof.** (heuristic) The reader should notice that when \( i = 0 \) (i.e., we have a skew shape), this becomes

\[
\mathbf{X} = \begin{pmatrix}
p_1, \ldots, p_{n-2}, p_{n-1} + t_{n-1} + 1, p_{n} - t_{n-1} - 1; \\
t_1, \ldots, t_{n-3}, t_{n-2} + t_{n-1} + 1, -(t_{n-1} + 1)
\end{pmatrix}.
\]

The proof proceeds very much as in the three-rowed case (which makes it equally “valid”). Assume we have an \( n \)-letter positive alphabet, and we impose the same frugality conditions on our tableaux that we did in the previous case. So, we assume we have a tableau in \([p_1, p_2, \ldots, p_n; t_1, \ldots, t_{n-1} + 1]\) generically of the form

\[
\begin{pmatrix}
\begin{array}{c|c}
& 1 + \bar{\sigma}_1, \ldots, p_1 + \bar{\sigma}_1 \\
\hline
b_{(p_1)}^{(p_1)} & \\
& \vdots \\
b_{(p_{n-i})}^{(p_{n-i})} & 1 + \bar{\sigma}_{n-i}, \ldots, p_{n-i} + \bar{\sigma}_{n-i} \\
& \vdots \\
b_{(p_{n-1})}^{(p_{n-1})} & 1, \ldots, \bar{p}_{n-1} \\
b_{(p_n)}^{(p_n)} & 1 - (t_{n-1} + 1), \ldots, p_n - (t_{n-1} + 1)
\end{array}
\end{pmatrix}
\]

where \( \bar{x} = x - \tau' \), \( \tau' = \min(0, -(t_{n-1} + 1)) \), and we ask under what conditions it lies in the kernel of \( \partial \).

To make things a little clearer, let’s first take the case where \( t_{n-1} \geq 0 \), i.e., a skew shape. In that case, we may replace \( b_{(p_n)}^{(p_n)} \) in the bottom row by \( b_{(p_{n-1}+1)}^{(p_{n-1})}b_{(p_{n-1})}^{(p_{n-1})} \) and we see that, by sliding this row one box to the right (this is essentially the map \( \partial \)), we immediately get two overlapping \( (b_{n-1})'s, \) and thus zero. We also see, in the same way that we saw for the three-rowed case, that this tableau is the ‘generic’ tableau for the shape \( \mathbf{X} \).

Now we assume that \( t_{n-1} < 0 \), and that our original shape is of type \( i > 0 \). Then our bottom row is completely covered by row \( n - i \), but row \( n - i - 1 \) doesn’t completely cover it. In that case, the bottom row of \([p_1, p_2, \ldots, p_n; t_1, \ldots, t_{n-1} + 1]\) does protrude from under row \( n - i - 1 \) (it might have been flush before) but doesn’t protrude beyond row \( n - i \). Thus, if we replace the bottom row of \((*)\) by \( b_{(p_{n-i+1}+t_{n-1}+1)}^{(p_{n-i+1})}b_{(p_{n-i+1})}^{(p_{n-i+1})} \), we have a non-zero tableau which lies in the kernel of \( \partial \) due to the fact that when it is slid one box to the right, we have an overlap of \( (b_{n-i-1})'s. A
calculation identical to that done in the three-rowed case tells us that this tableau is equal to the generic tableau for the shape $X$, and we’re done.

4. Bar Complexes

One of the fundamental constructions of homological algebra is the bar resolution, which has been successfully applied to obtain projective resolutions in several contexts. By putting together a simple bar complex with our place polarizations, we were able to produce projective resolutions of Weyl modules associated to two-rowed skew-shapes (see [5]).

In trying to extend the method we used in the two-rowed case to obtain a characteristic-free projective resolution of the general Weyl module, we were led to consider a very wide generalization of the bar construction (see [6]). This construction, which we called the differential bar complex, is of independent interest, and may potentially be applicable to a large variety of contexts. For the purposes of this paper, we will briefly summarize what appeared in [6], modify some of the definitions, and expand on some of the details that were omitted in that short note. Before that, however, we will review the definition of some of the basic complexes that have been used in connection with resolutions. Then we will discuss the two-rowed case that was dealt with in [5], and conclude the section with a description of the syzygies of the two-rowed skew-shape.

4.1. The (Arithmetic) Koszul and Bar Complexes. The usual Koszul complex is a complex associated to a map, $f$, from an $R$-module, $M$, to the commutative ring, $R$, and is often denoted $K(f)$. It is formed by taking, for each non-negative integer, $l$, the $l$th exterior power of $M$, $\Lambda^l(M)$, and mapping it into $\Lambda^{l-1}(M)$ as follows:

$$\partial_f : \Lambda^l(M) \to \Lambda^{l-1}(M) \text{ sends } m_1 \wedge \cdots \wedge m_l \text{ to }$$

$$\sum_{j=1}^{l} (-1)^{j-1} f(m_j) m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_l \text{ if } l > 0.$$ 

Here the circumflex over the $m_j$ indicates, as usual, that $m_j$ is to be omitted. Note that if $l = 1$, then $\Lambda^{l-1}(M) = \Lambda^0(M) = R$, and that in this case, the map $\partial_f$ is $f$ itself. If $M$ is a free $R$-module of rank $n$ with basis $\{x_1, \ldots, x_n\}$, then a map $f$ is completely determined by where the elements $x_i$ are sent, and it’s enough to define the boundary map on the elements of the form $x_{i_1} \wedge \cdots \wedge x_{i_l}$, with $i_1 < \cdots < i_l$ as above, namely:

$$x_{i_1} \wedge \cdots \wedge x_{i_l} \to \sum_{j=1}^{l} (-1)^{j-1} f(x_{i_j}) x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_l}.$$ 

The arithmetic Koszul complex is one that is associated with two integers, $n$ and $k$, in the following way. The integer $n$ is the rank of the free $\mathbb{Z}$-module $\mathbb{Z}^n$ of rank $n$, and the integer $k$ is what introduces the ‘arithmetic’. The
terms of the complex we are about to define are the exterior powers of \( \mathbb{Z}^n \), but the boundary maps are a bit peculiar and are defined as follows:

\[
\delta_f : \Lambda^l(\mathbb{Z}^n) \to \Lambda^{l-1}(\mathbb{Z}^n) \text{ sends } x_{i_1} \wedge \cdots \wedge x_{i_l} \text{ to }
\]

\[
\sum_{j=1}^l (-1)^{l-j} \left( i_{j+1} - i_{j-1} \right) x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_l} \text{ if } l > 0.
\]

Since the index of summation, \( j \), is running from 1 to \( l \), we have to agree to set \( i_{j+1} = k \), and \( i_0 = 0 \). Notice that when \( l = 1 \), the map \( \delta_f : \mathbb{Z}^n \to \mathbb{Z} \) is the one that sends the canonical basis element \( x_i \) to the binomial coefficient \( \binom{k}{i} \). Thus, the boundary map isn’t really the one that we would obtain as \( \partial_f \); it’s a slight variation that depends on the arithmetic of the binomial coefficients.

Finally, we come to the bar complex which is even more classical than the Koszul Complex. It arises in the following way: If \( \Lambda \) is an algebra over the commutative ring, \( R \), and \( M \) a \( \Lambda \)-module, then one can form the following complex:

\[
\cdots \to \Lambda \otimes \cdots \otimes \Lambda \otimes M \to \Lambda \otimes \cdots \otimes \Lambda \otimes M \to \cdots \to \Lambda \otimes M \to M,
\]

where the map \( \Lambda \otimes M \to M \) is simply the action of \( \Lambda \) on \( M \) and, in general, the map \( \Lambda \otimes \cdots \otimes \Lambda \otimes M \to \Lambda \otimes \cdots \otimes \Lambda \otimes M \) is defined by

\[
\lambda_1 \otimes \cdots \otimes \lambda_l \otimes m \to \sum_{j=1}^{l-1} (-1)^{j-1} \lambda_1 \otimes \cdots \lambda_j \lambda_{j+1} \otimes \cdots \otimes \lambda_l \otimes m + (-1)^{l-1} \lambda_1 \otimes \cdots \otimes \lambda_{l-1} \otimes \lambda_l m.
\]

In the case of a graded algebra, \( \Lambda \), one usually restricts the degrees of the \( \lambda_j \) to be positive (it is usually also assumed that all the \( \lambda \)'s are homogeneous and that the module, \( M \), is graded). Observe, too, that in the graded case, we get a subcomplex by stipulating that the degree of the \( \lambda_1 \) should always be greater than or equal to some fixed integer, \( s \). Another useful observation is that we may always take as our module \( M \) the algebra \( \Lambda \) itself.

There is a natural connection between the arithmetic Koszul complex and the bar complex. Consider the divided power algebra, \( \Lambda = D(\varepsilon) \), on one generator, \( \varepsilon \), over the integers; this is a graded algebra. It is the algebra which, in each degree, \( k \), is free on one generator, denoted \( \varepsilon^{(k)} \), and with multiplication given by \( \varepsilon^{(k)} \varepsilon^{(l)} = \binom{k+l}{k} \varepsilon^{(k+l)} \). Then for every pair of integers, \( t, s \), with \( s \leq t \), we have the sub bar complex given by

\[
\cdots \to \sum_{t-|q| \geq s, q_i > 0} D_{t-|q|}(\varepsilon) \otimes D_{q_1}(\varepsilon) \otimes D_{q_2}(\varepsilon) \to \sum_{t-|q| \geq s, q > 0} D_{t-q}(\varepsilon) \otimes D_{q}(\varepsilon) \to D_t(\varepsilon).
\]
Since each $D_r(\varepsilon)$ is of rank one, we see that the rank of $\sum_{t-q \geq s,q>0} D_{t-q}(\varepsilon) \otimes D_q(\varepsilon)$ is $t-s$. This suggests that we might consider the arithmetic Koszul complex associated to the integers $t-s$ and $k$ where $k$ is equal to $t$ (since the elements $\varepsilon^{(t-q)} \otimes \varepsilon^{(q)}$ map to $(\varepsilon_1^{(t)})$), and see if it is indeed isomorphic to this bar complex. It’s fairly straightforward to see that the correspondence

$$x_{i_1} \wedge \cdots \wedge x_{i_t} \rightarrow \varepsilon^{(t-i_1)} \otimes \varepsilon^{(i_1-i_2)} \otimes \cdots \otimes \varepsilon^{(i_{t-1}-i_t)} \otimes \cdots \otimes \varepsilon^{(i_1)}$$

is an isomorphism between the two complexes.

### 4.2. The Differential Bar Complex

The main components of the construction of this complex are the exterior algebra over $\mathbb{Z}$, $\Lambda(S)$, on a set of free generators, $S$, called the *separators*, an algebra, $A$, an $A$-module, $M$, and the free product of $\Lambda(S)$ with $A$. We will recall the definition given in [6].

Let $A$ be an associative algebra with identity, and let $S$ be a set, whose elements will be called *separators*. The algebra $\Lambda(S)$ has a natural $\mathbb{Z}_2$-grading: if $m$ is a monomial in $\Lambda(S)$, that is, a product of generators, we set $|m| = 0$ if $m$ is the product of an even number of generators, and $|m| = 1$ if $m$ is the product of an odd number of generators.

**Definition 4.1.** The free product of the algebra $A$ and the algebra $\Lambda(S)$ will be called the *bar* algebra on the Algebra $A$, with set of separators $S$, and denoted by $\text{Bar}(A; S)$.

The algebra $\text{Bar}(A; S)$ inherits a $\mathbb{Z}_2$-grading defined as follows. Every element of $\text{Bar}(A; S)$ is a linear combination of elements of the form

$$W = w_1 m_1 w_2 m_2 \cdots w_k m_k$$

where the $m_i$ are non-zero monomials in $\Lambda(S)$, and where the $w_i$ are elements of $A$. (The monomial $m_k$ can, of course, be equal to a scalar, and $w_1$ may be the identity element of $A$.) We set $|W| = 0$ if $|m_1 m_2 \cdots m_k| = 0$ and $|W| = 1$ if $|m_1 m_2 \cdots m_k| = 1$. One extends this definition by linearity to a $\mathbb{Z}_2$-grading of the algebra $\text{Bar}(A; S)$. Notice that $\text{Bar}(A; S)$ is a two-sided $A$-module in a natural way.

For every finite subset $T$ of the set $S$ of separators, the underlying module of the algebra $\text{Bar}(A; S)$ has a grading, which will be called the *$T$-grading* of $\text{Bar}(A; S)$, and which is defined as follows. The submodule $\text{Bar}(A; S; T, i)$ of $T$-degree $i$ is spanned by all elements of the form $(\ast)$ where the integer $i$ equals the total number of occurrences of separators in the set $T$ in the sequence $(m_1, m_2, \ldots, m_k)$. Clearly, the submodule $\text{Bar}(A; S; T, i)$ is a two-sided $A$-module.

Recall that for every separator $x$, there exists a unique antiderivation $\partial_x$ of the algebra $\Lambda(S)$, such that $\partial_x(x) = 1$ (where 1 is the identity of the exterior algebra $\Lambda(S)$), and $\partial_x(y) = 0$ for $y$ in $S$, and $y \neq x$. Recall also that $(\partial_x)^2 = 0$ and $\partial_x \partial_y = -\partial_y \partial_x$.

The antiderivation $\partial_x$ uniquely extends to an antiderivation of the $\mathbb{Z}_2$-graded algebra $\text{Bar}(A; S)$, again denoted by $\partial_x$, defined as follows. If $W$ is
as in (*), set $\partial_x(x) = 1$ (where 1 is now the identity of the algebra $\text{Bar}(A; S)$), and

$$
\partial_x(W) = w_1\partial_x(m_1)w_2m_2 \cdots w_km_k \\
+ (-1)^{|m_1|}w_1m_1w_2\partial_x(m_2) \cdots w_km_k + \cdots \\
+ (-1)^{\sum_{i=1}^{k-1}|m_i|}w_1m_1w_2m_2 \cdots w_k\partial_x(m_k).
$$

The antiderivation $\partial_x$ is well-defined. Again, we have $(\partial_x)^2 = 0$ and $\partial_x\partial_y = -\partial_y\partial_x$.

**Definition 4.2.** If $T$ is a non-empty finite subset of $S$, the operator $\partial_T = \sum_{x \in T} \partial_x$, is called the $T$-boundary operator.

The boundary operator $\partial_T$ maps $\text{Bar}(A; S; T, i + 1)$ into $\text{Bar}(A; S; T, i)$, for $i = 0, 1, 2, \ldots$

Now let $M$ be a left $A$-module. If $w$ is an element of $A$, we denote the action of $w$ on an element $v \in M$ by $w(v)$.

**Definition 4.3.** The free bar module of the $A$-module $M$, with set of separators $S$, denoted by $\text{Bar}(M; A; S)$, is the $\text{Bar}(A; S)$-module $\text{Bar}(A; S) \otimes_A M$.

**Remark 4.1.** The following two observations are easy, but worth underlining:

1. The module $\text{Bar}(M; A; S)$ is spanned by all elements of the form

$$w_1m_1w_2m_2 \cdots w_km_k \otimes v.$$

If $m_k = 1$, then

$$w_1m_1w_2m_2 \cdots w_k \otimes v = w_1m_1w_2m_2 \cdots m_{k-1} \otimes w_k(v),$$

since the tensor product is taken over $A$.

2. For each separator, $x$, there is a well-defined antiderivation on $\text{Bar}(M; A; S)$, again denoted by $\partial_x$, defined as follows:

$$\partial_x(w_1m_1w_2m_2 \cdots w_km_k \otimes v) = \partial_x(w_1m_1w_2m_2 \cdots w_km_k) \otimes v.$$

It’s worth noting from the equation above that, when $m_k = x$, we get (setting $W = w_1m_1w_2m_2 \cdots w_kx$):

$$\partial_x(W \otimes v) = w_1\partial_x(m_1)w_2m_2 \cdots w_kx \otimes v \\
+ (-1)^{|m_1|}w_1m_1w_2\partial_x(m_2) \cdots w_kx \otimes v + \cdots \\
+ (-1)^{\sum_{i=1}^{k-1}|m_i|}w_1m_1w_2m_2 \cdots m_{k-1} \otimes w_k(v).$$

In short, we see that if all of the $m_i$ were equal to $x$, the operator $\partial_x$ would be essentially the boundary operator of the standard bar complex.

That the antiderivation $\partial_x$ is well-defined, is clear. Again, we have $(\partial_x)^2 = 0$ and $\partial_x\partial_y = -\partial_y\partial_x$.

If $T$ is a non-empty finite subset of $S$, the operator $\partial_T = \sum_{x \in T} \partial_x$, is called the $T$-boundary operator. As with $\text{Bar}(A; S; T, i)$, one can also define $\text{Bar}(M, A; S; T, i)$.
The boundary operator $\partial_T$ maps $\text{Bar}(M, A; S; T, i+1)$ into $\text{Bar}(M, A; S; T, i)$, for $i = 0, 1, 2, \ldots$.

**Example 4.1.** Let $S$ be a one-element set, containing the element $x$. Then the module $\text{Bar}(M, A; S, i)$ is spanned by all elements of the form

$$w_1xw_2x\ldots w_i x \otimes v \quad (***)$$

and the derivation $\partial_x$ is computed as follows:

$$\partial_x(w_1xw_2x\ldots w_i x \otimes v) = w_1w_2x\ldots w_i x \otimes v + \cdots - w_1xw_2xw_3x\ldots w_i x \otimes v + \cdots + (-1)^{i-1}w_1xw_2x\ldots xw_i v.$$ 

Thus, the free bar module on $M$ with a single separator gives rise to the classical bar complex, as one sees by replacing the symbol $x$ by the symbol ‘|’. What we have called $\text{Bar}(M, A; S, i)$ are simply the chains in dimension $i$.

**Example 4.2.** Let $A$ be a connected graded algebra with identity. The submodule of the module of the example above, spanned by elements of the form (***) where $w_1, w_2, \ldots, w_i$ are all of positive degree gives the classical normalized bar construction of $A$ and $M$ with set of separators $S = \{x\}$.

**Example 4.3.** With $A$ and $M$ as above, assume that the degree of $w_1$ is $> t$. One then obtains the $(t^+)$-submodule (and hence, also, the $(t^+)$-complex) of the normalized bar construction. If in addition $M$ is a graded module, if a positive integer $n$ is fixed, and if in (***) it is assumed that $\deg(w_1) + \deg(w_2) + \cdots + \deg(w_i) + \deg(v) = n$, one obtains a bar module which will be called the $(t^+)$-graded strand of degree $n$.

### 4.3. The Two-Rowed Case.

In this subsection, we will briefly reconstruct the resolution for the two-rowed Weyl module much as it is described in [5]. We do this for two reasons: to illustrate the use of the classical bar complex in our context, and to make explicit the fact that, using letter-place techniques, one can do a bit more than was originally thought. That is, we can not only describe the resolution easily, but equally easily describe a contracting homotopy for the non-negative part of the resolution and a basis for the syzygies.

Recall that the Weyl module associated to the skew-shape

$$\begin{array}{c|c|c|c|c|c|c|c|c}
| & | & | & | & t & | & \cdots & | & p \\hline
& & & & & & & & q
\end{array}$$

is the image of $D_p \otimes D_q$ under the Weyl map. The ‘box map’ referred to earlier in this paper was described in [4] as the map $\sum_{k \geq t} D_{p+k} \otimes D_{q-k} \to D_p \otimes D_q$ which sends an element $x \otimes y$ of $D_{p+k} \otimes D_{q-k}$ to $\sum x_p \otimes x_k y$, where $\sum x_p \otimes x_k$ is the component of the diagonal of $x$ in $D_{p+k} \otimes D_k$. To put this all in letter-place perspective, we see that if we take a double standard
stands for the sum of the indices \( k \) where the symbol ‘bar complex of this algebra acting on \( \sum \) Thus, we may take the (\( t \) module: the second factor determines the grading) to get a complex over the Weyl module? In other words, that it is in fact a resolution. One way is

{1, 2} to negative places \( \{1', \ldots, (p + t)'\} \), as follows: \( \partial_{q2} \cdots \partial_{12} \partial_{(p+t)1'} \cdots \partial_{(t+1)1'} \), where \( \partial_{uv} \) stands for the place polarization from (positive) place \( v \) to (negative) place \( u' \). (This is just the two-rowed version of our discussion in section 3.) If we take an element of \( D_{p+k} \otimes D_{q-k} \), the ‘box map’ sending this to \( D_p \otimes D_q \)

is simply the sum, over positive \( k \), of the \( k^{th} \) divided power of the place polarization taking place 1 to place 2. It is now easy to see that a standard tableau \( \begin{pmatrix} w & w' \\ w & w' \end{pmatrix} \), with \( k > t \), gets carried to zero by the Weyl map (a fact that we mentioned in section 3 without proof).

If we let \( Z_{2,1} \) stand for the generator of a divided power algebra in one free generator, we see that \( Z_{2,1}^{(k)} \) acts on \( D_{p+k} \otimes D_{q-k} \) and carries it to \( D_p \otimes D_q \). Thus, we may take the \( (t^{+})\)-graded strand of degree \( q \) of the normalized bar complex of this algebra acting on \( \sum D_{p+k} \otimes D_{q-k} \) (where the degree of the second factor determines the grading) to get a complex over the Weyl module:

\[
\cdots \rightarrow \sum_{k_i > 0} Z_{2,1}^{(t+k_1)} x Z_{2,1}^{(k_2)} x \cdots x Z_{2,1}^{(k_{i+1})} x \otimes (D_{t+p+|k|} \otimes D_{q-t-|k|}) \rightarrow \\
\sum_{k_i > 0} Z_{2,1}^{(t+k_1)} x Z_{2,1}^{(k_2)} x \cdots x Z_{2,1}^{(k_{i+1})} x \otimes (D_{t+p+|k|} \otimes D_{q-t-|k|}) \rightarrow \\
\sum_{k > 0} Z_{2,1}^{(t+k)} x \otimes (D_{t+p+k} \otimes D_{q-t-k}) \rightarrow D_p \otimes D_q,
\]

where the symbol ‘\( x \)' is our separator variable as described above, and \( |k| \) stands for the sum of the indices \( k_i \). Here, the boundary operator is \( \partial_x \) or, what is the same thing, is obtained by polarizing the variable \( x \) to the element \( 1 \). This, then, describes a left complex over the Weyl module in terms of bar complexes and letter-place algebra. We also know from the fact that the Weyl module is the cokernel of the box map, that the zero-dimensional homology of this complex is the Weyl module itself. Now the question is: how do we show that this complex is an exact left complex over the Weyl module? In other words, that it is in fact a resolution. One way is to produce a splitting contracting homotopy, which is what we will do here. Another way is to use our fundamental exact sequences and a mapping cone argument; we will do that later.

Definition 4.4. With our complex given as above, we define the homotopy as follows:

\[
s_0 : D_p \otimes D_q \rightarrow \sum_{k > 0} Z_{2,1}^{(t+k)} x D_{t+p+k} \otimes D_{q-t-k}
\]
is defined by sending the double standard tableau \( \begin{pmatrix} w & 1^{(p)}_2(k) \\ w' & 2(q-k) \end{pmatrix} \) to zero if \( k \leq t \), and to \( Z_{2,1}^{(k)} x \begin{pmatrix} w & 1^{(p+k)}_2(k) \\ w' & 2(q-k) \end{pmatrix} \) if \( k > t \). For higher dimensions,

\[
\sum_{k_i > 0} Z_{2,1}^{(t+k_1)} xZ_{2,1}^{(k_2)} x \cdots xZ_{2,1}^{(k_t)} x D_{t+p+|k|} \otimes D_{q-t-|k|} \to \sum_{k_i > 0} Z_{2,1}^{(t+k_1)} xZ_{2,1}^{(k_2)} x \cdots xZ_{2,1}^{(k_{i+1})} x D_{t+p+|k|} \otimes D_{q-t-|k|}, \quad l > 0,
\]

is defined by sending \( Z_{2,1}^{(t+k_1)} xZ_{2,1}^{(k_2)} x \cdots xZ_{2,1}^{(k_t)} x \begin{pmatrix} w & 1^{(t+p+k)}_2(m) \\ w' & 2(q-t-|k|-m) \end{pmatrix} \) to zero if \( m = 0 \), and to \( Z_{2,1}^{(t+k_1)} xZ_{2,1}^{(k_2)} x \cdots xZ_{2,1}^{(k_t)} x Z_{2,1}^{(m)} x \begin{pmatrix} w & 1^{(t+p+k)}_2(m) \\ w' & 2(q-t-|k|-m) \end{pmatrix} \) if \( m > 0 \).

**Proposition 4.1.** The collection of maps \( \{s_1\}_{l \geq 0} \) provides a splitting contracting homotopy for the complex above.

**Proof.** We must show two things:

\[
\partial_x s_l + s_{l-1} \partial_x = 1 \quad \text{for} \quad l \geq 1;
\]

\[
s_l s_{l-1} = 0 \quad \text{for} \quad l \geq 1.
\]

That \( s_l s_{l-1} = 0 \) for \( l \geq 1 \) is easy to see since the image of the homotopy map always is zero or involves a tableau having no 2 in the first row.

To establish the first identity, we first consider the case \( l = 1 \). In that case, we have

\[
\partial_x s_1 \begin{pmatrix} Z_{2,1}^{(t+k)} x \begin{pmatrix} w & 1^{(t+p+k)}_2(r) \\ w' & 2(q-t-k-r) \end{pmatrix} \\ \end{pmatrix} = \partial_x Z_{2,1}^{(t+k)} x Z_{2,1}^{(r)} x \begin{pmatrix} w & 1^{(t+p+k+r)}_2(r) \\ w' & 2(q-t-k-r) \end{pmatrix} = Z_{2,1}^{(t+k)} x \begin{pmatrix} w & 1^{(t+p+k)}_2(r) \\ w' & 2(q-t-k-r) \end{pmatrix} - \left( t + k + r \right) \begin{pmatrix} w & 1^{(t+p+k+r)}_2(r) \\ w' & 2(q-t-k-r) \end{pmatrix}
\]

while

\[
s_0 \partial_x \begin{pmatrix} Z_{2,1}^{(t+k)} x \begin{pmatrix} w & 1^{(t+p+k)}_2(r) \\ w' & 2(q-t-k-r) \end{pmatrix} \\ \end{pmatrix} = s_0 \begin{pmatrix} w & 1^{(p)}_2(t+k+2r) \\ w' & 2(q-t-k-r) \end{pmatrix} = \left( t + k + r \right) s_0 \begin{pmatrix} w & 1^{(p)}_2(t+k+2r) \\ w' & 2(q-t-k-r) \end{pmatrix} = \left( t + k + r \right) Z_{2,1}^{(t+k+r)} x \begin{pmatrix} w & 1^{(t+p+k+r)}_2(r) \\ w' & 2(q-t-k-r) \end{pmatrix}.
\]

The sum of these two terms gives us the term we started with. Of course, we have assumed that \( r > 0 \) in the above. If \( r = 0 \), then the image of \( \partial_x s_1 \)
is zero, and the image of \( s_0 \partial_x \) is the term we started with. For \( l > 0 \), the proof proceeds in exactly the same way; the only difference is that there are more terms coming from the boundary maps that have to be matched up.

**Theorem 4.2.** The complex above is a projective resolution of the Weyl module associated to the shape \((A)\), over the Schur algebra of appropriate weight.

**Proof.** Since we know that the Weyl module is the cokernel of the box map, the fact that we have a contracting homotopy for the rest of the complex gives us a quick proof of the fact that the above is a resolution of the Weyl module. Moreover, this homotopy provides us with an explicit description of the basis for the syzygies: we see that in dimension 0 the basis for the syzygies can be taken to be the set:

\[
\left\{ Z_{(t+r)}^{2,1} x \begin{bmatrix} w \\ w' \end{bmatrix} \left| \frac{1(p+t+r)}{2(q-t-r)} \right.; \quad r > 0 \right\},
\]

while in positive dimension, \( l > 0 \), the basis can be taken to be the set:

\[
\left\{ Z_{(t+r_1)}^{2,1} \cdots Z_{(r_l)}^{2,1} x \begin{bmatrix} w \\ w' \end{bmatrix} \left| \frac{1(p+t+|r|)}{2(q-t-|r|)} \right.; \quad r > 0 \right\}.
\]

5. Some Easy Three-Rowed Examples

In this section, we review the resolution of the three-rowed skew-shape having one triple overlap that was written, in slightly different form, in [6]. We then describe the resolutions of the three-rowed almost skew-shape having one triple overlap (quite a different story form the previous case), and the three-rowed skew-shape having two triple overlaps. A full understanding of these cases will help understand the difficulties still inherent in the general case, but will also anticipate the procedure by which we discover the terms of the general resolution.

**5.1. Skew-Shape With No More Than One Triple Overlap.** The first of these easy examples deals with the resolution of the Weyl module associated to the skew-shape

\[
\begin{array}{c}
\text{t}_2 \\
\hline
\text{t}_1 \\
\hline
\text{p}_1 \\
\text{p}_2, \\
\text{p}_3 \\
\end{array}
\]

where we assume that the number of triple overlaps is \( \leq 1 \). This means that \( p_3 \leq t_1 + t_2 + 1 \). The important thing to notice about this class of shapes is that our fundamental exact sequence is very special in that it doesn't entail any almost skew-shapes. Namely, this exact sequence reduces to:

\[
(5.1) \quad 0 \rightarrow [p_1, p_2 + t_2 + 1, p_3 - t_2 - 1; t_1 + t_2 + 1, -(t_2 + 1)]
\]

\[
\rightarrow [p_1, p_2, p_3; t_1, t_2 + 1] \rightarrow [p_1, p_2, p_3; t_1, t_2] \rightarrow 0.
\]
But, since the third row of \([p_1, p_2 + t_2 + 1, p_3 - t_2 - 1; t_1 + t_2 + 1, -(t_2 + 1)]\) doesn’t overlap with the top row, we see that the Weyl module associated to this shape is isomorphic to that associated to the skew-shape \([p_1, p_2 + t_2 + 1, p_3 - t_2 - 1; t_1 + t_2 + 1, 0]\) (see [4]). Hence we have the exact sequence of skew-shapes:

\[
\begin{align*}
(5.2) & \quad 0 \rightarrow [p_1, p_2 + t_2 + 1, p_3 - t_2 - 1; t_1 + t_2 + 1, 0] \\
& \rightarrow [p_1, p_2, p_3; t_1, t_2 + 1] \rightarrow [p_1, p_2, p_3; t_1, t_2] \rightarrow 0.
\end{align*}
\]

We know that the Weyl module is presented by the box map ([4]):

\[
\begin{align*}
\sum_{k>0} D_{p_1+t_1+k} \otimes D_{p_2-t_1-k} \otimes D_{p_3} & \rightarrow D_{p_1} \otimes D_{p_2} \otimes D_{p_3}.
\end{align*}
\]

As in the two-rowed case, the maps \(D_{p_1+t_1+k} \otimes D_{p_2-t_1-k} \otimes D_{p_3} \rightarrow D_{p_1} \otimes D_{p_2} \otimes D_{p_3}\) may be interpreted as the \(k^{th}\) divided power of the place polarization from place 1 to place 2, and the maps \(\sum_{t>0} D_{p_1} \otimes D_{p_2+t_2+t} \otimes D_{p_3-t_2-t} \rightarrow D_{p_1} \otimes D_{p_2} \otimes D_{p_3}\) may be interpreted as the \(t^{th}\) divided power of the place polarization from place 2 to place 3. Again taking our cue from the two-rowed case, we now introduce two generators, \(Z_{2,1}\) and \(Z_{3,2}\) with their divided powers, and we write, in place of the above,

\[
\begin{align*}
\sum_{k>0} Z_{2,1}^{(t_1+k)} x D_{p_1+t_1+k} \otimes D_{p_2-t_1-k} \otimes D_{p_3} & \rightarrow D_{p_1} \otimes D_{p_2} \otimes D_{p_3}, \\
\sum_{t>0} Z_{3,2}^{(t_2+t)} y D_{p_1} \otimes D_{p_2+t_2+t} \otimes D_{p_3-t_2-t} & \rightarrow D_{p_1} \otimes D_{p_2} \otimes D_{p_3},
\end{align*}
\]

where \(x\) and \(y\) stand for separator variables, and the boundary map is the sum of polarizing \(x\) and \(y\) to 1. In short, we see that we have the makings of the free bar module on the set, \(S\), consisting of two separators \(x\) and \(y\), the free associative (non-commutative) algebra, \(A\), generated by \(Z_{2,1}\) and \(Z_{3,2}\), and their divided powers, and the module, \(M\), which is the appropriate direct sum of tensor products of divided power modules \(D_{p} \otimes D_{q} \otimes D_{r}\) for suitable \(p, q\) and \(r\), with the action of \(Z_{2,1}\) and \(Z_{3,2}\) (and their powers) being that of the indicated place polarizations. The boundary map that we use is then \(\partial_x + \partial_y\). However, we do not take the free bar module on these data, but the quotient after dividing out by the following identities (and it is here that we have the first stirrings of the Capelli identities):

\[
\begin{align*}
(5.3) & \quad Z_{32} x = x Z_{32}; \\
(5.4) & \quad Z_{32}^{(a)} Z_{21}^{(b)} x = \sum_{k<b} Z_{21}^{(b-k)} x Z_{32}^{(a-k)} \partial_{31}^{(k)},
\end{align*}
\]

where the symbol \(\partial_{31}^{(k)}\) means the divided power of the usual place polarization, and its presence on the right means that it is to be brought to operate on the element of \(M\) that is to the right of the tensor product sign. For example, if \(W = W' \otimes m\), with \(m \in M\), then \(\partial_{31}^{(k)} W\) means \(W' \otimes \partial_{31}^{(k)} (m)\).
To illustrate: if we have $W = Z_{21}^{(c)} x \otimes m$, then
\[ Z_{32}^{(a)} Z_{21}^{(b)} x W = \sum_{k<b} Z_{21}^{(b-k)} x Z_{32}^{(a-k)} Z_{21}^{(c)} x \otimes \partial_{31}^{(k)} (m). \]

Notice that this in turn becomes
\[
\sum_{k<b,l<c} Z_{21}^{(b-k)} x Z_{21}^{(c-l)} x Z_{32}^{(a-k-l)} \partial_{31}^{(l)} \otimes \partial_{31}^{(k)} (m) = \\
\sum_{k<b,l<c} \left( \frac{k+l}{l} \right) Z_{21}^{(b-k)} x Z_{21}^{(c-l)} x \otimes \partial_{32}^{(a-k-l)} \partial_{31}^{(k+l)} (m),
\]
the last equality being due to the first observation of Remark 4.1 (as well as to multiplication of divided powers).

**Remark 5.1.** The reader may wonder at the fact that we are restricting the index of summation, $k$, to be less than $b$. We do this because in the applications, we want our divided powers of the $Z_{21}$ terms to be positive (so that we remain in a ‘normalized’ complex).

**Definition 5.1.** If we denote by $V$ the element $Z_{32}^{(a)} Z_{21}^{(b)} x W$, where $W$ is in the submodule spanned by all terms of the form $Z_{21}^{(c_1)} x \cdots x Z_{21}^{(c_q)} x \otimes m$, denote by $\nabla$ the term $\sum_{k<b} Z_{21}^{(b-k)} x Z_{32}^{(a-k)} \partial_{31}^{(k)} W$. The module generated by all terms of the form $V - \nabla$ will be denoted by $\text{Cap}(1,2)$.

This last is an inductive definition that requires a little bit of explanation. If $q = 0$, then $\nabla$ is simply $\sum_{k<b} Z_{21}^{(b-k)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(k)} (m)$ (where we are assuming that $W$ is just the element $m$ contained in a tensor product of divided powers). If $W = Z_{21}^{(c)} x \otimes m$, then
\[
\nabla = \sum_{k<b} Z_{21}^{(b-k)} x Z_{32}^{(a-k)} Z_{21}^{(c)} \otimes \partial_{31}^{(k)} (m) = \\
\sum_{k<b} \sum_{l<c} Z_{21}^{(b-k)} x Z_{21}^{(c-l)} x \otimes \partial_{32}^{(a-k-l)} \partial_{31}^{(l)} \partial_{31}^{(k)} (m).
\]

In order to show that we still get a complex if we factor out by the submodule, $\text{Cap}(1,2)$, generated by the relations of the type $V - \nabla$ we have just been discussing, we must show that these relations are carried into $\text{Cap}(1,2)$ by the boundary map. Since the boundary map, $\partial$, is just $\partial_x + \partial_y$, and since the only separator in the terms of the relations is $x$, we need only show that the relations are stable under $\partial_x$.

**Proposition 5.1.** Let $W$ be an element of the bar module of the form $Z_{21}^{(c_1)} x \cdots x Z_{21}^{(c_q)} x \otimes m$, and let $V = Z_{32}^{(a)} Z_{21}^{(b)} x W$. Denote by $\nabla$ the element $\sum_{k<b} Z_{21}^{(b-k)} x Z_{32}^{(a-k)} \partial_{31}^{(k)} W$. Then we have
\[
(5.5) \quad \partial (\nabla) = \partial (V) - Z_{32}^{(a-b)} W.
\]
Consequently, if \( a < b \), then \( \partial(V - \overline{V}) \in \text{Cap}(1, 2) \). More precisely, we have

\[
\partial(V - \overline{V}) = Z_{32}^{(a)} \partial \left( Z_{21}^{(b)} xW \right) - Z_{32}^{(a)} \partial \left( Z_{21}^{(b)} xW \right)
\]

\[
= \partial(V) - \partial(\overline{V}),
\]

and we have the necessary stability of the relations.

**Proof.** If \( q = 0 \), then \( V = Z_{32}^{(a)} Z_{21}^{(b)} x \otimes m \) and \( \overline{V} = \sum_{k < b} Z_{21}^{(b-k)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(k)} (m) \).

Taking \( \partial(V) \), we get

\[
\partial(V) = \partial_{32}^{(a)} \partial_{21}^{(b)} (m).
\]

By 2.1, we know that

\[
\partial_{21}^{(b)} (m) = \sum_{k < b} \partial_{21}^{(b-k)} \partial_{32}^{(a-k)} \partial_{31}^{(k)} (m) + \partial_{32}^{(a-b)} \partial_{31}^{(b)} (m)
\]

\[
= \partial(V) + \partial_{32}^{(a-b)} \partial_{31}^{(b)} (m).
\]

This establishes 5.5 for \( q = 0 \).

We will now consider the case \( q = 1 \); the general case proceeds along similar lines. In this case we have \( V = Z_{32}^{(a)} Z_{21}^{(b)} x Z_{21}^{(c)} x \otimes m \) and

\[
\partial(V) = \sum_{k < b} \sum_{l \leq c} Z_{21}^{(b-k)} Z_{21}^{(c-l)} x \otimes \partial_{32}^{(a-k-l)} \partial_{31}^{(k)} \partial_{31}^{(l)} (m)
\]

\[
- \sum_{k < b} \sum_{l \leq c} Z_{21}^{(b-k)} x \otimes \partial_{21}^{(c-l)} \partial_{32}^{(a-k-l)} \partial_{31}^{(k)} \partial_{31}^{(l)} (m).
\]

We have allowed \( l \) to equal \( c \) in both sums since the extra term added to the first sum is equal to the similar term subtracted in the second. Now

\[
\sum_{k < b} \sum_{l \leq c} Z_{21}^{(b-k)} Z_{21}^{(c-l)} x \otimes \partial_{32}^{(a-k-l)} \partial_{31}^{(k)} \partial_{31}^{(l)} (m) =
\]

\[
\sum_{a \leq b+c-1} \sum_{k < b} Z_{21}^{(b-k)} Z_{21}^{(c-a)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(a)} (m) =
\]

\[
\sum_{a \leq b+c-1} \sum_{b < k} Z_{21}^{(b+c-a)} Z_{21}^{(c-a)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(a)} (m) -
\]

\[
\sum_{a=b} x \otimes \partial_{32}^{(a-a)} \partial_{31}^{(a)} (m).
\]

But

\[
\sum_{a=b} Z_{21}^{(b+c-a)} x \otimes \partial_{32}^{(a-a)} \partial_{31}^{(a)} (m) =
\]

\[
\sum_{a=b} Z_{21}^{(b+c-a)} x \otimes \partial_{32}^{(a-a)} \partial_{31}^{(a)} (m) =
\]

\[
\sum_{a=b} Z_{21}^{(b+c-a)} x \otimes \partial_{32}^{(a-a)} \partial_{31}^{(a)} (m) =
\]

\[
Z_{32}^{(a-b)} Z_{21}^{(c)} x \otimes \partial_{31}^{(b)} (m),
\]

so that we have

\[
\sum_{k < b} \sum_{l \leq c} Z_{21}^{(b-k)} Z_{21}^{(c-l)} x \otimes \partial_{32}^{(a-k-l)} \partial_{31}^{(k)} \partial_{31}^{(l)} (m) =
\]

\[
\sum_{a \leq b+c-1} \sum_{b < k} Z_{21}^{(b+c-a)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(a)} (m) -
\]

\[
Z_{32}^{(a-b)} Z_{21}^{(c)} x \otimes \partial_{31}^{(b)} (m).
\]
Furthermore, induction gives us the fact that
\[
\sum_{k<b} \sum_{l \leq c} Z_{21}^{(b-k)} \partial_{21}^{(c-l)} \partial_{32}^{(a-k-l)} \partial_{31}^{(l)} (m) =
\sum_{k<b} Z_{21}^{(b-k)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(c)} (m) =
Z_{32}^{(a)} Z_{21}^{(b)} x \otimes \partial_{31}^{(c)} (m).
\]
Thus, since
\[
\vartheta (V) = \left( b + c \right) Z_{32}^{(a)} Z_{21}^{(b+c)} x \otimes m - Z_{32}^{(a)} Z_{21}^{(b)} x \otimes \partial_{31}^{(c)} (m),
\]
we get our desired equality for \( q = 1 \).

Moving to the general case, we let
\[
V = \sum_{k<b} \sum_{k_1 < c_1} Z_{21}^{(b-k)} x Z_{21}^{(c_1-k_1)} x Z_{21}^{(a-k_1)} x Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes m,
\]
with \( q \geq 2 \). Then
\[
V = \sum_{k<b} \sum_{k_1 < c_1} Z_{21}^{(b-k)} Z_{21}^{(c_1-k_1)} x Z_{32}^{(a-k_1)} x Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(k_1)} \partial_{31}^{(k)} (m)
\]
and
\[
\vartheta (V) = \sum_{k<b} \sum_{k_1 < c_1} Z_{21}^{(b-k)} Z_{21}^{(c_1-k_1)} x Z_{32}^{(a-k_1)} x Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(k_1)} \partial_{31}^{(k)} (m)
\]
\[
- \sum_{k<b} \sum_{k_1 < c_1} Z_{21}^{(b-k)} x Z_{21}^{(c_1-k_1)} Z_{32}^{(a-k_1)} x Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(k_1)} \partial_{31}^{(k)} (m)
\]
\[
+ \sum_{k<b} \sum_{k_1 < c_1} Z_{21}^{(b-k)} x Z_{21}^{(c_1-k_1)} x \partial \left( Z_{32}^{(a-k_1)} Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(k_1)} \partial_{31}^{(k)} (m) \right)
\]
where, as in the above case, we are allowed to let the index \( k_1 \) run up to \( c_1 \). Now, arithmetic considerations identical to the kind used in that earlier case, tell us that
\[
\sum_{k<b} \sum_{k_1 \leq c_1} Z_{21}^{(b-k)} Z_{21}^{(c_1-k_1)} x Z_{32}^{(a-k_1)} x Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(k_1)} \partial_{31}^{(k)} (m) =
\left( b + c \right) \sum_{a<b} \sum_{\alpha < c-a} Z_{21}^{(b+a-\alpha)} x Z_{32}^{(a-\alpha)} x Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(\alpha)} (m) -
Z_{32}^{(a-b)} Z_{21}^{(c_1)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(b)} (m).
\]
Furthermore, induction gives us the fact that the term
\[
\sum_{k<b} \sum_{k_1 < c_1} Z_{21}^{(b-k)} x Z_{21}^{(c_1-k_1)} x \partial \left( Z_{32}^{(a-k_1)} Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(k_1)} \partial_{31}^{(k)} (m) \right)
\]
is equal to
\[
\sum_{k<b} \sum_{k_1 < c_1} Z_{21}^{(b-k)} x Z_{21}^{(c_1-k_1)} x \partial \left( Z_{32}^{(a-k_1)} Z_{21}^{(c_2)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(k_1)} \partial_{31}^{(k)} (m) \right) -
\sum_{k<b} \sum_{k_1 < c_1} Z_{21}^{(b-k)} x Z_{21}^{(c_1-k_1)} x Z_{32}^{(a-k_1-c_2)} Z_{21}^{(c_3)} x \cdots x Z_{21}^{(c_q)} x \otimes \partial_{31}^{(c_2)} \partial_{31}^{(k_1)} \partial_{31}^{(k)} (m) \]
which in turn is equal to

\[
\sum_{k<b} \sum_{k_1 \leq c_1} Z^{(b-k)}_{21} x Z^{(c_1-k_1)}_{21} x \partial \left( Z^{(a-k-k_1)}_{32} Z^{(c_2)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(k_1)}_{31} \partial^{(k)}_{31} (m) \right) - \\
\frac{Z^{(a-c_2)}_{32} Z^{(b)}_{21} x Z^{(c_1)}_{21} x Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(c_2)}_{31} (m)}{Z^{(a-c_2)}_{32} Z^{(b)}_{21} x Z^{(c_1)}_{21} x Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(c_2)}_{31} (m)}.
\]

Thus, we finally obtain

\[
\partial \left( \mathcal{V} \right) = \left( \begin{array}{c} b+c_1 \\ b \end{array} \right) \sum_{\alpha<b+c} Z^{(b+c-a)}_{21} x Z^{(c_1)}_{32} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(a)}_{31} (m) - \\
\frac{Z^{(a-b)}_{32} Z^{(c_1)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(b)}_{31} (m)}{Z^{(a-b)}_{32} Z^{(c_1)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(b)}_{31} (m)} - \\
\sum_{k<b} \sum_{k_1 \leq c_1} Z^{(b-k)}_{21} x Z^{(c_1-k_1)}_{21} x Z^{(a-k-k_1)}_{32} Z^{(c_2)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(k_1)}_{31} \partial^{(k)}_{31} (m) + \\
\sum_{k<b} \sum_{k_1 \leq c_1} Z^{(b-k)}_{21} x Z^{(c_1-k_1)}_{21} x \partial \left( Z^{(a-k-k_1)}_{32} Z^{(c_2)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(k_1)}_{31} \partial^{(k)}_{31} (m) \right) - \\
\sum_{k<b} \sum_{k_1 \leq c_1} Z^{(b-k)}_{21} x Z^{(c_1-k_1)}_{21} x \frac{Z^{(a-k-k_1-c_2)}_{32} Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(c_2)}_{31} \partial^{(k_1)}_{31} \partial^{(k)}_{31} (m)}{Z^{(a-b)}_{32} Z^{(c_1)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(b)}_{31} (m)}.
\]

On the other hand, it’s easy to see that

\[
\partial \left( \mathcal{V} \right) = \left( \begin{array}{c} b+c_1 \\ b \end{array} \right) \frac{Z^{(a)}_{32} Z^{(b+c_1)}_{21} x Z^{(c_1)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes m - \\
\frac{Z^{(a)}_{32} Z^{(b)}_{21} x Z^{(c_1)}_{21} x Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes m + \\
\frac{Z^{(a)}_{32} Z^{(b)}_{21} x Z^{(c_1)}_{21} x \partial \left( Z^{(c_2)}_{32} x Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes m \right)}{Z^{(a)}_{32} Z^{(b)}_{21} x Z^{(c_1)}_{21} x \partial \left( Z^{(c_2)}_{32} x Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes m \right)}.
\]

Thus in evaluating the difference between \( \partial \left( \mathcal{V} \right) \) and \( \partial \left( \mathcal{V} \right) \), we must evaluate

\[
-Z^{(a-b)}_{32} Z^{(c_1)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(b)}_{31} (m) - \\
\sum_{k<b} \sum_{k_1 \leq c_1} Z^{(b-k)}_{21} x Z^{(c_1-k_1)}_{21} x Z^{(a-k-k_1)}_{32} Z^{(c_2)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(k_1)}_{31} \partial^{(k)}_{31} (m) + \\
\sum_{k<b} \sum_{k_1 \leq c_1} Z^{(b-k)}_{21} x Z^{(c_1-k_1)}_{21} x Z^{(a-k-k_1-c_2)}_{32} Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(c_2)}_{31} \partial^{(k_1)}_{31} \partial^{(k)}_{31} (m) + \\
\frac{Z^{(a)}_{32} Z^{(b)}_{21} x Z^{(c_1)}_{21} x Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes m.}
\]

If our stated result is to be true, we must show that the last three terms add up to zero. Since we have two terms that involve products of \( Z_{21} \) terms, we have to do a little more addition of the type we have already done a good deal of. First of all, we know that

\[
\frac{Z^{(a)}_{32} Z^{(b)}_{21} x Z^{(c_1)}_{21} x Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes m \cdot \\
\left( \begin{array}{c} c_1 + c_2 \\ c_1 \end{array} \right) \sum_{k<b} \sum_{\beta \leq c_1 + c_2} Z^{(b-k)}_{21} x Z^{(c_1+c_2-\beta)}_{21} x Z^{(a-k-\beta)}_{32} Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes \partial^{(\beta)}_{31} \partial^{(k)}_{31} (m)}{Z^{(a)}_{32} Z^{(b)}_{21} x Z^{(c_1)}_{21} x Z^{(c_3)}_{21} x \cdots x Z^{(c_9)}_{21} x \otimes m.
\]

while
\[
\sum_{k < b} \sum_{k_1 \leq c_1} \sum_{c_2} Z_2^{(b-k)} x Z_2^{(c_1-k_1)} Z_3^{(a-k-k_1)} Z_2^{(c_2)} x \cdots x Z_2^{(c_3)} x \otimes \partial_3^{(k_1)} \partial_3^{(k)} (m)
\]

\[
= \sum_{k < b} \sum_{k_1 \leq c_1} \sum_{c_2} Z_2^{(b-k)} x Z_2^{(c_1-k_1)} Z_3^{(a-k-k_1)} Z_2^{(c_2)} x \cdots x Z_2^{(c_3)} x \otimes \partial_3^{(k_1)} \partial_3^{(k)} (m).
\]

By our now familiar argument, using the fact that our index \( k_1 \) is not restricted, we arrive at the fact that
\[
\sum_{k < b} \sum_{k_1 \leq c_1} \sum_{c_2} Z_2^{(b-k)} x Z_2^{(c_1-k_1)} Z_3^{(a-k-k_1)} Z_2^{(c_2)} x \cdots x Z_2^{(c_3)} x \otimes \partial_3^{(k_1)} \partial_3^{(k)} (m) =
\]
\[
\sum_{k > 0} \sum_{k_1 \leq c_1} \sum_{c_2} Z_2^{(b-k)} x Z_2^{(c_1-k_1)} Z_3^{(a-k-k_1)} Z_2^{(c_2)} x \cdots x Z_2^{(c_3)} x \otimes \partial_3^{(k_1)} \partial_3^{(k)} (m) -
\]
\[
Z_3^{(a-c_2)} Z_2^{(b)} x Z_2^{(c_1)} x \cdots x Z_2^{(c_3)} x \otimes \partial_3^{(c_2)} (m).
\]

From here it is an easy step to get our basic equality 5.5.

The final statement of our proposition, for \( a < b \), is clear. \( \blacksquare \)

**Definition 5.2.** We define the module \( M_n ([a_1, p_2, p_3; t_1, t_2]) \), for \( n > 0 \), to be the submodule of the quotient module described above, freely spanned by all elements of the form

\[
\sum_{l > 0} Z_3^{(l_1)} y Z_3^{(l_2)} y \cdots y Z_3^{(l_p)} y Z_2^{(k_1)} x Z_2^{(k_2)} x \cdots x Z_2^{(k_q)} x \otimes \begin{pmatrix}
\scriptstyle w \\
\scriptstyle w' \\
\scriptstyle w''
\end{pmatrix}
\]

where

\[
l_1 > t_2; k_1 > t_1 + |l|;
\]
\[
l_i > 0; k_j > 0, \quad 0 < i \leq p; \quad 0 < j \leq q
\]
\[
|l| = \sum l_i; \quad |k| = \sum k_j;
\]
\[
\pi = p_1 + |k|;
\]
\[
\sigma_1 + \sigma_2 = p_2 + |l| - |k|;
\]
\[
\rho_1 + \rho_2 + \rho_3 = p_3 - |l|;
\]
\[
p + q = n.
\]

We define \( M_0 ([a_1, p_2, p_3; t_1, t_2]) = D_{p_1} \otimes D_{p_2} \otimes D_{p_3} \), and \( M_{-1} ([a_1, p_2, p_3; t_1, t_2]) = [a_1, p_2, p_3; t_1, t_2] \).

As special cases, it should be remarked that when \( p = 0 \), we have the terms of the resolution of \([a_1, p_2; t_1] \) tensored with \( D_{p_3} \). When \( q = 0 \), we have \( D_{p_1} \) tensored with the terms of the resolution of \([p_2, p_3; t_2] \). Notice that the tableau, with the conditions on \( \pi, \sigma \)'s and \( \rho \)'s, simply represents an element of \( D_{p_1+t_1+t_2+0} \otimes [0] \otimes D_{p_2} \otimes D_{p_3-t_2-0} \). Notice too that the map \( \partial_x + \partial_y \) maps \( M_n ([a_1, p_2, p_3; t_1, t_2]) \) to \( M_{n-1} ([a_1, p_2, p_3; t_1, t_2]) \) for \( n > 0 \). To see that, we observe that when we apply \( \partial_x + \partial_y \), we are removing either an \( x \) or a \( y \) between \( Z \)'s or immediately to the left of the tensor product. It
is easy to see that in the latter case, our term remains one of the type we allow. Also, when we remove an $x$ between two $Z_{2i}$'s or a $y$ between two $Z_{32}$'s, we simply multiply the like terms, and we still have a term of the type allowed. The only questionable term is one in which the $y$ between a $Z_{32}$ and a $Z_{21}$ has been removed. It is precisely in this case that we invoke the identity 5.4 above. Of course, we have to show that when we apply that identity, the resulting summands are of the allowed type again. But notice the identity 5.4 above. Of course, we have to show that when we apply that identity, the resulting summands are of the allowed type again. But notice that

$$Z_{3,2} \cdot yZ_{3,2} \cdot y \cdots yZ_{3,2} \cdot Z_{2,1} \cdot xZ_{2,1} \cdot x$$

$$\cdots xZ_{2,1} \cdot x \otimes \begin{pmatrix} w & 1^{(\pi)} & 2^{(\sigma_1)} & 3^{(\rho_1)} \\ w' & 2^{(\sigma_2)} & 3^{(\rho_2)} \\ w'' & 3^{(\rho_3)} \end{pmatrix}$$

is simply

$$\sum_{\alpha=0}^{l_p} Z_{3,2}^{(l_1)} \cdot yZ_{3,2}^{(l_2)} \cdot y \cdots yZ_{3,2}^{(l_\alpha-\alpha)} \cdot Z_{2,1}^{(l_\alpha-\alpha)} \cdot Z_{2,1} \cdot x$$

$$\cdots xZ_{2,1}^{(l_\alpha)} \cdot x \otimes \begin{pmatrix} w & 1^{(\pi-\alpha)} & 2^{(\sigma_1)} & 3^{(\alpha)}3^{(\rho_1)} \\ w' & 2^{(\sigma_2)} & 3^{(\rho_2)} \\ w'' & 3^{(\rho_3)} \end{pmatrix}.$$ 

If we set $|l'| = \sum_{i=1}^{p-1} l_i$, we see that the exponent on the first $Z_{21}$ is greater than $t_1 + |l'|$, so that condition is preserved. The rest of the conditions are easy to check; we point out that it is here that we use the fact that the index of summation in the identity 5.4 is constrained.

**Theorem 5.2.** Let $[p_1, p_2, p_3; t_1, t_2]$ be a Weyl module with $p_3 \leq t_1 + t_2 + 1$. The sequence of modules and maps

$$\cdots \to M_n ([p_1, p_2, p_3; t_1, t_2]) \to M_{n-1} ([p_1, p_2, p_3; t_1, t_2]) \to \cdots$$

$$\cdots \to M_1 ([p_1, p_2, p_3; t_1, t_2]) \to M_0 ([p_1, p_2, p_3; t_1, t_2])$$

is a projective resolution of $[p_1, p_2, p_3; t_1, t_2]$, where the map from $M_0 ([p_1, p_2, p_3; t_1, t_2])$ to $[p_1, p_2, p_3; t_1, t_2]$ is the Weyl map, and the arrows are the maps described above i.e., the maps induced by $\partial = \partial_x + \partial_y$. We will denote this resolution by $\text{Res}( ([p_1, p_2, p_3; t_1, t_2]),$ and continue to denote the boundary maps by $\partial$.

**Proof.** Since we know that the map $M_1 ([p_1, p_2, p_3; t_1, t_2]) \to M_0 ([p_1, p_2, p_3; t_1, t_2])$ is the box map, and that the Weyl module is the cokernel of this map, we will have the fact that $\text{Res}([p_1, p_2, p_3; t_1, t_2])$ is a left complex over $[p_1, p_2, p_3; t_1, t_2]$ once we know that it is a complex. But the boundary map on $\text{Res}([p_1, p_2, p_3; t_1, t_2])$ is just that induced on the bar module modulo $\text{Cap}(1, 2)$, so that immediately gives us that $\partial^2 = 0$. Hence what we have to show is that the complex is acyclic.
We will define a filtration \( \{ \mathbf{F}_j \} \) on \( \text{Res}(\{p_1, p_2, p_3; t_1, t_2\}) \) by placing restrictions on the terms of type (5) as follows:

\[
\mathbf{F}_0 = \text{the subcomplex spanned by those terms with } p = 0;
\]

For \( j > 0 \), we define

\[
\mathbf{F}_j = \text{the subcomplex spanned by } \mathbf{F}_{j-1} \text{ and those terms with } t_1 \geq p_3 - j + 1.
\]

Since all the terms of type (3) must have \( t_1 \geq t_2 + 1 \), we see that \( j \) runs from 0 to \( p_3 - t_2 \). We have the filtration, then:

\[
\mathbf{F}_0 \subset \mathbf{F}_1 \subset \cdots \subset \mathbf{F}_{p_3 - t_2} = \text{Res}(\{p_1, p_2, p_3; t_1, t_2\}).
\]

The first thing to notice about this filtration is that \( \mathbf{F}_j = \text{Res}(\{p_1, p_2, p_3; t_1, p_3 - j\}) \). The next is to observe that \( \mathbf{F}_j/\mathbf{F}_{j-1} \approx \text{Res}(\{p_1, p_2 + p_3 - j + 1, j - 1; t_1 + p_3 - j + 1, 0\}) \) for \( j > 0 \), and that this isomorphism is of degree \(-1\). That is, we have

\[
(\mathbf{F}_j/\mathbf{F}_{j-1})_n \approx (\text{Res}(\{p_1, p_2 + p_3 - j + 1, j - 1; t_1 + p_3 - j + 1, 0\}))_{n-1}.
\]

We see this by noting that the terms of \( \mathbf{F}_j/\mathbf{F}_{j-1} \) are spanned by elements of the form \( \mathbb{Z}^{p_3-j+1}_{32} W \), where \( W \) is an element of \( \text{Res}(\{p_1, p_2 + p_3 - j + 1, j - 1; t_1 + p_3 - j + 1, 0\}) \). The isomorphism is defined by mapping \( \mathbb{Z}^{p_3-j+1}_{32} W \) to \( W \). This explains the shift in dimension.

Whether one proceeds with the proof of the theorem now by resorting to the spectral sequence associated to this filtration, or simply goes step by step, the crucial ingredients for us are the same: an induction on \( p_3 - t_2 \), and exactness of the sequence 5.1 or 5.2. We have already remarked that \( \mathbf{F}_0 \) is acyclic since it is \( \text{Res}(p_1, p_2; t_1) \otimes D_{p_3} \). We further notice that \( \mathbf{F}_1/\mathbf{F}_0 \approx \text{Res}(\{p_1, p_2 + p_3, 0; t_1 + p_3, 0\}) \), and this too is acyclic, since it is the resolution of the two-rowed shape \( \{p_1, p_2 + p_3; t_1 + p_3\} \). The acyclicity of these, together with the exact sequence of complexes (with dimension shift!)

\[
0 \rightarrow \mathbf{F}_0 \rightarrow \mathbf{F}_1 \rightarrow \mathbf{F}_1/\mathbf{F}_0 \rightarrow 0
\]
gives us the vanishing of the homology of \( \mathbf{F}_1 \) in dimensions greater than 1. In low dimensions we have the exact sequence

\[
0 \rightarrow H_1(\mathbf{F}_1) \rightarrow H_0(\mathbf{F}_1/\mathbf{F}_0) \rightarrow H_0(\mathbf{F}_0) \rightarrow H_0(\mathbf{F}_1) \rightarrow 0.
\]

Reinterpreting the homologies of \( \mathbf{F}_0 \) and \( \mathbf{F}_1/\mathbf{F}_0 \), we have the exactness of

\[
0 \rightarrow H_1(\mathbf{F}_1) \rightarrow [p_1, p_2 + p_3; t_1 + p_3] \rightarrow [p_1, p_2; t_1] \otimes D_{p_3} \rightarrow H_0(\mathbf{F}_1) \rightarrow 0.
\]

But we know that the kernel of \( [p_1, p_2 + p_3; t_1 + p_3] \rightarrow [p_1, p_2; t_1] \otimes D_{p_3} \) is zero, and the cokernel is \( [p_1, p_2, p_3; t_1, p_3 - 1] \). This tells us that \( H_1(\mathbf{F}_1) = 0 \), and that \( H_0(\mathbf{F}_1) \approx [p_1, p_2, p_3; t_1, p_3 - 1] \), i.e., \( \mathbf{F}_1 \) is indeed a resolution of \( [p_1, p_2, p_3; t_1, p_3 - 1] \). Proceeding in this way, one gets the result of the theorem.

- The condition that \( p_3 \leq t_1 + t_2 + 1 \) is critical to this argument because otherwise we wouldn’t be able to invoke the exactness of the sequence 5.2. We will see how important this is in our next examples.
**Additional Observation:** Since the description of the terms of our complex, 5.6, seems a bit arbitrary and cumbersome, let us take a little time to give an alternate description. To do this we first introduce some more notation. In general we let $Z_{n,m}$ stand for polarizations from place $m$ to place $n$, and we’re going to suppress the various separators that come into play (as a rule, there will be a new separator introduced for each operator $Z_{n,m}$; in the case at hand, it will be $Z_{32}$ with separator $y$). For a fixed $n, m, \tau$ and $l$, we’ll denote by

$$Z^{\tau}_{n,m} \otimes Z^{(l)}_{n,m}$$

the homogeneous strand of the bar complex of total degree $\tau + l$ with initial term of degree $\geq \tau$. For example, $Z^{(0)}_{32} \otimes Z^{(0)}_{32}$ is just the complex concentrated in dimension zero consisting of the element $Z^{(0)}_{32}$, while $Z^{(2)}_{32} \otimes Z^{(2)}_{32}$ is the complex

$$0 \rightarrow Z^{(0)}_{32} \otimes Z^{(1)}_{32} \otimes Z^{(0)}_{32} \rightarrow Z^{(0)}_{32} \otimes Z^{(1)}_{32} \otimes Z^{(0)}_{32} \rightarrow Z^{(1)}_{32} \rightarrow 0.$$

If we simply write $Z^{(l)}_{n,m}; l > 0$, this will mean the homogeneous strand of the normalized bar complex of degree $l$. Now a typical term of 5.6 arises in the following way. We choose $\tau = t_2 + 1$, and any non-negative integer $l$. We then form the complex $Z^{\tau+1}_{32} \otimes Z^{(l)}_{32}$ and tensor it with

$$0 \rightarrow [p_1, p_2; t_1 + t_2 + 1 + l] \otimes D_{p_3 - (t_2 + 1 + l)}.$$

If one looks at the terms of 5.7, one sees that these are the terms described in 5.6. So our complex, $\text{Res} ([p_1, p_2, p_3; t_1 + t_2 + 1 + l]) \otimes D_{p_3 - (t_2 + 1 + l)}$, may be described as

$$\text{Res} ([p_1, p_2; t_1]) \otimes D_{p_3} \oplus \sum_{l \geq 0} Z^{t_2+1}_{32} \otimes Z^{(l)}_{32} \otimes \text{Res} ([p_1, p_2 + t_2 + 1 + l; t_1 + t_2 + 1 + l] \otimes D_{p_3 - (t_2 + 1 + l)}.$$

### 5.2. Almost Skew-Shape With One Triple Overla.

The proof of the last theorem partially disguises one crucial feature of this investigation: how to find the resolutions we are looking for in the first place. We say ‘partially disguises’, because the last few lines of the proof do indicate how one might have gone about discovering the resolution. Namely, we have the exact sequence

$$0 \rightarrow [p_1, p_2 + p_3; t_1 + p_3] \rightarrow [p_1, p_2; t_1] \otimes D_{p_3} \rightarrow [p_1, p_2, p_3; t_1, p_3 - 1] \rightarrow 0,$$

and we have the resolutions of the Weyl modules $[p_1, p_2 + p_3; t_1 + p_3]$ and $[p_1, p_2; t_1] \otimes D_{p_3}$. By suitable induction, we know the resolutions of these two modules, and by Lemma 3.1, we obtain a resolution of $[p_1, p_2, p_3; t_1, p_3 - 1]$.
if we can define a map between these two resolutions and take the mapping cone of this map. Now, as pointed out in the above proof, $F_1/F_0$ is spanned by elements of the form $Z_{32}^{p_3-j+1}yW$, where $W$ is an element of Res($[p_1, p_2 + p_3; t_1 + p_3]$). It is also fairly easy to see that if we take the element $\partial_y \left( Z_{32}^{p_3-j+1}yW \right)$, we obtain an element of Res($[p_1, p_2; t_1] \otimes D_{p_3}$). This is the map of resolutions that we want, and it is clear that $F_1$ is the mapping cone of the map, and that the exact sequence

$$0 \rightarrow F_0 \rightarrow F_1 \rightarrow F_1/F_0 \rightarrow 0$$

is the one associated to the mapping cone (see 3.1). In point of fact, this is the way these resolutions were found originally.

Our reason for giving this detailed discussion is that we will apply this technique to find the resolution of the almost skew-shape with one triple overlap, namely $[p_1, p_2, p_3; t_1, t_2]$ with $t_2 < 0$, $t_1 + t_2 \geq 0$, and $p_3 = t_1 + t_2 + 1$:

If we look at the fundamental exact sequence starting with this shape, we get

$$0 \rightarrow [p_1 + p_3, p_2; t_1 - p_3] \stackrel{\partial^{(p_3)}}{\rightarrow} [p_1, p_2, p_3; t_1, 0] \rightarrow [p_1, p_2, p_3; t_1, t_2] \rightarrow 0.$$

The Weyl module $[p_1, p_2, p_3; t_1, 0]$ is really the shape $[p_1, p_2, p_3; t_1, t_1 + 1]$, but since there is no triple overlap in this shape, we may push the bottom row all the way to the left, which is what $[p_1, p_2, p_3; t_1, 0]$ represents. Now $[p_1 + p_3, p_2; t_1 - p_3]$ is a two-rowed shape and $[p_1, p_2, p_3; t_1, 0]$ is a three-rowed shape, both of whose resolutions we know (the latter due to the last theorem). What we should look for is a map between these resolutions over $\partial^{(p_3)}$, take the mapping cone, and produce the resolution of $[p_1, p_2, p_3; t_1, t_2]$.

We have used the notation $\partial^{(p_3)}$ for the map between Weyl modules, because that map is induced by the divided power of the indicated place polarization on the generators. That is, the map between the Weyl modules is induced by

$$\partial^{(p_3)} : D_{p_1+p_3} \otimes D_{p_2} \rightarrow D_{p_1} \otimes D_{p_2} \otimes D_{p_3}.$$

The elements of Res($[p_1 + p_3, p_2; t_1 - p_3]$) in dimension $m$ are spanned by terms of the form

$$Z_{21}^{(t_1-p_3+k_1)} x \cdots x Z_{21}^{(k_m)} x \otimes v,$$

where all the $k_j > 0$, $|k| = \sum k_j$, and $v \in D_{p_1+t_1+|k|} \otimes D_{p_2+p_3-t_1-|k|}$, while those of Res($[p_1, p_2, p_3; t_1, 0]$) in dimension $m$ are spanned by terms of the form

$$\sum_{l_i, k_j > 0} Z_{3,2}^{(l_1)} y Z_{3,2}^{(l_2)} y \cdots y Z_{3,2}^{(l_k)} y Z_{2,1}^{(t_1+|l|+k_1)} x Z_{2,1}^{(k_2)} x \cdots x Z_{2,1}^{(k_q)} x \otimes v,$$
where \( p + q = m, \ |l| = \sum i, |k| = \sum k, \) and \( v \in D_{p_1+t_1+\|l\|+\|k\|} \otimes D_{p_2-t_1-\|k\|} \otimes D_{p_3-\|l\|}. \) If we take our cue from the previous section, we should preface all the terms \( W \) of \( 5.8 \) by a new formal operator and separator \( Z_{31}^{(p_3)} z, \) and define an identity of the type \( Z_{31}^{(p_3)} W = X, \) where \( X \) will be some element spanned by terms of the form \( 5.9. \) We would then define the map from \( \text{Res} ([p_1 + p_3, p_2; t_1 - p_3]) \) to \( \text{Res} ([p_1, p_2, p_3; t_1, 0]) \) to be the one that sends \( Z_{31}^{(p_3)} z W \) to \( X. \) This will have the following effect: if we succeed in doing this, we would take the mapping cone of this map, and the total boundary map of the resulting complex would look like \( \partial = \partial_x + \partial_y + \partial_z. \) In other words, we would have expressed the mapping cone as a bar module with respect to an algebra, \( A, \) generated by \( Z_{21}, Z_{32}, \) and \( Z_{31} \) and set of separators, \( S = \{ x, y, z \}. \) The module, \( M, \) is a suitable direct sum of \( D_p \otimes D_q \otimes D_r \) (where the action of all the \( Z_{ij} \) on \( M \) is by place polarization).

**Definition 5.3.** With \( A, S \) and \( M \) as above, define the submodule, \( \text{Cap}'(1, 2, 3), \) of the free bar module on the above data to be the module spanned by \( \text{Cap}(1, 2) \) as well as all terms of the form

\[
Z_{31}^{(a)} Z_{21}^{(b)} x \otimes v - \left( (-1)^a Z_{21}^{(a+b)} x \otimes \partial_{32}^{(a)} (v) \right) + \sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(a-u)} \partial_{21}^{(b+u)} (v)
\]

\[
Z_{31}^{(a)} Z_{21}^{(b_1)} x Z_{21}^{(b_2)} x W' - \left( (-1)^a Z_{21}^{(a+b_1)} x \otimes Z_{32}^{(a)} Z_{21}^{(b_2)} x W' - \sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes Z_{31}^{(a-u)} Z_{21}^{(b_1+b_2+u)} x W' \right)
\]

where \( W' \) is of the form \( Z_{21}^{(b_3)} x \cdots x Z_{21}^{(b_m)} x \otimes v. \) If we let \( V \) be of the form \( Z_{31}^{(a)} Z_{21}^{(b)} x W, \) where \( W \) is of the form \( Z_{21}^{(b_1)} x \cdots x Z_{21}^{(b_m)} x \otimes v, \) denote by \( \hat{V} \) the element

\[
\hat{V} = (-1)^a Z_{21}^{(a+b)} x Z_{32}^{(a)} W - \sum_{u>0} \left( b + b_1 \right) Z_{32}^{(a-u)} Z_{21}^{(b+b_1+u)} x W',
\]

where \( W' = Z_{21}^{(b_2)} x \cdots x Z_{21}^{(b_m)} x \otimes v. \) If \( V \) is of the form \( Z_{31}^{(a)} Z_{21}^{(b)} x W, \) where \( W \) is in the submodule spanned by all elements of the form \( Z_{21}^{(b_1)} x \cdots x Z_{21}^{(b_m)} x \otimes v, \) we extend the notation \( \hat{V} \) by linearity.

Before going further, we should explain where the above relations come from. Recall the Capelli identity 2.2 for the places 1, 2 and 3:

\[
\partial_{21}^{(a)} \partial_{32}^{(r)} = \sum_{\alpha \geq 0} (-1)^{\alpha} \partial_{32}^{(r-\alpha)} \partial_{21}^{(s-\alpha)} \partial_{31}^{(a)}. \]

Let \( s = a + b, \) and \( r = a. \) Then the above identity gives us

\[
\partial_{21}^{(a+b)} \partial_{32}^{(a)} = \sum_{\alpha \geq 0} (-1)^{\alpha} \partial_{32}^{(a-\alpha)} \partial_{21}^{(a+b-\alpha)} \partial_{31}^{(a)}. \]
If we set \( a - \alpha = u \), then the above becomes
\[
\partial^{(a+b)}_{21} \partial^{(a)}_{32} = \sum_{u \geq 0} (-1)^{a-u} \partial^{(a-u)}_{32} \partial^{(b+u)}_{21} \partial^{(a)}_{21} \partial^{(b)}_{21},
\]
or,
\[
(-1)^a \partial^{(a)}_{31} \partial^{(b)}_{21} = \partial^{(a+b)}_{21} \partial^{(a)}_{32} - (-1)^a \sum_{u > 0} (-1)^u \partial^{(a-u)}_{32} \partial^{(b+u)}_{21};
\]
(5.12)
\[
\partial^{(a)}_{31} \partial^{(b)}_{21} = (-1)^a \partial^{(a+b)}_{21} \partial^{(a)}_{32} - \sum_{u > 0} (-1)^u \partial^{(a-u)}_{32} \partial^{(b+u)}_{21}.
\]

This identity is what informed the selection of the relation 5.10 above. Notice that the terms in 5.10 that replace \( Z^{(a)}_{31} Z^{(b)}_{21} x \otimes v \) just involve operators \( Z_{21} \) and \( Z_{32} \), and these are the only kinds of terms that appear in the complex to which we are mapping. This is the reason for looking at the Capelli identity and \( Z \) which we are mapping. This is the reason for looking at the Capelli identity that we did: it is appropriate to our aims. One might ask the natural question: Why didn’t we simply choose the relation
\[
Z^{(a)}_{31} Z^{(b)}_{21} x W - (-1)^a Z^{(a+b)}_{21} x \otimes Z^{(a)}_{32} W - \sum_{u > 0} (-1)^u Z^{(a)}_{32} y \otimes Z^{(a-u)}_{31} Z^{(b+u)}_{21} W
\]
for any term \( W \) of the appropriate sort? The simple answer is that it didn’t work; that is, we could not prove that the submodule spanned by those relations was stable under the boundary operator. We will return to this puzzling phenomenon at the end of this section.

As in the previous section, we want to show:

**Proposition 5.3.** The module \( \text{Cap}'(1,2,3) \) is stable under the boundary operator \( \partial = \partial_x + \partial_y + \partial_z \). More precisely, if \( V \) is of the form \( Z^{(a)}_{31} Z^{(b)}_{21} x W \), where \( W \) is of the form \( Z^{(b_1)}_{21} x \cdots x Z^{(b_m)}_{21} x \otimes v \), then
\[
\partial \left( \overrightarrow{V} \right) = \overrightarrow{\partial(V)},
\]
so that
\[
\partial \left( V - \overrightarrow{V} \right) = \partial(V) - \overrightarrow{\partial(V)}.
\]

**Proof.** We first look at the case \( m = 0 \). In this case,
\[
\partial \left( \overrightarrow{V} \right) = \partial \left( \overrightarrow{Z^{(a)}_{31} Z^{(b)}_{21} x \otimes v} \right) = \partial \left( (-1)^a Z^{(a+b)}_{21} x \otimes \partial^{(a)}_{32} W - \sum_{u > 0} (-1)^u Z^{(a)}_{32} y \otimes \partial^{(a-u)}_{31} \partial^{(b+u)}_{21} (v) \right) = (-1)^a \partial^{(a+b)}_{21} \partial^{(a)}_{32} (v) - \sum_{u > 0} (-1)^u \partial^{(a)}_{32} \partial^{(a-u)}_{31} \partial^{(b+u)}_{21} (v) = \partial^{(a)}_{31} \partial^{(b)}_{21} (v) = \partial \left( \overrightarrow{Z^{(a)}_{31} Z^{(b)}_{21} x \otimes v} \right) = \partial(V) = \overrightarrow{\partial(V)}.
\]
so that we have the result.
Assume next that $m = 1$.

\[
\partial \left( Z_{31}^{(a)} Z_{21}^{(b)} x Z_{21}^{(b_1)} x \otimes v \right) = \left( b + b_1 \right) Z_{31}^{(a)} Z_{21}^{(b+b_1)} x \otimes v - Z_{31}^{(a)} Z_{21}^{(b)} x \otimes \partial_{21}^{(b_1)}(v);
\]

while

\[
\partial \left( \widehat{V} \right) = \partial \left( -1 \right) a Z_{21}^{(a+b)} x \sum_{u>0} (-1)^u \left( b + b_1 \right) \frac{Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x \otimes v}{Z_{32}^{(a+b)} x \otimes v} - \\
\sum_{u>0} (-1)^u \left( b + b_1 \right) \frac{\partial \left( Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x \otimes v \right)}{Z_{32}^{(a+b)} x \otimes v} = \\
(5.14)
\]

\[
\partial \left( -1 \right) a Z_{21}^{(a+b)} x \sum_{k \leq b_1} Z_{21}^{(b_1-k)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(k)}(v) - \\
\sum_{u>0} (-1)^u \left( b + b_1 \right) \frac{\partial \left( Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b_1-k)} x \otimes v \right)}{Z_{32}^{(a+b)} x \otimes v} = \\
(5.15)
\]

\[
= (-1)^a \sum_{k \leq b_1} \left( a + b + b_1 - k \right) Z_{21}^{(a+b+b_1-k)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(k)}(v) - \\
(5.16)
\]

\[
(-1)^a Z_{21}^{(a+b)} x \otimes \sum_{k \leq b_1} \partial_{21}^{(b_1-k)} \partial_{32}^{(a-k)} \partial_{31}^{(k)}(v) - \\
\sum_{u>0} (-1)^u \left( b + b_1 \right) \frac{\partial \left( Z_{32}^{(u)} y \left( -1 \right)^{a-u} Z_{21}^{(a+b+b_1)} x \otimes \partial_{32}^{(a-u)}(v) \right)}{Z_{32}^{(a+b)} x \otimes v} - \\
\sum_{u>0} (-1)^u \left( b + b_1 \right) \frac{\partial \left( \sum_{w>0} Z_{32}^{(u)} y \left( -1 \right)^{w} Z_{32}^{(w)} y \otimes \partial_{31}^{(a-u-w)} \partial_{21}^{(b+b_1+u+w)}(v) \right)}{Z_{32}^{(a+b)} x \otimes v}
\]
\[(5.17)\]
\[= (-1)^a \sum_{k \leq b_1} \left( \frac{a + b + b_1 - k}{b_1 - k} \right) \mathcal{Z}_{21}^{a+b+b_1-k} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(k)} (v) - \]

\[(5.18)\]
\[(-1)^a \mathcal{Z}_{21}^{a+b} x \otimes \partial_{32}^{(a)} \partial_{21}^{(b_1)} (v) - \]

\[(5.19)\]
\[(-1)^a \sum_{u > 0} \left( \frac{b + b_1}{b_1 - u} \right) \mathcal{Z}_{32}^{(a)} \mathcal{Z}_{21}^{(a+b+b_1)} x \otimes \partial_{32}^{(a-u)} (v) + \]

\[(5.20)\]
\[\sum_{u > 0} \sum_{w \geq 0} (-1)^{u+w} \left( \frac{b + b_1}{b_1 - u} \right) \left( \frac{u + w}{u} \right) \mathcal{Z}_{32}^{(u+w)} y \otimes \partial_{31}^{(a-u-w)} \partial_{21}^{(b+b_1+u+w)} (v) + \]

\[(5.21)\]
\[\sum_{u > 0} (-1)^u \left( \frac{b + b_1}{b_1 - u} \right) \sum_{w \geq 0} (-1)^w \mathcal{Z}_{32}^{(u)} y \otimes \partial_{32}^{(w)} \partial_{31}^{(a-u-w)} \partial_{21}^{(b+b_1+u+w)} (v) \]
\begin{align}
(5.22) & \quad = (-1)^a \sum_{k \leq b_1} \binom{a + b + b_1 - k}{b_1 - k} Z_{21}^{(a+b+b_1-k)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(k)} (v) - \\
(5.23) & \quad (-1)^a Z_{21}^{(a+b)} x \otimes \partial_{32}^{(a)} \partial_{21}^{(b_1)} (v) - \\
(5.24) & \quad (-1)^a \sum_k \binom{a + b + b_1 - k}{b_1 - k} Z_{21}^{(a+b+b_1-k)} x \otimes \partial_{32}^{(a-k)} \partial_{31}^{(k)} (v) + \\
(5.25) & \quad (-1)^a \binom{b + b_1}{b_1} Z_{21}^{(a+b+b_1)} x \otimes \partial_{32}^{(a)} (v) - \\
(5.26) & \quad (-1)^a \sum_{u>0} \binom{b + b_1 - u}{b_1 - u} Z_{32}^{(u)} y \otimes \partial_{32}^{(a+b+b_1)} \partial_{31}^{(a-u)} (v) - \\
(5.27) & \quad \sum_{\alpha>0} (-1)^{\alpha} \left\{ \binom{b + b_1 + \alpha}{b_1} - \binom{b + b_1}{b_1} \right\} Z_{32}^{(\alpha)} y \otimes \partial_{31}^{(a-\alpha)} \partial_{21}^{(b+b_1+\alpha)} (v) + \\
(5.28) & \quad (-1)^a \sum_{u>0} (-1)^u \binom{b + b_1}{b_1 - u} Z_{32}^{(u)} y \otimes \partial_{21}^{(a+b+b_1)} \partial_{32}^{(a-u)} (v).
\end{align}

Before proceeding further with more calculations, we should point out that we went from 5.14 to 5.15 and 5.16 by playing the standard game with the boundary: adding and subtracting appropriate terms. We got from 5.16 to 5.17 by applying 2.1, from 5.19 to 5.26 and from 5.18 to 5.24 and 5.25 by applying standard identities on binomial coefficients. Finally, we got from 5.21 to 5.28 by applying 2.2.

Notice that we get some cancellation: the term 5.22 cancels 5.24, and 5.26 cancels 5.28.
We have to compare the remaining terms with

\[
\hat{\partial} (V) = \left( \frac{b + b_1}{b_1} \right) \left( Z_{31}^{(a)} Z_{21}^{(b+b_1)} x \otimes v - Z_{31}^{(a)} Z_{21}^{(b)} x \otimes \partial_{21}^{(b_1)} (v) \right) = (-1)^a \left( \frac{b + b_1}{b_1} \right) Z_{21}^{(a+b+b_1)} x \otimes \partial_{32}^{(a)} v - \]

\[
\sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(a-u)} \partial_{21}^{(b+b_1+u)} (v) - \]

\[
\sum_{u>0} (-1)^u \left( \frac{b + b_1 + u}{b_1} \right) Z_{32}^{(u)} y \otimes \partial_{31}^{(a-u)} \partial_{21}^{(b+b_1+u)} (v). \]

We notice that 5.23 cancels 5.31, 5.27 cancels 5.30 and 5.32, while 5.25 cancels 5.29. This proves the result for \(m = 1\).

Now we turn to the general case with \(m > 1\), so we may assume that

\[
V = Z_{31}^{(a)} Z_{21}^{(b)} x Z_{21}^{(b_1)} x W',
\]

where \(W'\) is of dimension \(m - 1 > 0\). As usual, we have

\[
\hat{\partial} (V) = \left( \frac{b + b_1}{b_1} \right) Z_{31}^{(a)} Z_{21}^{(b+b_1)} x W' - Z_{31}^{(a)} Z_{21}^{(b)} x Z_{21}^{(u)} W' + Z_{31}^{(a)} Z_{21}^{(b)} x Z_{21}^{(b_1)} x \partial (W'),
\]

so that

\[
\hat{\partial} (V) = (-1)^a \left( \frac{b + b_1}{b_1} \right) Z_{21}^{(a+b+b_1)} x Z_{32}^{(a)} W'- \]

\[
\sum_{u>0} (-1)^u \left( \frac{b + b_1 + b_2}{b_2} \right) \left( \frac{b + b_1 + u}{b_1 + b_2 - u} \right) Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+b_2+u)} x W' - \]

\[
\sum_{u>0} (-1)^u \left( \frac{b + b_1 + u}{b_1 - u} \right) Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x \partial W'.
\]

We also have

\[
\hat{\partial} V = (-1)^a Z_{21}^{(a+b)} Z_{32}^{(a)} Z_{21}^{(b_1)} x W' - \sum_{u>0} (-1)^u \left( \frac{b + b_1}{b_1 - u} \right) Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x W',
\]
so that

\begin{align*}
(5.39) \quad \partial \left( \frac{\partial}{\partial V} \right) &= (-1)^a Z_{21}^{(a+b)} Z_{32}^{(a)} Z_{21}^{(b_1)} x W' - (-1)^a Z_{21}^{(a+b)} x \partial \left( Z_{32}^{(a)} Z_{21}^{(b_1)} x W' \right) - \\
(5.40) \quad \sum_{u>0} (-1)^u \left( \frac{b+b_1}{b_1-u} \right) Z_{32}^{(u)} \frac{Z_{31}^{(a-u)} Z_{21}^{(b_1+u)} x W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'} = \\
(5.41) \quad \sum_{u>0} (-1)^u \left( \frac{b+b_1}{b_1-u} \right) Z_{32}^{(u)} y \partial \left( \frac{Z_{31}^{(a-u)} Z_{21}^{(b_1+u)} x W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'} \right).
\end{align*}

By 5.5 we know that

\[ \partial \left( \frac{Z_{32}^{(a)} Z_{21}^{(b_1)} x W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'} \right) = \partial \left( \frac{Z_{32}^{(a)} Z_{21}^{(b_1)} x W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'} \right) - \frac{x Z_{32}^{(a-b_1)} \partial Z_{21}^{(b_1)} W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'}, \]

and since

\[-(-1)^a Z_{21}^{(a+b)} \frac{x Z_{32}^{(a)} Z_{21}^{(b_1)} W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'} + (-1)^a Z_{21}^{(a+b)} \frac{x Z_{32}^{(a)} Z_{21}^{(b_1)} x \partial W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'} =
\]

\[-(-1)^a Z_{21}^{(a+b)} \frac{x Z_{32}^{(a)} Z_{21}^{(b_1)} x W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'},\]

we see that the difference between 5.35 plus 5.36 and the second term of 5.39 is \(-(-1)^a Z_{21}^{(a+b)} \frac{x Z_{32}^{(a-b_1)} \partial Z_{21}^{(b_1)} W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'}\). We are thus left with having to prove that the sum of the terms remaining in the expression for \(\partial (V)\):

\begin{align*}
(5.42) \quad (-1)^a \left( \frac{b+b_1}{b_1} \right) Z_{21}^{(a+b+b_1)} x Z_{32}^{(a)} W' - \\
(5.43) \quad \left( \frac{b+b_1}{b_1} \right) \sum_{u>0} (-1)^u \left( \frac{b+b_1+b_2}{b_2-u} \right) Z_{32}^{(u)} y \frac{Z_{31}^{(a-u)} Z_{21}^{(b_1+b_2+u)} x W''}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'} + \\
(5.44) \quad \sum_{u>0} (-1)^u \left( \frac{b_1+b_2}{b_2} \right) \left( \frac{b+b_1+b_2}{b_1+b_2-u} \right) Z_{32}^{(u)} y \frac{Z_{31}^{(a-u)} Z_{21}^{(b_1+b_2+u)} x W''}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'} - \\
(5.45) \quad \sum_{u>0} (-1)^u \left( \frac{b+b_1}{b_1-u} \right) Z_{32}^{(u)} y \frac{Z_{31}^{(a-u)} Z_{21}^{(b_1+u)} x \partial W'}{Z_{21}^{(a)} Z_{21}^{(b_1)} x W'}.\end{align*}
is equal to that for $\partial \left( \widehat{V} \right)$:

\[(5.46) \quad (-1)^a Z_{21}^{(a+b)} Z_{32}^{(a)} Z_{21}^{(b_1)} x W' - (-1)^a Z_{21}^{(a+b)} Z_{32}^{(a-b_1)} Z_{21}^{(b_1)} W' - \]

\[(5.47) \quad \sum_{u > 0} (-1)^u \left( b + b_1 \right) \frac{Z_{32}^{(u)} Z_{31}^{(a-u)} Z_{21}^{(b_1+u)}}{Z_{32}^{(a)} Z_{21}^{(b_1+u)} x W'} \]

\[(5.48) \quad \sum_{u > 0} (-1)^u \left( b + b_1 \right) \frac{Z_{32}^{(u)} y \partial \left( Z_{31}^{(a-u)} Z_{21}^{(b_1+u)} x W' \right)}{Z_{32}^{(a)} Z_{21}^{(b_1+u)} x W'} \]

Now in the expression for $\partial \left( \widehat{V} \right)$, the term 5.42 is the only one that involves the separator $x$ exclusively, while from $\partial \left( \widehat{V} \right)$ such terms are contributed by both 5.46 and 5.47. In fact, we have

$$Z_{32}^{(u)} Z_{31}^{(a-u)} Z_{21}^{(b_1+u)} x W' = (-1)^a Z_{32}^{(a)} Z_{21}^{(b_1+u)} x W' -$$

$$Z_{32}^{(u)} \sum_{w > 0} (-1)^w \left( b + b_1 + b_2 + u \right) \frac{Z_{32}^{(u+w)} y \partial \left( Z_{31}^{(a-u-w)} Z_{21}^{(b_1+b_2+u+w)} x W'' \right)}{Z_{31}^{(a)}} . -$$

\[(b + b_1 + b_2 + u) Z_{32}^{(u)} y \frac{Z_{31}^{(a-u)} Z_{21}^{(b_1+b_2+u)} x W''}{y} \]

where we have not restricted $w$ to positive values in the summation, but have subtracted the extra terms that correspond to $w = 0$. Thus our term

$$\sum_{u > 0} (-1)^u \left( b + b_1 \right) \frac{Z_{32}^{(u)} Z_{31}^{(a-u)} Z_{21}^{(b_1+u)} x W'}{Z_{32}^{(a)} Z_{21}^{(b_1+u)} x W'}$$

gives us

$$(-1)^a \sum_{u > 0} \left( b + b_1 \right) \sum_{k} Z_{21}^{(a+b+b_1-k)} x Z_{32}^{(u-k)} Z_{31}^{(a-u)} \partial_{31}^{(k)} W' -$$

$$\sum_{u+w > 0} (-1)^{u+w} \left( b + b_1 \right) \left( b + b_1 + b_2 + u \right) \frac{Z_{32}^{(u+w)} y Z_{31}^{(a-u-w)} Z_{21}^{(b_1+b_2+u+w)} x W''}{Z_{31}^{(a)}} . -$$

$$\sum_{u > 0} (-1)^u \left( b + b_1 \right) \left( b + b_1 + b_2 + u \right) Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b_1+b_2+u)} x W''$$
which in turn gives us
\[
(-1)^a \sum_{k \leq b_1} \binom{a + b + b_1 - k}{b_1 - k} Z_{21}^{(a+b+b_1-k)} x Z_{32}^{(a-k)} \partial_3^{(k)} W' = Z_{21}^{(a+b+b_1)} x Z_{32}^{(a)} W' - \left( b + b_1 \right) Z_{21}^{(a+b+b_1)} x Z_{32}^{(a)} W' - \]
\[
\sum_{u>0} \binom{b_1 + b_2}{b_2} \binom{b + b_1 + b_2}{b_1 + b_2 - u} Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+b_2+u)} x W'' + \]
\[
\sum_{u>0} \binom{b_1 + b_2}{b_1} \binom{b + b_1 + b_2}{b_2 - u} Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+b_2+u)} x W'' - \]
\[
\sum_{u>0} (-1)^u \binom{b + b_1}{b_1 - u} \binom{b + b_1 + b_2 + u}{b_2} Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+b_2+u)} x W''.
\]

To arrive at these last terms, we have used the fact (easily proved) that
\[
\frac{Z_{32}^{(u-k)} Z_{32}^{(a-u)} \partial_3^{(k)} W'}{Z_{32}^{(a-k)} \partial_3^{(k)} W'} = \binom{a - k}{u - k} Z_{32}^{(a-k)} \partial_3^{(k)} W',
\]
and the identity on products of binomial coefficients (also easily proved, by induction on \(d\)):
\[
\binom{n}{m} \binom{b}{n - u} = \sum_{u \geq 0} \binom{b - m}{n - m - u} \binom{b + u}{m - d + u} \binom{d}{u}.
\]

In addition, the index of summation, \(k\), is allowed to run up to \(b_1\) in the first sum because we have incorporated into that sum the term \((-1)^a Z_{21}^{(a+b)} x Z_{32}^{(a-b_1)} \partial_3^{(b_1)} W'\).

Next we observe that
\[
(-1)^a \sum_{k} \binom{a + b + b_1 - k}{b_1 - k} Z_{21}^{(a+b+b_1-k)} x Z_{32}^{(a-k)} \partial_3^{(k)} W' = (-1)^a Z_{21}^{(a+b)} Z_{32}^{(a)} W',
\]
so that, when the smoke clears, we see that what remains to be verified is that
\[
\sum_{u>0} (-1)^u \binom{b + b_1}{b_1 - u} Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x W' \partial W' = \]
\[
\sum_{u>0} (-1)^u \binom{b + b_1}{b_1 - u} Z_{32}^{(u)} y \partial \left( Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x W' \right) .
\]
But by induction, we know that
\[
\partial \left( Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x W' \right) = \partial \left( Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x W' \right)
\]
and this in turn equals
\[
= Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} W' - Z_{31}^{(a-u)} Z_{21}^{(b+b_1+u)} x \partial W'.
\]
This completes the proof of our proposition. \(\blacksquare\)
As a result of this proposition, we see that our strategy can work. That is, we define the map

\[ f : \text{Res} \left( \{p_1 + p_3, p_2; t_1 - p_3\} \right) \to \text{Res} \left( \{p_1, p_2, p_3; t_1, 0\} \right) \]
as follows: if \( W \) is a term of \( \text{Res} \left( \{p_1 + p_3, p_2; t_1 - p_3\} \right) \) in dimension \( n \), then we define

\[ f(W) = Z(p_3) \]

which is an element of dimension \( n \) in \( \text{Res} \left( \{p_1, p_2, p_3; t_1, 0\} \right) \) as can be checked easily. If we take the mapping cone of this map, but agree to label the terms, \( W \), of \( \text{Res} \left( \{p_1 + p_3, p_2; t_1 - p_3\} \right) \) as \( Z(p_3) \) \( W \), then the mapping cone of the map is a resolution of \( \{p_1, p_2, p_3; t_1, t_2\} \) whose terms look like this:

\[
\begin{align*}
\text{dim} 0 & : D_{p_1} \otimes D_{p_2} \otimes D_{p_3} ; \\
\text{dim} 1 & : \left\{ \begin{array}{l}
Z_{21}^{(t_1+k)} x ; v \in D_{p_1+t_1+k} \otimes D_{p_2-(t_1+k)} \otimes D_{p_3} ; k > 0; \\
Z_{32}^{(l)} y ; v \in D_{p_1} \otimes D_{p_2+l} \otimes D_{p_3-l} ; l > 0; \\
Z_{31}^{(p_3)} z ; v \in D_{p_1+p_3} \otimes D_{p_2};
\end{array} \right. \\
\text{dim} 2 & : \left\{ \begin{array}{l}
Z_{21}^{(t_1+k_1)} x \ Z_{21}^{(k_2)} x ; v \in D_{p_1+t_1+|k|} \otimes D_{p_2-(t_1+|k|)} \otimes D_{p_3} ; k_j > 0; \\
Z_{32}^{(l)} y \ Z_{32}^{(l_1+l+k)} x ; v \in D_{p_1+t_1+l+k} \otimes D_{p_2-(t_1+k)} \otimes D_{p_3-l} ; l, k > 0; \\
Z_{32}^{(l_1)} y \ Z_{32}^{(l_2)} y ; v \in D_{p_1} \otimes D_{p_2+|l|} \otimes D_{p_3-|l|} ; l_i > 0; \\
Z_{31}^{(p_3)} z \ Z_{21}^{(t_1+p_3+k)} x ; v \in D_{p_1+t_1+k} \otimes D_{p_2+p_3-t_1-k} ; k > 0;
\end{array} \right. \\
\vdots
\end{align*}
\]

\[
\text{dim} n \geq 3 : \left\{ \begin{array}{l}
\text{Res} \left( \{p_1, p_2; t_1\} \right)_{n} \otimes D_{p_3}; \\
\left( Z_{32}^{(l)} \right)_{q} y \ (\text{Res} \left( \{p_1, p_2 + l; t_1 + l\} \right))_{n-q} \otimes D_{p_3-l} ; l > 0; \\
Z_{31}^{(p_3)} z \ (\text{Res} \left( \{p_1 + p_3, p_2; t_1 - p_3\} \right))_{n-1}.
\end{array} \right.
\]

We have used the separator symbols \( y \) and \( z \) to indicate that the terms that we take in the bar complex are as indicated. In brief, we have the result:

**Theorem 5.4.** The resolution of \( \{p_1, p_2, p_3; t_1, t_2\} \), where \( t_2 < 0, t_1 + t_2 \geq 0 \), and \( p_3 = t_1 + t_2 + 1 \), is the subcomplex of the quotient of the bar complex on
the data described above modulo $\Cap'(1, 2, 3)$ freely generated by the terms of

$$\Res([p_1, p_2; t_1]) \otimes D_{p_3} \oplus \sum_{l > 0} Z_{32}^{(l)} y \Res([p_1, p_2 + l; t_1 + l]) \otimes D_{p_3 - l}$$

$$\oplus Z_{31}^{(p_3)} z \Res([p_1 + p_3, p_2; t_1 - p_3]) \, .$$

Proof. We simply point out that since these terms describe the terms of the mapping cone constructed above, our result on mapping cones (Lemma 3.1) does the rest.

In discussing the almost skew-shape, we have insisted that $t_2 < 0$. Notice, though, that if we had allowed $t_2$ to be equal to 0, we would have had $t_1 - p_3 < 0$, and therefore the term $Z_{31}^{(p_3)} z \Res([p_1 + p_3, p_2; t_1 - p_3])$ would not appear. In that case, we would have the resolution of a skew-shape with one triple overlap, and the third row flush with the second. Although this doesn’t appear to be of much relevance at this juncture, we will have to be aware of this fact when we get to the section on stems and homotopies.

5.3. Skew-Shape With Two Triple Overlaps. In this section we deal with the shape that was mainly responsible for seeking a more conceptual route into the construction of resolutions of Weyl modules. As the methods used here are very similar to the methods used in the previous sections, we will not go into as much detail; we will simply point out the new considerations that enter in this case.

We start with the Weyl module $[p_1, p_2, p_3; t_1, t_2]$ with $t_1, t_2 \geq 0$ and $p_3 = t_1 + t_2 + 2$. Writing the fundamental exact sequence

$$0 \rightarrow [p_1, p_2 + t_2 + 1, p_3 - (t_2 + 1); t_1 + t_2 + 1, -(t_2 + 1)]$$

$$\partial_{32}^{(t_2 + 1)} : [p_1, p_2, p_3; t_1, t_2 + 1] \rightarrow [p_1, p_2, p_3; t_1, t_2] \rightarrow 0,$$

we see that the middle term is a skew-shape with one triple overlap, while the term at the top (the kernel) is an almost skew-shape with one triple overlap. Since we know the resolutions of these two modules, we must simply define a map from the one to the other, take the mapping cone, and we have our desired resolution. The problem, then, is to define the map and then to identify the resulting mapping cone.

At this point we will use the shorthand way, developed in the preceding subsection, of describing our resolutions. The resolution of $[p_1, p_2 + t_2 + 1, p_3 - (t_2 + 1); t_1 + t_2 + 1, -(t_2 + 1)]$ is

(5.49) $\Res([p_1, p_2 + t_2 + 1; t_1 + t_2 + 1]) \otimes D_{p_3 - (t_2 + 1)}$

(5.50) $\oplus \sum_{l > 0} Z_{32}^{(l)} y \Res([p_1, p_2 + t_2 + 1 + l; t_1 + t_2 + 1 + l]) \otimes D_{p_3 - (t_2 + 1) - l}$

(5.51) $\oplus Z_{31}^{(t_2 + 1)} z \Res([p_1 + t_1 + 1, p_2 + t_2 + 1; t_2]) \, ,
while that of $[p_1, p_2, p_3; t_1, t_2 + 1]$ is
\begin{equation}
\text{Res}([p_1, p_2; t_1]) \otimes D_{p_3, \oplus}.
\end{equation}
(5.52)

\begin{equation}
\sum_{l \geq 0} Z_{32}^{t_2 + 2} \otimes Z_{32}^{(l)} \otimes \text{Res}([p_1, p_2 + t_2 + 2 + l; t_1 + t_2 + 2 + l]) \otimes D_{p_3 - (t_2 + 2 + l)}.
\end{equation}
(5.53)

It is clear, from previous discussions, how to go from terms in 5.49 and 5.50 to the terms 5.52 and 5.53: since we are covering the map $\partial_{32}^{(2)}$, we preface all terms in 5.49, 5.50 and 5.51 with $Z_{32}^{(t_2 + 1)} y$ and in the case of the first two, just polarize the $y$ to 1. Our problem arises when we want to polarize the $y$ to 1 in the terms of the form
\begin{equation}
Z_{32}^{(t_2 + 1)} y Z_{31}^{(t_1 + 1)} z \text{ Res}([p_1 + t_1 + 1, p_2 + t_2 + 1; t_2]),
\end{equation}

since we have no terms of the form $Z_{32}^{(t_2 + 1)} Z_{31}^{(t_1 + 1)} z$. What we see is that we must end up with terms of the form we find in 5.52 and 5.53, that is we have to express $Z_{32}^{(t_2 + 1)} Z_{31}^{(t_1 + 1)} z$ in terms of $Z_{32}^{(l)} y$ and $Z_{21}^{(k)} x$. As in the previous case, we take our clue from certain Capelli identities. We notice that
\begin{equation}
\partial_{21}^{(a)} \partial_{32}^{(a+b)} = \sum_{\alpha \geq 0} (-1)^{\alpha} \partial_{32}^{(a+b-\alpha)} \partial_{21}^{(a+\alpha)} \partial_{31}^{(\alpha)}
\end{equation}
so that
\begin{equation}
(-1)^{\alpha} \partial_{32}^{(b)} \partial_{31}^{(a)} = \partial_{21}^{(a)} \partial_{32}^{(a+b)} - \sum_{0 \leq \alpha < a} (-1)^{\alpha} \partial_{32}^{(b+a-\alpha)} \partial_{21}^{(a-\alpha)} \partial_{31}^{(\alpha)}
\end{equation}
or, equivalently,
\begin{equation}
\partial_{32}^{(b)} \partial_{31}^{(a)} = (-1)^{\alpha} \partial_{21}^{(a)} \partial_{32}^{(a+b)} - \sum_{u > 0} (-1)^{u} \partial_{32}^{(b+u)} \partial_{21}^{(u)} \partial_{31}^{(a-u)}.
\end{equation}

The reason for selecting this particular identity is that it not only expresses $\partial_{32}^{(b)} \partial_{31}^{(a)}$ in terms that begin with $\partial_{21}^{(a)}$, but also terms that begin with divided powers of $\partial_{32}$ that are greater than $b$. These are just the conditions that we need, since the initial $Z_{32}$ terms in 5.53 all involve divided powers that exceed $t_2 + 1$, while the divided powers of the initial $Z_{21}$ terms of 5.52 are bounded below by $t_1 + 1$. This prompts us to make the following definition, which will take into account the extra relations that we need.

**Definition 5.4.** With $A, S$ and $M$ our data as above, define $\text{Cap}'(1, 2, 3)$ to be the submodule of the free bar module on that data to be the submodule spanned by $\text{Cap}'(1, 2, 3)$ as well as all terms of the form
\begin{equation}
(5.54) Z_{32}^{(b)} Z_{31}^{(a)} Z_{21}^{(c_1)} x \cdots x Z_{21}^{(c_q)} x v - (-1)^{a} Z_{21}^{(a)} x Z_{32}^{(a+b)} Z_{21}^{(c_1)} x \cdots x Z_{21}^{(c_q)} x v \sum_{u > 0} (-1)^{u} \binom{c_1 - a}{u} Z_{32}^{(a+u)} y Z_{31}^{(b-u)} Z_{21}^{(c_1+u)} x \cdots x Z_{21}^{(c_q)} x v.
\end{equation}
We will not introduce more decoration to define these relations, but we make the by now familiar remarks about them. Namely:

**Proposition 5.5.** The submodule, \( \text{Cap}\'(1, 2, 3) \), of the free bar module is stable under the boundary map. The resolution of the three-rowed skew-shape with two triple overlaps, \([p_1, p_2, p_3; t_1, t_2], p_3 = t_1 + t_2 + 2\), is the mapping cone of the map described above. Its terms, using the usual shorthand, are

\[
(5.55) \quad \text{Res} ([p_1, p_2; t_1]) \otimes D_{p_3} + \sum_{l \geq 0} Z_{32}^{t_1+1} \otimes Z_{32}^{(l)} \otimes \text{Res} ([p_1, p_2 + t_2 + 1 + l; t_1 + t_2 + 1 + l]) \otimes D_{p_3-(t_2+1+l)} + Z_{32}^{(t_2+1)} y Z_{31}^{(t_1+1)} z \text{Res} ([p_1 + t_1 + 1, p_2 + t_2 + 1; t_2]).
\]

**Proof.** We know from the mapping cone property that the resolution of \([p_1, p_2, p_3; t_1, t_2], p_3 = t_1 + t_2 + 2\), is

\[
(5.56) \quad \text{Res} ([p_1, p_2; t_1]) \otimes D_{p_3} + \sum_{l \geq 0} Z_{32}^{t_2+1} \otimes Z_{32}^{(l)} \otimes \text{Res} ([p_1, p_2 + t_2 + 2 + l; t_1 + t_2 + 2 + l]) \otimes D_{p_3-(t_2+2+l)} + Z_{32}^{(t_2+1)} y \text{Res} ([p_1 + t_1 + 1, p_2 + t_2 + 1; t_2]) \otimes D_{p_3-(t_2+1)} + \sum_{l > 0} Z_{32}^{(l)} y \text{Res} ([p_1 + t_1 + 1, p_2 + t_2 + 1; t_2]) \otimes D_{p_3-(t_2+1) - l}.
\]

We also observe that if we add up the terms in 5.56, 5.57 and 5.58, we get 5.55. For in 5.55, the term corresponding to \( l = 0 \) is obtained from 5.57, and the rest is clear.

From the foregoing examples we see how we can, by means of standard types of Capelli-like identities, write the boundary maps of these complexes without resorting to horrendous \textit{ad hoc} calculations, although it must be admitted that some of our proofs do require no small amount of work.

**Puzzling Observation:** We should point out that when we define our relations 5.54, we have to do the same thing we did before when we defined 5.10 and 5.11 (although here we didn’t break it up into two cases). Namely, we had to explicitly put the binomial coefficients into the definition of the relations, because the ones that would emerge just from a straight operational definition don’t work. However, the difference between the binomial coefficient that we write and the one
that would appear ‘operationally’ is systematic. For example, the coefficient that we write above is \((\lambda_u^1 - a) = (\lambda_{c_1-a}^1)\), while the one that we would get from the operational approach is \((\lambda_{c_1-a+u}^1)\). In the relation 5.11, the coefficient that we wrote is \((b_1+b_2^u)\), while the operational approach would have given us \((b_1+b_2+u)\). In each case, then, we’re off by \(u\) both above and below. In some mysterious way, we feel that this may be explainable by the fact that the term is in each case prefixed by \(Z_{32}^y\), that is, a term of degree \(u\). However, so far we have been unable to give a rationale for this peculiar behavior. We do feel that it is significant, and that its explanation would be of some importance.

We’ll end this section with the description of the resolution of the three-rowed partition \((p, q, 2)\).

**Example 5.1.** Let \(X_n\) be the chains of dimension \(n\). Then

\[
X_0 = D_p \otimes D_q \otimes D_2;
\]

\[
X_1 = \sum_{k>0} Z_{2,1}^{(k)} x D_{p+k} \otimes D_{q-k} \otimes D_2 \oplus \sum_{1 \leq l \leq 2} Z_{3,2}^{(l)} y D_p \otimes D_{q+l} \otimes D_{2-l};
\]

\[
X_2 = \sum_{k_1>0} Z_{2,1}^{(k_1)} x Z_{2,1}^{(k_2)} x D_{p+k} \otimes D_{q-k} \otimes D_2 \oplus \sum_{l=1} Z_{3,2}^{(l)} y Z_{2,1}^{(k_1)} x D_{p+k} \otimes D_{q+l-k} \otimes D_{2-l}
\]

\[
\oplus Z_{3,2}^{(1)} y D_p \otimes D_{q+1} \oplus Z_{3,1}^{(1)} z D_{p+1} \otimes D_{q+1};
\]

\[
\ldots
\]

\[
X_n = \sum_{k_1>0} Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n)} x D_{p+|k|} \otimes D_{q-|k|} \otimes D_2
\]

\[
\oplus \sum_{l=1} Z_{3,2}^{(l)} y Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n-1)} x D_{p+|k|} \otimes D_{q+l-|k|} \otimes D_{2-l}
\]

\[
\oplus \sum_{l=1} Z_{3,2}^{(l)} y Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n-2)} x D_{p+|k|} \otimes D_{q+2-|k|}
\]

\[
\oplus \sum_{k_i>0} Z_{3,2}^{(1)} y Z_{3,1}^{(1)} z Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n-1)} x D_{p+1+|k|} \otimes D_{q+1-|k|}.
\]

The boundary map on the terms of the form

\[
\sum_{k_i>0} Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n)} x D_{p+|k|} \otimes D_{q-|k|} \otimes D_2
\]

is the one used on the two-rowed resolution, with the \(D_2\) just coming along for the ride. Also, the boundary map on the terms of the form

\[
\sum_{k_i>0} Z_{3,2}^{(l)} y Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n-1)} x D_{p+|k|} \otimes D_{q+l-|k|} \otimes D_{2-l}
\]

\[
\sum_{l=1} Z_{3,2}^{(l)} y Z_{3,1}^{(1)} z Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n-2)} x D_{p+1+|k|} \otimes D_{q+1-|k|}
\]

and

\[
\sum_{l=1} Z_{3,2}^{(l)} x Z_{3,2}^{(1)} y Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n-2)} x D_{p+|k|} \otimes D_{q+2-|k|}
\]

are also of the form we’re very familiar with: the same as the map described in [6]. One has only to recall that we use the identity:

\[
Z_{3,2}^{(l)} Z_{2,1}^{(k_1)} x \cdots x Z_{2,1}^{(k_n)} x v = \sum Z_{2,1}^{(k_1-u)} x Z_{3,2}^{(l-u)} Z_{2,1}^{(k_2)} x \cdots x Z_{2,1}^{(k_n)} x \partial_{3,1}^{(u)} v.
\]

The only ‘new’ map in this set-up is the one on the terms of the form

\[
Z_{3,2}^{(1)} y Z_{3,1}^{(1)} x \cdots x Z_{2,1}^{(k_n-2)} x D_{p+1+|k|} \otimes D_{q+1-|k|}.
\]
Since the boundary map on this term sends it to
\[ Z_{3,2}^{(1)} y Z_{3,1}^{(1)} = \partial \left\{ Z^{(k_1)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x D_{p+1+|k|} \otimes D_{q+1-|k|} \right\} \]
\[ \pm Z_{3,2}^{(1)} y \left\{ Z^{(1)}_{3,1} Z^{(k_1)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x D_{p+1+|k|} \otimes D_{q+1-|k|} \right\} \]
\[ \mp Z_{3,2}^{(1)} z \left\{ Z^{(k_1)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x D_{p+1+|k|} \otimes D_{q+1-|k|} \right\} \],
we have to define the terms
\[ Z_{3,1}^{(1)} Z^{(k_1)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x D_{p+1+|k|} \otimes D_{q+1-|k|} \]
and
\[ Z_{3,1}^{(1)} \left\{ Z^{(k_1)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x D_{p+1+|k|} \otimes D_{q+1-|k|} \right\} \]
with \( n \geq 2 \).

If \( n = 2 \), \( Z_{3,1}^{(1)} v = \partial_{3,1}(v) \), while \( Z_{3,2}^{(1)} Z^{(k_1)}_{3,1} z = v = -Z_{2,1}^{(1)} x \partial_{2,1}^{(2)}(v) + Z_{3,2}^{(2)} y \partial_{2,1}^{(1)}(v) \).
For \( n > 2 \), we have
\[ Z_{3,1}^{(1)} Z^{(k_1)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x v = -Z_{2,1}^{(k_1+1)} x Z_{3,2}^{(1)} Z^{(k_2)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x v + \]
\[ \left( k_1 + k_2 \right) Z_{3,2}^{(1)} y Z_{2,1}^{(1)} x \cdots x Z^{(k_n-2)}_{2,1} x v, \]
and
\[ Z_{3,2}^{(1)} Z_{3,1}^{(1)} Z^{(k_1)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x v = -Z_{2,1}^{(1)} x Z_{3,2}^{(2)} Z^{(1)}_{2,1} x \cdots x Z^{(k_n-2)}_{2,1} x v + (k_1 - 1) Z_{3,2}^{(2)} y Z_{2,1}^{(1)} x \cdots x Z^{(k_n-2)}_{2,1} x v. \]

6. Action on Stems and Homotopy-Transitivity

6.1. Stems and Resolutions of Three-Rowed Shapes. In all that we have so far seen, at least for three-rowed shapes, the resolutions that we’ve constructed all have a common subcomplex, which we will call the Stem. Namely, if \( [p_1, p_2, p_3; t_1, t_2] \) is such a shape, then, if we define \( \tau = \max(0, t_2) \), all of our resolutions have the following subcomplex:

\[ \left\{ \begin{array}{l}
\text{Res } ([p_1, p_2; t_1]) \otimes D_{p_3} \\
\sum_{l \geq 0} Z_{32}^{(\tau+1)} \otimes Z_{32}^{(l)} y \text{ Res } ([p_1, p_2 + \tau + l + 1; t_1 + \tau + l + 1]) \otimes D_{p_3-(\tau+l+1)}.
\end{array} \right. \]

Note that if \( \tau = 0 \), then \( Z_{32}^{(\tau+1)} \otimes Z_{32}^{(l)} = Z_{32}^{(l+1)} \).

**Definition 6.1.** If \( [p_1, p_2, p_3; t_1, t_2] \) is a three-rowed shape of the type we have been discussing, We define Stem ([p_1, p_2, p_3; t_1, t_2]) to be the complex 6.1 above.

One important observation to make here is that for almost skew-shapes, as well as for skew-shapes with \( t_2 = 0 \), the stem is independent of the value of \( t_2 \).

If we want to proceed with the construction of resolutions inductively, our next step would be to find the resolution of the three-rowed almost
skew-shape with two triple overlaps. For suppose we were to try to go immediately to the three-rowed skew-shape with three triple overlaps, that is, \( t_2 \geq 0, p_3 = t_1 + t_2 + 3 \), and build the mapping cone coming from the fundamental exact sequence:

\[
(6.2) \quad 0 \rightarrow [p_1, p_2 + t_2 + 1, p_3 - (t_2 + 1); t_1 + t_2 + 1, -(t_2 + 1)] \overset{\partial_{t+1}^{(t_2+1)}}{\rightarrow} [p_1, p_2, p_3; t_1, t_2 + 1] \rightarrow [p_1, p_2, p_3; t_1, t_2] \rightarrow 0.
\]

In that case, \([p_1, p_2, p_3; t_1, t_2 + 1]\) is a skew-shape with two triple overlaps whose resolution we know, but \([p_1, p_2 + t_2 + 1, p_3 - (t_2 + 1); t_1 + t_2 + 1, -(t_2 + 1)]\) is an almost skew-shape with two triple overlaps (unless \( t_2 = -1 \)), and we would not yet have in hand its resolution. We therefore consider the resolution of the three-rowed almost skew-shape with two overlaps: \([p_1, p_2, p_3; t_1, t_2]\), \( t_2 < 0, t_1 + t_2 \geq 0, p_3 = t_1 + t_2 + 2 \), and start with the corresponding exact sequence

\[
(6.3) \quad 0 \rightarrow [p_1 + t_1 + t_2 + 1, p_2, p_3 - (t_1 + t_2 + 1); -(t_2 + 1), t_2 + 1] \overset{\partial_{t+1}^{(t_1+t+1)}}{\rightarrow} [p_1, p_2, p_3; t_1, t_2 + 1] \rightarrow [p_1, p_2, p_3; t_1, t_2] \rightarrow 0
\]

The two new shapes both have just one triple overlap, and they are both skew-shapes in the special case that \( t_2 = -1 \), or both almost skew-shapes. We saw at the end of the section on almost skew-shapes with one triple overlap that the resolution of such a shape in the degenerate case, that is, when \( t_2 = 0 \), is the same as that for the skew-shape with \( t_2 = 0 \), so we may proceed as though our shapes were all almost skew-shapes. Our procedure is now transparent: we take the resolutions of the terms we already know, prefix every term of the resolution of the kernel in the exact sequence 6.3 with \( Z_{31}^{(t_1+t+1)}z \), and define a map of complexes over \( \partial_{31}^{(t_1+t+1)} \). Now the resolution of \([p_1 + t_1 + t_2 + 1, p_2, p_3 - (t_1 + t_2 + 1); -(t_2 + 1), t_2 + 1]\), after prefixing all of its terms as we indicated, is

\[
(6.4) \quad Z_{31}^{(t_1+t+1)}z \text{ Res } ([p_1 + t_1 + t_2 + 1, p_2; -(t_2 + 1)]) \otimes D_1 \oplus Z_{31}^{(t_1+t+1)}zZ_{32}^{(1)}y \text{ Res } ([p_1 + t_1 + t_2 + 1, p_2 + 1; -t_2]) \oplus Z_{31}^{(t_1+t+1)}zZ_{31}^{(1)}z \text{ Res } ([p_1 + t_1 + t_2 + 2, p_2; -(t_2 + 2)]);
\]

the resolution of \([p_1, p_2, p_3; t_1, t_2 + 1]\) is

\[
(6.5) \quad \text{ Res } ([p_1, p_2; t_1]) \otimes D_{p_3} \oplus \sum_{l>0} Z_{31}^{(l)}y \text{ Res } ([p_1, p_2 + l; t_1 + l]) \otimes D_{p_3-l} \oplus Z_{31}^{(t_1+t+2)}z \text{ Res } ([p_1 + t_1 + t_2 + 2, p_2; -(t_2 + 2)]) .
\]

The simplification in 6.4 is due to the fact that \( p_3 = t_1 + t_2 + 2 \), so that \( p_3 - (t_1 + t_2 + 1) = 1 \). This doesn’t leave much leeway for the divided powers of \( Z_{32} \). Notice here too that if we had \( t_2 = -1 \), then neither the terms

\[
Z_{31}^{(t_1+t+1)}zZ_{31}^{(1)}z \text{ Res } ([p_1 + t_1 + t_2 + 2, p_2; -(t_2 + 2)]);
\]

...
nor

$$Z_{31}^{(p_3)} z \Res ([p_1 + p_3, p_2; t_1 - p_3])$$

would survive (since $p_3 = t_1 + 1$ in this case), while all the other terms would be those of the resolutions of skew-shapes.

If we assumed we had a map from 6.4 to 6.5, then we could take its mapping cone and, doing the same kind of consolidation that we have done before, we would see that the resolution of $[p_1, p_2, p_3; t_1, t_2]$ would be

$$\Stem ([p_1, p_2, p_3; t_1, t_2]) \oplus$$

$$\sum_{l_2=0}^{1} \sum_{l_1=0}^{1} \left( Z_{31}^{l_1+t_2+1} \otimes Z_{31}^{(l_1)} \right) z \left( Z_{32}^{(l_2)} \right) y \Res \left( [p_1 + t_1 + t_2 + 1 + l_1, p_2 + l_2; -(t_2 + 1) + l_2 - l_1] \right) \otimes D_{1-t_1-l_2}.$$ 

In this very special case, we observe that $l_1 + l_2 \leq 1$, so again there aren’t too many summands to deal with.

The big “if” is the assumption that we have a description of the map between the resolutions. We have this when we consider the terms

$$Z_{31}^{(t_1+t_2+1)} z \Res ([p_1 + t_1 + t_2 + 1, p_2; -(t_2 + 1)]) \otimes D_1,$$

for if we simply remove the separator $z$ that we arbitrarily inserted, then each term that we get would be recognizable as one of the relations 5.10 or 5.11. For terms of the form

$$Z_{31}^{(t_1+t_2+1)} z Z_{32}^{(1)} y \Res ([p_1 + t_1 + t_2 + 1, p_2 + 1; -t_2]),$$

we see that if we remove the $z$, we have a new entity, $Z_{31}^{(t_1+t_2+1)} Z_{32}^{(1)} y W$, where $W$ is of the form

$$W = Z_{21}^{(c_1)} x \cdots x Z_{21}^{(c_s)} x \otimes m,$$

which we haven’t dealt with yet. (We’re using the fact that $Z_{32}^{(1)}$ is just the complex with $Z_{32}^{(1)}$ in dimension zero.) In addition, we have terms of the form $Z_{31}^{t_1+t_2+1} Z_{31}^{(1)} z W$ coming from the last of the terms of 6.4, which we would think could be handled simply as $(t_1 + t_2 + 2)Z_{31}^{t_1+t_2+2} z W$. However, this doesn’t work, in the sense that this would not give us a map from one resolution to the other which commutes with the boundary. We will illustrate this fact after we have handled the first-mentioned new situation, that is, the definition of the term $Z_{31}^{(a)} Z_{32}^{(b)} y W$. To anticipate a bit, we will actually deal with terms of the form

$$Z_{31}^{(a)} Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_s)} y W.$$
In the usual way, but again without adding new decorations to the symbols, we define the relation

\[(6.7)\]

\[Z_{31}^{(a)} Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_s)} y \otimes m = Z_{32}^{(b_1)} y Z_{32}^{(b_2)} y \cdots y Z_{32}^{(b_s)} y Z_{21}^{(c)} y \otimes m + \]

\[(-1)^{s+1} \sum_{a>0} (-1)^{u} Z_{32}^{(a)} y Z_{31}^{(a-a)} Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_s)} y \otimes \partial_{21}^{(c+u)} (m);\]

\[Z_{31}^{(a)} Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_s)} y Z_{21}^{(c)} yW = Z_{32}^{(b_1)} y Z_{31}^{(a)} Z_{32}^{(b_2)} y \cdots y Z_{32}^{(b_s)} y Z_{21}^{(c)} y Z_{21}^{(c_2)} xW'\]

\[+ (-1)^{s+1} \sum_{a>0} (-1)^{u} (c_1+c_2) Z_{32}^{(a)} y Z_{31}^{(a-a)} Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_s)} y Z_{21}^{(c_1+c_2+u)} xW'.\]

We have distinguished three cases to indicate that when \(W\) is of degree zero, the operation by \(Z_{31}^{(a)}\) essentially commutes. The second case is to emphasize the fact that when \(W\) is of dimension one, no binomial coefficient enters the picture. The last case, the general one, is much of the form of 5.11. Now that we have taken care of this case, we can update the definition of our submodule of relations and make the

**Definition 6.2.** With \(A, S\) and \(M\) our data as above, define \(\text{Cap}(1, 2, 3)\) to be the submodule of the free bar module on that data to be the submodule spanned by \(\text{Cap}''(1, 2, 3)\) as well as all terms of the form 6.7 above.

In case the reader was wondering why our previous definitions of the relations module had all the primes and double primes, this was the reason: it’s clear that we had to get to the relations in 6.7 before we had the full definition of the relations submodule.

As in the previous section, the proof that these relations are compatible with the boundary map is complicated, but not very different from what has gone before. We will therefore omit that proof here; the only caveat that must be given at this point is that we assume that \(c_1\) is larger than all the \(b\)'s. Since we are assuming that all our terms that we are operating on are in Stem, this is a natural assumption. What is perhaps more surprising is the statement made above that the simple multiplication of the \(Z_{31}^{(a)}\) terms that seems as though it should work, doesn’t. Obviously it does work in dimension zero. That is, if we take \(Z_{31}^{(t_1-1)} z Z_{31}^{(1)} z \otimes m\) and map it to \(t_1 Z_{31}^{(t_1)} z \otimes m\) and then take the boundary, we get \(t_1 \partial_{31}^{(t_1)} (m)\). Similarly, if we first go down to \(Z_{31}^{(t_1-1)} z \otimes \partial_{31}^{(1)} (m)\) and then over to \(\partial_{31}^{(t_1-1)} \partial_{31}^{(1)} (m)\), we again get \(t_1 \partial_{31}^{(t_1)} (m)\). But once we operate on terms of dimension greater than zero, we hit some trouble. We will illustrate this with an example.

Consider the following situation. Suppose that \(t_2 = -2\), so that \([p_1 + t_1 + t_2 + 2, p_2; -(t_2 + 2)] = [p_1 + t_1, p_2; 0]\). We want to map from

\[Z_{31}^{(t_1-1)} z Z_{31}^{(1)} z \text{ Res } ([p_1 + t_1, p_2; 0])\]
to
\[
\text{Res}([p_1, p_2; t_1]) \otimes D_{p_3} + \\
\sum_{l>0} Z_{32}^{(l)} y \text{ Res}([p_1, p_2 + l; t_1 + l]) \otimes D_{p_3-l}, \]
\[Z_{31}^{(t_1)} z \text{ Res}([p_1 + t_1, p_2; 0]).\]

presumably by taking an element of the form \(Z_{31}^{(t_1-1)} z Z_{31}^{(1)} z Z_{21}^{(k)} x \otimes m\) to
\(t_1 Z_{31}^{(t_1)} z Z_{21}^{(k)} x \otimes m\), where \(m \in D_{p_1+t_1+k} \otimes D_{p_2-k}\). But if this is to do the job, we must have that
\[
\begin{align*}
& t_1 Z_{31}^{(t_1)} z Z_{21}^{(k)} x \otimes m - t_1 Z_{31}^{(t_1)} z \otimes \partial_{21}^{(k)} (m) = \\[ \]
& Z_{31}^{(t_1-1)} z \otimes m - Z_{31}^{(t_1-1)} Z_{31}^{(1)} z \otimes \partial_{21}^{(k)} (m).
\end{align*}
\]
Evaluating the top line we get
\[
(-1)^{t_1} t_1 Z_{21}^{(t_1+k)} x \otimes \partial_{32}^{(t_1)} (m) - \\
 t_1 \sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(t_1-u)} \partial_{21}^{(k+u)} \partial_{32} (m) - t_1 Z_{31}^{(t_1)} z \otimes \partial_{21}^{(k)} (m),
\]
while the bottom line gives us
\[
-t_1 Z_{31}^{(t_1)} z \otimes \partial_{32} (m) + Z_{31}^{(t_1)} Z_{32}^{(1)} y \otimes \partial_{21}^{(k+1)} (m) - t_1 Z_{31}^{(t_1)} z \otimes \partial_{21}^{(k)} (m) = \\
(-1)^{t_1} t_1 Z_{21}^{(t_1+k)} x \partial_{32}^{(t_1)} (m) + \\
\sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(t_1-u)} \partial_{21}^{(k+u+1)} \partial_{32} (m) + \\
Z_{32}^{(1)} y \otimes \partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)} (m) - \\
t_1 Z_{31}^{(t_1)} z \otimes \partial_{21}^{(k)} (m).
\]
Notice that we have used the lowest dimensional case of 6.7 in evaluating
\(Z_{31}^{(t_1-1)} Z_{32}^{(1)} y \otimes \partial_{21}^{(k+1)} (m)\) and have set
\[
Z_{31}^{(t_1)} z \otimes \partial_{21}^{(k)} (m) = t_1 Z_{31}^{(t_1)} z \otimes \partial_{21}^{(k)} (m).
\]
It is here that we have to start to study the operator \(Z_{31}\) under composition. The first thing we see is that the difference between the two lines is
\[
-t_1 \sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(t_1-u)} \partial_{21}^{(k+u)} (m)
\]
\[
- \sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(t_1-u-1)} \partial_{21}^{(k+u+1)} \partial_{32} (m) - \\
Z_{32}^{(1)} y \otimes \partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)} (m).
\]
If we take the boundary of this difference, we get
\[
- t_1 \sum_{u>0} (-1)^u \partial_{32}^{(u)} \partial_{31}^{(t_1-u)} \partial_{21}^{(k+u)}(m) - \\
\sum_{u>0} (-1)^u \partial_{32}^{(u)} \partial_{31}^{(t_1-u-1)} \partial_{21}^{(k+u+1)}(m) - \\
\partial_{32}^{(1)} \partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)}(m).
\]

Using our Capelli identities, we have:
\[
- t_1 \sum_{u>0} (-1)^u \partial_{32}^{(u)} \partial_{31}^{(t_1-u)} \partial_{21}^{(k+u)}(m) = -(-1)^t_1 t_1 \partial_{21}^{(t_1+k)} \partial_{32}^{(t_1)} + t_1 \partial_{31}^{(t_1)} \partial_{21}^{(k)};
\]
\[
- \sum_{u>0} (-1)^u \partial_{32}^{(u)} \partial_{31}^{(t_1-u-1)} \partial_{21}^{(k+u+1)}(m) = (-1)^t_1 t_1 \partial_{21}^{(t_1+k)} \partial_{32}^{(t_1)} + \partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)} \partial_{32};
\]
\[
- \partial_{32}^{(1)} \partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)}(m) = -\partial_{31}^{(t_1-1)} \left( \partial_{21}^{(k+1)} \partial_{32}^{(1)} + \partial_{21}^{(k)} \partial_{31}^{(1)} \right)
\]  
\[
= -\partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)} \partial_{32}^{(1)} - t_1 \partial_{31}^{(t_1)} \partial_{21}^{(k)}.
\]

If we sum up all these lines, we get zero, that is, the boundary of this difference is zero. Thus, although our initial difference is not zero, it is a cycle and therefore, bounds. In our special case, we actually see that 6.8 is the boundary of
\[
U = \sum_{u>0} (-1)^u Z_{32}^{(u)} y Z_{32}^{(1)} y \otimes \partial_{31}^{(t_1-1-u)} \partial_{21}^{(k+u+1)}(m).
\]

For
\[
\partial (U) = \sum_{u>0} (-1)^u (u+1) Z_{32}^{(u+1)} y \otimes \partial_{31}^{(t_1-1-u)} \partial_{21}^{(k+u+1)}(m) - \\
\sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{32}^{(1)} \partial_{31}^{(t_1-1-u)} \partial_{21}^{(k+u+1)}(m),
\]

which equals
\[
\sum_{w>1} (-1)^w w Z_{32}^{(w)} y \otimes \partial_{31}^{(t_1-w)} \partial_{21}^{(k+w)}(m) + \\
\sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(t_1-1-u)} \partial_{21}^{(k+u+1)} \partial_{32}^{(1)}(m) + \\
\sum_{u>0} (-1)^u (t_1-u) Z_{32}^{(u)} y \otimes \partial_{31}^{(t_1-u)} \partial_{21}^{(k+u)}(m),
\]
which gives us
\[
\sum_{u>0} (-1)^w t_1 Z_{32}^{(w)} y \otimes \partial_{31}^{(t_1-w)} \partial_{21}^{(k+w)}(m) + \]
\[
t_1 Z_{32}^{(1)} y \otimes \partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)}(m) - \]
\[
(t_1 - 1) Z_{32}^{(1)} y \otimes \partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)}(m) + \]
\[
\sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(t_1-1-u)} \partial_{21}^{(k+u+1)} \partial_{32}^{(1)}(m),
\]
from which we finally get
\[
Z_{32}^{(1)} y \otimes \partial_{31}^{(t_1-1)} \partial_{21}^{(k+1)}(m) + \]
\[
\sum_{u>0} (-1)^w t_1 Z_{32}^{(w)} y \otimes \partial_{31}^{(t_1-w)} \partial_{21}^{(k+w)}(m) + \]
\[
\sum_{u>0} (-1)^u Z_{32}^{(u)} y \otimes \partial_{31}^{(t_1-1-u)} \partial_{21}^{(k+u+1)} \partial_{32}^{(1)}(m).
\]

This is the negative of the difference, 6.8.

What this example serves to show is that the action of \(Z_{31}\) on words of the form \(W = Z_{32}^{(b_1)} \cdots y Z_{21}^{(a_1)} x \cdots x Z_{21}^{(a_q)} x \otimes m\) is not transitive, but is homotopy-transitive. This means that \(Z_{31}^{(a_1)} Z_{31}^{(a_2)} W\) is not equal to \((a_1 + a_2) Z_{31}^{(a_1 + a_2)} W\), but that there is an element, which we will call
\[
s_{a_1,a_2}^{*}(W),
\]
where the \(\ast\) stands for the suitable dimension (in this instance, \(s + q\)), such that
\[
Z_{31}^{(a_1)} Z_{31}^{(a_2)} W - \left(\frac{a_1 + a_2}{a_2}\right) Z_{31}^{(a_1 + a_2)} W = \partial s_{a_1,a_2}^{*}(W) + s_{a_1,a_2}^{*} \partial(W).
\]

Notice that our word, \(W\), is a typical term in the complex that we have called Stem \(\left[\cdots \right]\). In the next subsection, we will discuss the action of \(Z_{31}^{(a)}\) on the complex Stem \(\left[\cdots \right]\), and indicate where we have gotten to in defining the homotopy that we mention above.

An immediate question that may come to mind is why we have to study the action on stems; after all, we have only seen the problem of this action appear in the case of words, \(W\), of the form \(Z_{21}^{(a_1)} x \cdots x Z_{21}^{(a_q)} x \otimes m\). However, as we iterate the mapping cone procedure in the study of shapes with more triple overlaps, we will clearly have to consider resolutions of the kernel of the fundamental exact sequence in which we already have a prefix of \(Z_{31}^{(a)}\) on a stem, and then will have to affix another \(Z_{31}^{(a')}\) in front of that. We already saw this in the case of our term
\[
\sum_{l_2=0}^{1} \sum_{l_1=0}^{1} \left( Z_{31}^{l_1 + l_2 + 1} \otimes Z_{31}^{l_1} \right) z \left( Z_{32}^{l_2} \right) y \Res(\cdots)
\]
We then let \( p \) know the resolutions of these two modules. The resolution of the kernel is an almost skew-shape also with \( r \).

For three-rowed shapes we already have the necessary tools and notation to write down the terms of the resolution of the general skew-shape and the general almost skew-shape, so we will conclude this subsection with a complete description of these terms.

**Theorem 6.1.**

1. Let \([p_1, p_2, p_3; t_1, t_2]\) be a skew-shape. Then the terms of its resolution are
   \[
   \text{Stem} ([p_1, p_2, p_3; t_1, t_2]) \oplus
   \sum_{l_i \geq 0} \left( Z^{l_2+1}_{32} \otimes Z^{(l_1)}_{32} \right) y \left( Z^{l_1+1}_{31} \otimes Z^{(l_1)}_{31} \right) z
   \]
   \[
   \text{Res} ([p_1 + t_1 + l_1 + 1, p_2 + t_2 + l_2 + 1; t_2 + l_2 - l_1]) \otimes D_{p_3 - (|t_1 + l_1| + 2)}.
   \]

2. Let \([p_1, p_2, p_3; t_1, t_2]\) be an almost skew-shape. Then the terms of its resolution are
   \[
   \text{Stem} ([p_1, p_2, p_3; t_1, t_2]) \oplus
   \sum_{l_i \geq 0} \left( Z^{l_1+t_2+1}_{31} \otimes Z^{(l_1)}_{31} \right) z \left( Z^{l_2}_{32} \right) y
   \]
   \[
   \text{Res} ([p_1 + t_1 + t_2 + l_1 + 1, p_2 + t_2 + 1; -(t_2 + 1) + l_2 - l_1]) \otimes D_{p_3 - (|t_1 + l_1| + 2)}.
   \]

**Proof.** The proof proceeds by induction on the number of triple overlaps of the shape; we have already verified the theorem for small \( r \). Suppose that we know the theorem to be true for all shapes with at most \( r \) triple overlaps. We then let \([p_1, p_2, p_3; t_1, t_2]\) be a skew-shape with \( r + 1 \) triple overlaps, and consider the exact sequence 6.2:

\[
0 \rightarrow [p_1, p_2 + t_2 + 1, p_3 - (t_2 + 1); t_1 + t_2 + 1, -(t_2 + 1)]_{p_3^{(t_2+1)}} \rightarrow [p_1, p_2, p_3; t_1, t_2 + 1] \rightarrow [p_1, p_2, p_3; t_1, t_2] \rightarrow 0.
\]

The shape in the middle is again a skew-shape with \( r \) triple overlaps, while the kernel is an almost skew-shape also with \( r \) triple overlaps. In either case, we know the resolutions of these two modules. The resolution of \([p_1, p_2 + t_2 + 1, p_3 - (t_2 + 1); t_1 + t_2 + 1, -(t_2 + 1)]\), with each term prefixed with \( Z^{(t_2+1)}_{32} \) is

\[
Z^{(t_2+1)}_{32} y \text{Res} ([p_1, p_2 + t_2 + 1; t_1 + t_2 + 1]) \otimes D_{p_3 - (t_2+1)} \oplus
\]
\[
Z^{(t_2+1)}_{32} y \sum_{l > 0} Z^{(l)}_{32} y
\]
\[
\text{Res} ([p_1, p_2 + t_2 + 1 + l; |t_1| + 1 + l]) \otimes D_{p_3 - (t_2+1+l)} \oplus
\]
\[
Z^{(t_2+1)}_{32} y \left( Z^{l_1+1}_{31} \otimes Z^{(l_1)}_{31} \right) z Z^{(l)}_{32} y
\]
\[
\text{Res} ([p_1 + t_1 + l_1 + 1, p_2 + t_2 + l_2 + 1; t_2]) \otimes D_{p_3 - (|t_1 + l_1| + 2)}.
\]
and the resolution of \([p_1, p_2, p_3; t_1, t_2 + 1]\) is

\[
\begin{align*}
\text{Res} ([p_1, p_2; t_1]) \otimes D_{p_3} & \oplus \\
\sum_{t \geq 0} \left( Z_{t_2+2}^{(l)} \otimes Z_{t_2}^{(l)} \right) y \\
\text{Res} ([p_1, p_2 + t_2 + l + 2; |t| + l + 2]) \otimes D_{p_3 - (t_2 + l + 2)} & \oplus \\
\sum_{t \geq 0} \left( Z_{t_2+2}^{(l)} \otimes Z_{t_2}^{(l)} \right) y \left( Z_{t_3+1}^{(l)} \otimes Z_{t_3+1}^{(l)} \right) z \\
\text{Res} ([p_1 + t_1 + l_1 + 1, p_2 + t_2 + l_2 + 2; t_2 + 1 + l_2 - l_1]) & \otimes D_{p_3 - (|t| + |l| + 3)}.
\end{align*}
\]

Whatever the map between these two resolutions (and one exists, since these are projective resolutions of their respective Weyl modules over the Schur algebra of appropriate degree, ??), if we form the mapping cone and just enumerate the terms without regard to boundary maps, the resulting terms are:

\[
\begin{align*}
\left\{ \text{Res} ([p_1, p_2; t_1]) \otimes D_{p_3} \oplus \right. \\
Z_{t_2+1}^{(l)} y \sum_{t \geq 0} Z_{t_2}^{(l)} y \\
\left. \text{Res} ([p_1, p_2 + t_2 + l + 1; |t| + l + 1]) \otimes D_{p_3 - (t_2 + l + 1)} \oplus \\
\sum_{t \geq 0} \left( Z_{t_2+1}^{(l)} \otimes Z_{t_2}^{(l)} \right) y \left( Z_{t_3+1}^{(l)} \otimes Z_{t_3+1}^{(l)} \right) z \\
\text{Res} ([p_1 + t_1 + l_1 + 1, p_2 + t_2 + l_2 + 1; t_2 + l_2 - l_1]) & \otimes D_{p_3 - (|t| + |l| + 2)}
\right\} = \text{Stem} ([p_1, p_2, p_3; t_1, t_2])
\end{align*}
\]

which is the result we want. The sleight of hand used here in bringing \(\sum_{t \geq 0} Z_{t_2}^{(l)} y\) all the way to the left of \(Z_{t_3+1}^{(l)} \otimes Z_{t_3+1}^{(l)}\) will have to be justified if we want to say that this representation of the terms is also a representation of the complexes they connote, but for now, since we are just focusing on the terms of the resolutions, we are done with the first part of the theorem.

We now let \([p_1, p_2, p_3; t_1, t_2]\) be an almost skew-shape with \(r + 1\) triple overlaps, and consider the exact sequence 6.3:

\[
0 \rightarrow [p_1 + t_1 + t_2 + 1, p_2, p_3 - (t_1 + t_2 + 1); -(t_2 + 1), t_2 + 1] \rightarrow [p_1 + t_1 + t_2 + 1, p_2, p_3 - (t_1 + t_2 + 1); -(t_2 + 1), t_2 + 1] \rightarrow [p_1, p_2, p_3; t_1, t_2] \rightarrow 0.
\]

Then, as above, the shapes \([p_1 + t_1 + t_2 + 1, p_2, p_3 - (t_1 + t_2 + 1); -(t_2 + 1), t_2 + 1]\) and \([p_1, p_2, p_3; t_1, t_2 + 1]\) have \(r\) triple overlaps, and they are almost skew-shapes if \(t_2 < -1\). We will first deal with this case.

By our theorem for \(r\) triple overlaps, the resolution of \([p_1 + t_1 + t_2 + 1, p_2, p_3 - (t_1 + t_2 + 1); -(t_2 + 1), t_2 + 1]\), prefixed with \(Z_{t_3+1}^{(l)} \otimes Z_{t_3+1}^{(l)} z\), is

\[
\begin{align*}
Z_{t_3+1}^{(l)} z & \text{Res} ([p_1 + |t| + 1, p_2; -(t_2 + 1)]) \otimes D_{p_3 - (|t| + 1)} \oplus \\
Z_{t_3+1}^{(l)} z & \sum_{t \geq 0} Z_{t_2}^{(l)} y \\
\text{Res} ([p_1 + |t| + 1, p_2 + l; -(t_2 + 1) + l]) & \otimes D_{p_3 - (|t| + l + 1)} \oplus \\
Z_{t_3+1}^{(l)} z & \sum_{t \geq 0} \left( Z_{t_3+1}^{(l)} \otimes Z_{t_3+1}^{(l)} \right) z \left( Z_{t_2}^{(l)} \right) y \\
\text{Res} ([p_1 + |t| + l_1 + 2, p_2 + l_2 + 1; -(t_2 + 1) - 1 + l_2 - l_1]) & \otimes D_{p_3 - (|t| + |l| + 3)}.
\end{align*}
\]
The resolution of \([p_1, p_2, p_3; t_1, t_2 + 1]\) is

\[(6.12)\]
\[
\sum_{l_i \geq 0} \left( Z_{31}^{[l]+2} \otimes Z_{31}^{(t_i)} \right) \sum_{Y} \left( Z_{32}^{(l_2)} \right) y \cdot \\text{Res} \left( (p_1 + |l| + l_1 + 2, p_2 + l_2; -(t_2 + 2) + l_2 - l_1) \right) \otimes D_{p_3 - (|l|+|l|+2)}.
\]

Given a map from 6.11 to 6.12, the collection of terms for the mapping cone would be

\[
\sum_{l_i \geq 0} \left( Z_{31}^{[l]+1} \otimes Z_{31}^{(t_i)} \right) \sum_{Y} \left( Z_{32}^{(l_2)} \right) y \cdot \\text{Res} \left( (p_1 + |l| + l_1 + 1, p_2 + l_2; -(t_2 + 1) + l_2 - l_1) \right) \otimes D_{p_3 - (|l|+1)}.
\]

which is the desired result. We have here made use of some useful facts. First, we notice that if \(t_2 < 0\), then the stems of \([p_1, p_2, p_3; t_1, t_2]\) and \([p_1, p_2, p_3; t_1, t_2 + 1]\) are the same. We also see that \(Z_{31}^l \otimes Z_{31}^{(t_i)}\) is the same as \(Z_{31}^{(l+i+1)}\). The rest of the consolidation of terms is straightforward.

If \(t_2 = -1\), we have to see how far our description in 6.10 differs from that in 6.9 when \(t_2 \text{ 'degenerates' to } 0\). Our problem here is that the term \(\sum_{l_i \geq 0} \left( Z_{32}^{t_1+1} \otimes Z_{32}^{(l_2)} \right) y \left( Z_{31}^{t_1+1} \otimes Z_{31}^{(l_1)} \right) z \cdot \text{Res} (\cdots)\) appears in 6.9 while the term \(\sum_{l_i \geq 0} \left( Z_{31}^{t_1+t_2+1} \otimes Z_{31}^{(l_2)} \right) z \left( Z_{32}^{(l_2)} \right) y \cdot \text{Res} (\cdots)\) appears in 6.10. The two \(\text{Res} (\cdots)\) terms are the same if we assume \(t_2 = 0\) and for them to exist in the resolutions, we must have \(l_2 > l_1\). Also, if \(t_2 = 0\), the complex terms become \(\sum_{l_i \geq 0} \left( Z_{32} \otimes Z_{32}^{(l_2)} \right) y \left( Z_{31}^{t_1+1} \otimes Z_{31}^{(l_1)} \right) z \) and \(\sum_{l_i \geq 0} \left( Z_{31}^{t_1+1} \otimes Z_{31}^{(l_1)} \right) z \left( Z_{32}^{(l_2)} \right) y\), but with the restriction that \(l_2 > l_1\), we see that the index \(l_2\) in the latter term must be positive. Since \(Z_{32}^l \otimes Z_{32}^{(l_2)} = Z_{32}^{(l_2+1)}\), the indices that can occur in the first term are the same as those that occur in the second. Therefore, we see that just counting terms, the two complexes are the same. This concludes the proof of the theorem. \(\blacksquare\)

Let us look briefly at the question about what further relations in the bar complex will allow us to make the exchange of position as complexes that we asked in the above proof. It seems clear that the following will do the job:

\[
\begin{align*}
Z_{31}^{(a)} Z_{32}^{(b)} &= Z_{32}^{(b)} Z_{31}^{(a)}; \\
Z_{31}^{(a)} z Z_{32}^{(b)} y &= Z_{31}^{(a)} Z_{32}^{(b)} z y; \\
Z_{32}^{(b)} y Z_{31}^{(a)} z &= Z_{32}^{(b)} Z_{31}^{(a)} y z.
\end{align*}
\]
With these relations, we get, for example,

\[
\begin{align*}
Z^{(b)}_{32}yZ^{(a_1)}_{31}zZ^{(a_2)}_{31} & = \\
Z^{(a_1)}_{31}Z^{(b)}_{32}yzZ^{(a_2)}_{31} & = \\
-Z^{(a_1)}_{31}Z^{(b)}_{32}yzZ^{(a_2)}_{31} & = \\
-Z^{(a_1)}_{31}Z^{(a_2)}_{31}Z^{(b)}_{32}yz & = \\
Z^{(a_1)}_{31}Z^{(a_2)}_{32}zy & = \\
Z^{(a_1)}_{31}Z^{(a_2)}_{32}zZ^{(b)}_{32}y & = 
\end{align*}
\]

6.2. Homotopy Transitivity. In the previous subsection, we looked at a particular case of the action of \(Z^{(a)}_{31}\) on an element of the stem of a shape.

We saw that if \(m\) is an element of \(D \otimes D \otimes D\), then the action of \(Z^{(a)}_{31}\) is transitive, but if \(W\) is of the form \(Z^{(b)}_{21}x \otimes m\), it isn’t. We saw, however, in a special instance of that case, that we had an element whose boundary gave us the ‘deficiency’ in the transitivity; this led us to think of studying the homotopy that would describe the homotopy transitivity of this action.

What we shall do here is study a few more cases in which we can explicitly write down the homotopy, and indicate some alternative routes that might make this problem more tractable.

First, let us simplify our exposition. The elements of \(\text{Stem}(X)\) are spanned by elements of the form

\[
V = Z^{(b_1)}_{32}y \cdots yZ^{(b_r)}_{32}yZ^{(c_1)}_{21}x \cdots xZ^{(c_q)}_{21}x \otimes m,
\]

where the exponents are subjected to certain constraints, and \(m\) is contained in a suitable tensor product of divided powers. We will work in general without specifying these constraints. We will refer, though, to the elements, \(V\), of type 6.13a as though they were elements of some stem complex, and sometimes say \(V \in \text{Stem}\). Also, in our applications, the powers of \(Z^{(a)}_{31}\) are generally specified; in our discussion here, we will work with arbitrary powers of \(Z^{(a)}_{31}\). In all of this, we will be assuming that \(\text{Stem}\) is in the quotient bar module obtained as the quotient of the free bar module modulo \(\text{Cap}(1, 2, 3)\), and we will avoid decorating our elements with superlines and superbraces.

This means that when we write \(Z^{(a)}_{31}V\), we mean \(\widetilde{Z^{(a)}_{31}}V\), and so forth.

What we are looking for, then is a sequence of maps

\[
s^{n}_{a_1, a_2} : (\text{Stem})^n \to (\text{Stem})^{n+1}
\]

such that, if \(V\) is of dimension \(n\),

\[
Z^{(a_1)}_{31}Z^{(a_2)}_{31}V - \left(a_1 + a_2 \atop a_2\right)Z^{(a_1+a_2)}_{31}V = \partial s^{n}_{a_1, a_2} (V) + s^{n-1}_{a_1, a_2} \partial (V),
\]

where \(\partial\) stands for the boundary map of \(\text{Stem}\).
We have already seen that, since the action is transitive in dimension zero, we may define $s^{0}_{a_1, a_2}$ to be the zero map. Taking our cue from our particular example in the preceding subsection, and noting that the action of $Z_{31}$ commutes with terms of the form $V$ when $q = 0$, we are tempted to define

\[(6.15)\]

$$s_{a_1,a_2}^{1}\left(Z_{31}^{(c)} x \otimes m\right) = \sum_{a_i>0} (-1)^{|a_i|} Z_{32}^{(u_1)} y Z_{32}^{(u_2)} y \otimes \partial_{31}^{(a_1-u_1)} \partial_{31}^{(a_2-u_2)} \partial_{21}^{(c+|u|)} (m);$$

$$s_{a_1,a_2}^{1}\left(Z_{32}^{(b)} x \otimes m\right) = 0.$$

It is easy to prove the following proposition:

**Proposition 6.2.** The map $s_{a_1,a_2}^{1}$ defined above satisfies the condition 6.14.

**Proof.** In this low dimension, given that $s_{a_1,a_2}^{0}$ is the zero map, we have only to verify

$$Z_{31}^{(a_1)} Z_{31}^{(a_2)} V = \left(a_1 + a_2\right) Z_{31}^{(a_1+a_2)} V = \partial s_{a_1,a_2}^{1} (V),$$

where $V$ is either $Z_{21}^{(c)} x \otimes m$ or $Z_{32}^{(b)} x \otimes m$. In the latter case, we know that $Z_{31}^{(a_1)} Z_{31}^{(a_2)} V = \left(a_1+a_2\right) Z_{31}^{(a_1+a_2)} V = 0$, so we may concentrate on the former case. There we have

$$Z_{31}^{(a_1)} Z_{31}^{(a_2)} V =$$

$$Z_{31}^{(a_1)} \left((-1)^{a_2} Z_{21}^{(a_2+c)} x \otimes \partial_{32}^{(a_2)} (m) - \sum_{u>0} (-1)^{u} Z_{32}^{(u)} y \otimes \partial_{31}^{(a_2-u)} \partial_{21}^{(c+u)} (m)\right) =$$

$$(-1)^{a_2} \sum_{u>0} (-1)^{u} Z_{32}^{(u)} y \partial_{31}^{(a_1-u)} \partial_{21}^{(a_2+c+u)} \partial_{32}^{(a_2)} (m) -$$

$$\sum_{w>0} (-1)^{w} Z_{32}^{(w)} y \otimes \partial_{31}^{(a_1)} \partial_{31}^{(a_2-w)} \partial_{21}^{(c+w)} (m) =$$

$$(-1)^{|a|} \left(a_1 + a_2\right) Z_{21}^{(|a|+c)} x \otimes \partial_{32}^{(|a|)} (m) -$$

$$(-1)^{a_2} \sum_{u>0} (-1)^{u} Z_{32}^{(u)} y \partial_{31}^{(a_1-u)} \partial_{21}^{(a_2+c+u)} \partial_{32}^{(a_2)} (m) -$$

$$\sum_{w>0} (-1)^{w} Z_{32}^{(w)} y \otimes \partial_{31}^{(a_1)} \partial_{31}^{(a_2-w)} \partial_{21}^{(c+w)} (m).$$
Calculating $(a_1 + a_2)Z_{31}^{(a_1 + a_2)}V$, we get

$$(a_1 + a_2)Z_{31}^{(a_1 + a_2)}V = (-1)^{|a|/a_2}Z_{31}^{(a_1 + a_2)}V = \left(-1\right)^{|a|/a_2}Z_{21}^{(|a| + c)}x \otimes \partial_{32}^{(|a|)}(m) - \left(a_1 + a_2\right) \sum_{w > 0} (-1)^w Z_{32}^{(w)} y \otimes \partial_{31}^{(|a| - w)}\partial_{21}^{(c + w)}(m).$$

Thus the difference, $Z_{31}^{(a_1)}Z_{32}^{(a_2)}V - (a_1 + a_2)Z_{31}^{(a_1 + a_2)}V$ is equal to

$$(a_1 + a_2) \sum_{w > 0} (-1)^w Z_{32}^{(w)} y \otimes \partial_{31}^{(|a| - w)}\partial_{21}^{(c + w)}(m) - \sum_{w > 0} (-1)^w Z_{32}^{(w)} y \otimes \partial_{31}^{(a_1)}\partial_{31}^{(a_2 - w)}\partial_{21}^{(c + w)}(m) - (-1)^a_2 \sum_{w > 0} (-1)^w Z_{32}^{(w)} y \partial_{31}^{(a_1 - u)}\partial_{21}^{(a_2 + c + u)}\partial_{32}^{(a_2)}(m) - \sum_{w > 0} (-1)^w Z_{32}^{(w)} y \otimes \partial_{31}^{(|a| - w)}\partial_{21}^{(c + w)}(m).$$

To calculate $\partial_{s_{a_1,a_2}}(V)$, we have to evaluate

$$\partial \left(\sum_{u_1 > 0} (-1)^{|u_1|/u_1} Z_{31}^{(u_1)} y Z_{32}^{(u_2)} y \otimes \partial_{31}^{(a_1 - u_1)}\partial_{31}^{(a_2 - u_2)}\partial_{21}^{(c + |u_1|)}(m)\right).$$

But this is

$$(6.17)\sum_{u_1 > 0} (-1)^w Z_{31}^{(u_1)} y \sum_{u_2 \geq 0} (-1)^u_2 \partial_{31}^{(u_2)}\partial_{31}^{(a_1 - u_1)}\partial_{31}^{(a_2 - u_2)}\partial_{21}^{(c + |u_1|)}(m) = \sum_{w > 0} (-1)^{\left|a_1\right|-w/a_1} Z_{32}^{(w)} y \partial_{31}^{(a_1 - w)}\partial_{21}^{(c + w)}(m) - \sum_{w > 0} (-1)^w \left|a_1\right|-w/a_1 Z_{31}^{(w)} y \partial_{31}^{(a_1 - w)}\partial_{21}^{(c + w)}(m) - (-1)^{a_2} \sum_{u_1 > 0} (-1)^u_1 Z_{31}^{(u_1)} y \sum_{u_2 \geq 0} \partial_{31}^{(a_1 - u_1)}\partial_{21}^{(a_2 + c + u_1)}\partial_{32}^{(a_2)}(m),$$

where the last line is obtained by applying 2.2 to 6.17. These last three terms agree with the difference already calculated (since $\sum_{w > 0} (-1)^w Z_{32}^{(w)} y \otimes \partial_{31}^{(a_1)}\partial_{31}^{(a_2 - w)}\partial_{21}^{(c + w)}(m) = \sum_{w > 0} (-1)^w \left(\partial_{31}^{(|a| - w)}\partial_{21}^{(c + w)}(m)\right)$, and we have our proposition.

The next step would be to consider two-dimensional elements, $V$. We may exclude the pure $Z_{32}$ term, since our action of $Z_{31}$ on these terms is transitive, so that the homotopy, $s_{a_1,a_2}^n$, on such terms is always zero. In fact, because of the triviality of this action, it is not difficult to study the more general case of terms of the form

$$V = Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_i)} y \otimes m$$
and

\[ V = Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_r)} y Z_{21}^{(c)} x \otimes m. \]

In the first case, the homotopy value is easily seen to be zero. To handle the second type of term efficiently, we will introduce some notation. Say we have two sets of indices, \( b_1, \ldots, b_r \), and \( u_1, \ldots, u_k \). We will write \( Z_{32}^{(b)} y \) to signify the term \( Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_r)} y \) and we will write Shuf \( \left( Z_{32}^{(b)} y; Z_{32}^{(a)} y \right) \) to signify the signed sum of all the shuffles of \( Z_{32}^{(u)} y \) through \( Z_{32}^{(b)} y \). For example, if \( r = 2 \) and \( k = 1 \), Shuf \( \left( Z_{32}^{(b)} y; Z_{32}^{(a)} y \right) \) would be

\[ Z_{32}^{(b_1)} y Z_{32}^{(a)} y Z_{32}^{(b_2)} y - Z_{32}^{(b_1)} y Z_{32}^{(u)} y Z_{32}^{(b_2)} y + Z_{32}^{(u)} y Z_{32}^{(b_1)} y Z_{32}^{(b_2)} y. \]

The first observation we make is the following

**Lemma 6.3.** Let \( V = Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_r)} y Z_{21}^{(c)} x \otimes m \). Then

\[
Z_{31}^{(u)} V = (-1)^u Z_{32}^{(b)} y Z_{21}^{(a+c)} x \otimes \partial_{32}^{(a)} (m) - \sum_{u>0} (-1)^u \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(a)} y \right) \otimes \partial_{31}^{(a-u)} \partial_{21}^{(c+u)} (m).
\]

**Proof.** The proof is a straightforward application of induction on \( r \). If \( r = 1 \), it is easy to see that

\[
Z_{31}^{(u)} V = (-1)^u Z_{32}^{(b)} y Z_{21}^{(a+c)} x \otimes \partial_{32}^{(a)} (m) - \sum_{u>0} (-1)^u Z_{32}^{(u)} y Z_{32}^{(a)} y \partial_{31}^{(a-u)} \partial_{21}^{(c+u)} (m) + \sum_{u>0} (-1)^u Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a-u)} \partial_{21}^{(c+u)} (m).
\]

Then we take \( V = Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_{r-1})} y Z_{21}^{(c)} x \otimes m \), and we see that

\[
Z_{31}^{(u)} V = Z_{32}^{(b_1)} y Z_{31}^{(a)} Z_{32}^{(b_2)} y \cdots y Z_{32}^{(b_{r-1})} y Z_{21}^{(c)} x \otimes m - \sum_{u>0} (-1)^u Z_{32}^{(u)} y Z_{31}^{(a-u)} Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_{r-1})} y \otimes \partial_{21}^{(c+u)} (m).
\]

Applying induction to the top line, and using the fact that \( Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_{r-1})} y \otimes \partial_{21}^{(c+u)} (m) \) is a pure \( Z_{32} \) term, we get that

\[
Z_{31}^{(u)} V = (-1)^u Z_{32}^{(b_1)} y Z_{32}^{(a)} y Z_{21}^{(a+c)} x \otimes \partial_{32}^{(a)} (m) - Z_{32}^{(b_1)} y \sum_{u>0} (-1)^u \text{Shuf} \left( Z_{32}^{(b_1)} y; Z_{32}^{(a)} y \right) \otimes \partial_{31}^{(a-u)} \partial_{21}^{(c+u)} (m) + \sum_{u>0} (-1)^u Z_{32}^{(u)} y Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_{r-1})} y \otimes \partial_{31}^{(a-u)} \partial_{21}^{(c+u)} (m).
\]
where $b'$ is the sequence $b_2, \ldots, b_{r+1}$. The bottom line is what is needed to turn Shuf $\left( Z_{32}^{(b')} y; Z_{32}^{(u)} y \right)$ into Shuf $\left( Z_{32}^{(b)} y; Z_{32}^{(u)} y \right)$, and the proof is complete. $
abla$

Proposition 6.4. Let $V = Z_{32}^{(b_1)} y \cdots y Z_{32}^{(b_r)} y Z_{21}^{(c)} x \otimes m$, with $r \geq 0$. Define

$$s_{a_1, a_2}^{r+1}(V) = (-1)^r \sum_{u_i > 0} (-1)^{u_1+u_2} \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(u)} y \right) \otimes \partial_{31}^{(a_1-u)} \partial_{32}^{(a_2-u)} \partial_{21}^{(c+u)}(m).$$

Then

$$Z_{31}^{(a_1)} Z_{31}^{(a_2)} V - \left( \frac{a_1 + a_2}{a_2} \right) Z_{31}^{(a_1+a_2)} V = \partial s_{a_1, a_2}^{r+1}(V) + s_{a_1, a_2}^r(V).$$

Proof. The previous lemma makes it very easy to see that

$$Z_{31}^{(a_1)} Z_{31}^{(a_2)} V =$$

$$(-1)^{|a|} \left( \frac{|a|}{a_2} \right) Z_{32}^{(b)} y Z_{21}^{(|a|+c)} x \otimes \partial_{32}^{(|a|)}(m) -$$

$$(-1)^{a_2} \sum_{u > 0} (-1)^u \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(u)} y \right) \otimes \partial_{31}^{(a_1-u)} \partial_{21}^{(a_2-u)} \partial_{32}^{(c+u)}(m) -$$

$$\sum_{u > 0} (-1)^u \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(u)} y \right) \otimes \partial_{31}^{(a_1)} \partial_{31}^{(a_2-u)} \partial_{21}^{(c+u)}(m).$$

It is equally easy, if not more so, to see that

$$\left( \frac{a_1 + a_2}{a_2} \right) Z_{31}^{(a_1+a_2)} V =$$

$$(-1)^{|a|} \left( \frac{|a|}{a_2} \right) Z_{32}^{(b)} y Z_{21}^{(|a|+c)} x \otimes \partial_{32}^{(|a|)}(m) -$$

$$\left( \frac{|a|}{a_2} \right) \sum_{u > 0} (-1)^u \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(u)} y \right) \otimes \partial_{31}^{(|a|-u)} \partial_{21}^{(c+u)}(m).$$

Thus

$$Z_{31}^{(a_1)} Z_{31}^{(a_2)} V - \left( \frac{a_1 + a_2}{a_2} \right) Z_{31}^{(a_1+a_2)} V =$$

\[ (6.18) \] $$\left( \frac{|a|}{a_2} \right) \sum_{u > 0} (-1)^u \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(u)} y \right) \otimes \partial_{31}^{(|a|-u)} \partial_{21}^{(c+u)}(m) -$$

\[ (6.19) \] $$(-1)^{a_2} \sum_{u > 0} (-1)^u \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(u)} y \right) \otimes \partial_{31}^{(a_1-u)} \partial_{21}^{(a_2+c+u)} \partial_{32}^{(a_2)}(m) -$$

\[ (6.20) \] $$\sum_{u > 0} (-1)^u \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(u)} y \right) \otimes \partial_{31}^{(a_1)} \partial_{31}^{(a_2-u)} \partial_{21}^{(c+u)}(m).$$

Next we have to consider

$$\partial s_{a_1, a_2}^{r+1}(V) + s_{a_1, a_2}^r(V).$$
From the definition of $s_{a_1,a_2}^{r+1}(V)$, we must compute the boundary of

$$(-1)^r \sum_{u_i > 0} (-1)^{u_1 + u_2} \text{Shuf} \left( Z_{32}^{(a_1 + u_1 - u_i)}; Z_{32}^{(u_2)} \right) \partial_{31}^{(a_1 - u_1)} \partial_{31}^{(a_2 - u_2)} \partial_{21}^{(c + |u|)}(m).$$

The important thing to observe here is that there is a great deal of internal cancellation. The terms that remain after that cancellation can be described schematically as

$$(-1)^r \sum_{u_i > 0} (-1)^{u_1 + u_2} \text{Shuf} \left( \partial \left( Z_{32}^{(b)}; Z_{32}^{(u)} \right) \partial_{31}^{(a_1 - u_1)} \partial_{31}^{(a_2 - u_2)} \partial_{21}^{(c + |u|)}(m);$$

$$\sum_{u_1 > 0} \sum_{u_2 \geq 0} (-1)^{|u|} \binom{|a| - |u|}{a_1 - u_1} \binom{|u|}{u_1} \text{Shuf} \left( Z_{32}^{(b)}; Z_{32}^{(u_1 + u_2)} \right) \partial_{32}^{(a_1 - u_1)} \partial_{32}^{(a_2 - u_2)} \partial_{21}^{(c + |u|)}(m);$$

$$- \sum_{u_1 > 0} (-1)^{u_1} \text{Shuf} \left( Z_{32}^{(b)}; Z_{32}^{(u_1)} \right) \partial_{32}^{(a_1 - u_1)} \partial_{32}^{(a_2 - u_2)} \partial_{21}^{(c + |u|)}(m);$$

$$- \sum_{u_1 > 0} \sum_{u_2 \geq 0} (-1)^{|u|} \text{Shuf} \left( Z_{32}^{(b)}; Z_{32}^{(u_2)} \right) \partial_{32}^{(b)} \partial_{32}^{(a_1 - u_1)} \partial_{32}^{(a_2 - u_2)} \partial_{21}^{(c + |u|)}(m).$$

By $b'$ we mean the sequence $b_1, \ldots, b_{r-1}$; by $\text{Shuf} \left( \partial \left( Z_{32}^{(b)}; Z_{32}^{(u)} \right) \right)$ we mean that we take the sum of the formal boundary terms (with sign) of $Z_{32}^{(b)}y$, and take the shuffles of each with the sequence $u_1, u_2$. We would have, in general, the sequences $(-1)^{r+1} b_1, \ldots, b_1 + b_i + \ldots + 1, \ldots, b_r$ shuffled with $u_1, u_2$. We may allow $u_2 \geq 0$ for the usual reason.

One way to see that the internal cancellation leaves us with these terms is to partition the $(r+2)$ shuffles into the $(r+1)$ that have $u_2$ on the extreme right, and the $(r+1)$ that have $b_i$ on the extreme right. An easy argument gives the internal cancellation among the first kind, and a simple induction on $r$ gives the cancellations among those of the second kind. Finally, all terms in the boundary that have $a \cdots y Z_{32}^{(b)} Z_{32}^{(u_1 + u_2)} y$ from the first kind, cancel with the terms that have $a \cdots y Z_{32}^{(b)} Z_{32}^{(b)} y$ from the second kind.

Finally we must evaluate $s_{a_1,a_2}^{r}(V)$. Using our same schematics, we see that the terms that occur here are

$$(-1)^{r-1} \sum_{u_i > 0} (-1)^{u_1 + u_2} \text{Shuf} \left( \partial \left( Z_{32}^{(b)}; Z_{32}^{(u)} \right) \partial_{31}^{(a_1 - u_1)} \partial_{31}^{(a_2 - u_2)} \partial_{21}^{(c + |u|)}(m);$$

$$+ \sum_{u_1 > 0} \sum_{u_2 \geq 0} (-1)^{|u|} \text{Shuf} \left( Z_{32}^{(b)}; Z_{32}^{(u)} \right) \partial_{32}^{(b)} \partial_{32}^{(a_1 - u_1)} \partial_{32}^{(a_2 - u_2)} \partial_{21}^{(c + |u|)}(m).$$
Thus when we take the sum, \( \partial s_{a_1, a_2}^{r+1} (V) + s_{a_1, a_2}^r \partial (V) \), what we're left with is

\[
(6.21) \sum_{u_1 > 0} \sum_{u_2 \geq 0} (-1)^{|u|} \left( \left| a \right| - \left| u \right| \right) \left( \left| u \right| \right) \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(|u|)} y \right) \otimes \partial_{31}^{(a-|u|)} \partial_{21}^{(c+|u|)} (m) -
\]

(6.22) \sum_{u_1 > 0} (-1)^{|u|} \text{Shuf} \left( Z_{32}^{(b)} y; Z_{32}^{(|u_1|)} y \right) \otimes \sum_{u_2 \geq 0} (-1)^{|u_2|} \partial_{32}^{(u_2)} \partial_{31}^{(a_1 - u_1)} \partial_{31}^{(a_2 - u_2)} \partial_{21}^{(c+|u|)} (m).

The by now familiar arguments tell us that 6.21 is equal to 6.18 and 6.20; an application of the Capelli identity, 2.2, tells us that 6.22 is equal to 6.19. This completes the proof of the proposition.

At this point we should consider the two-dimensional term

\[ V = Z_{21}^{(c_1)} y Z_{21}^{(c_2)} x \otimes m. \]

Since the action of \( Z_{31} \) on these terms is more complicated, even involving some anomalous binomial coefficients, we can expect that our homotopy will start to become a bit more complicated. The first thing we have to evaluate is

\[
(6.23) Z_{31}^{(a_1)} Z_{31}^{(a_2)} V - \left( \frac{a_1 + a_2}{a_2} \right) Z_{31}^{(a_1 + a_2)} V.
\]

A straightforward calculation gives us

\[
Z_{31}^{(a_1)} Z_{31}^{(a_2)} V =
\]

\[
(-1)^{|a|} Z_{21}^{(|a|+c_1)} x \sum_{k+l \leq c_2} Z_{21}^{(c_2-k-l)} x \otimes \partial_{32}^{(a_1-l)} \partial_{32}^{(a_2-k)} \partial_{31}^{(l)} \partial_{31}^{(k)} (m) -
\]

\[
(-1)^{|a|} \sum \left( \begin{array}{c} |c| \end{array} \right) \left( \begin{array}{c} c_2 - u \end{array} \right) Z_{32}^{(|a|+|c|)} y Z_{21}^{(|a|+|c|)} x \otimes \partial_{32}^{(a_1)} \partial_{32}^{(a_2-u)} (m) +
\]

\[
(-1)^{|a_1|} \sum \left( \begin{array}{c} |c| \end{array} \right) \left( \begin{array}{c} c_2 - u_1 \end{array} \right) (-1)^{|u_2|} \text{Shuf} \left( Z_{32}^{(|u_1|)} y; Z_{32}^{(|u_2|)} y \right) \otimes \partial_{31}^{(a_1-u_2)} \partial_{21}^{(a_2+|c|+u_2)} \partial_{32}^{(a_2-u_1)} (m) +
\]

\[
(-1)^{|a_2|} \sum \left( \begin{array}{c} |c| \end{array} \right) \left( \begin{array}{c} c_2 - u_1 \end{array} \right) Z_{32}^{(|c|+|u|)} y Z_{32}^{(|c|+|u|)} y \otimes \partial_{31}^{(a_2-|u|)} \partial_{31}^{(a_2+|c|+|u|)} \partial_{21}^{(a_2-k)} \partial_{31}^{(k)} (m) -
\]

\[
(-1)^{|a_2|} \sum \frac{a_2 + |c| - k}{c_2 - k - u} Z_{32}^{(|a|+|c|-k)} y Z_{21}^{(|a|+|c|-k)} x \otimes \partial_{32}^{(a_1-u)} \partial_{32}^{(a_2-k)} \partial_{31}^{(k)} (m) +
\]

\[
(-1)^{|a_2|} \sum \frac{a_2 + |c| - k}{c_2 - k - u_1} Z_{32}^{(|a|+|c|-k)} y Z_{32}^{(|a|+|c|-k)} y \otimes \partial_{31}^{(a_1-|u|)} \partial_{21}^{(a_2+k+|c|+|u|-k)} \partial_{32}^{(a_2-k)} \partial_{31}^{(k)} (m),
\]

while a simpler calculation gives

\[
\left( \begin{array}{c} |a| \end{array} \right) Z_{31}^{(|a|)} V =
\]
way that the above difference is equal to

\( (\text{DIFF}) \)

Thus our difference, 6.23, is

\[
(-1)^{|a|} \frac{|a|}{a_1} \sum_{\alpha} Z_{21}^{(a)-a_1} x \sum_{\alpha} Z_{21}^{(c_{2}-\alpha)} x \otimes \partial_{32}^{(a)-a_1} \partial_{31}^{(c_{2})} (m) -
\]

\[
(-1)^{|a|} \frac{|a|}{a_1} \sum \left( |c| \right) \sum_{c_{2}-u} Z_{21}^{(u)} y Z_{21}^{(a)-u} x \otimes \partial_{32}^{(a)-u} \partial_{31}^{(c_{2}-u)} (m) +
\]

\[
\left( \frac{|a|}{a_1} \right) \sum_{u} (-1)^{u+w} \left( |c| \right) Z_{21}^{(u)} y Z_{21}^{(w)} x \otimes \partial_{32}^{(a)-u} \partial_{31}^{(c_{2}-u)} (m) +
\]

\[
\sum(-1)^{u+w} \left( |c| \right) Z_{21}^{(u)} y Z_{21}^{(w)} x \otimes \partial_{32}^{(a)-u} \partial_{31}^{(c_{2}-u)} (m) +
\]

\[
(-1)^{|a|} \sum_{u} \left( |c| \right) Z_{21}^{(u)} y Z_{21}^{(w)} x \otimes \partial_{32}^{(a)-u} \partial_{31}^{(c_{2}-u)} (m) +
\]

\[
(-1)^{|a|} \sum_{k} \left( a_{2}+a_{2}-k \right) \sum_{c_{2}-u} Z_{21}^{(a)-u} x \otimes \partial_{32}^{(a)-u} \partial_{31}^{(c_{2}-u)} (m) +
\]

\[
(-1)^{|a|} \sum_{k} \left( a_{2}+a_{2}-k \right) \sum_{c_{2}-u} Z_{21}^{(a)-u} x \otimes \partial_{32}^{(a)-u} \partial_{31}^{(c_{2}-u)} (m).
\]

It is this difference that we have to realize by defining \( s_{a_1,a_2}^2 (V) \) in such a way that the above difference is equal to

\[
\partial s_{a_1,a_2}^2 (V) + s_{a_1,a_2}^1 \partial (V).
\]

Since

\[
\partial (V) = \left( \frac{|c|}{c_2} \right) Z_{21}^{(c_{2})} x \otimes m - Z_{21}^{(c_{1})} x \otimes \partial_{21}^{(c_{2})} (m),
\]

we see easily that \( s_{a_1,a_2}^1 \partial (V) \) is equal to

\[
\left( \frac{|c|}{c_2} \right) \sum_{u_{1},>0} (-1)^{|u|} \frac{|u|}{u_1} \sum_{c_{2}-u} Z_{21}^{(u_{1})} y Z_{32}^{(a_{1})} y \otimes \partial_{31}^{(a_{1}-u_{1})} \partial_{31}^{(a_{2})} \partial_{31}^{(c_{2})} (m) -
\]

\[
\left( \frac{|c|}{c_2} \right) \sum_{u_{1},>0} (-1)^{|u|} \frac{|u|}{u_1} \sum_{c_{2}-u} Z_{21}^{(u_{1})} y Z_{32}^{(a_{1})} y \otimes \partial_{31}^{(a_{1}-u_{1})} \partial_{31}^{(a_{2})} \partial_{31}^{(c_{2})} (m).
\]

Looking at the terms of DIFF above, and taking a clue from what \( s_{a_1,a_2}^1 \) turned out to be, we would assume that a good first approximation to what we are after might be

\[
\sum_{u_{1},>0} (-1)^{|u|} \left( \frac{|c|}{c_2} \right) \left( \frac{|a|-|u|}{a_1-u_1} \right) Z_{21}^{(u_{1})} y Z_{32}^{(a_{1})} y \otimes \partial_{31}^{(a_{2})} \partial_{31}^{(c_{2})} (m).
\]
If we calculate the boundary of this term, we do eliminate all the terms in DIFF that involve a term having a 

$\sum_{u_i > 0} (-1)^{|u|} \left( \frac{|c| + u_3}{c_2 - u_1} \right) \left( \frac{|a| - |u|}{a_2 - u_3} \right) Z_{32}^{(u_1)} y Z_{32}^{(u_2)} Z_{32}^{(u_3)} \bigotimes \partial_{31}^{(|a| - |u|)} \partial_{21}^{(|c| + |u|)} (m) -$ 

(6.26) 

$\sum_{u_i > 0} (-1)^{|u|} \left( \frac{|c|}{c_2 - u_1} \right) \left( \frac{|a| - |u|}{a_2 - u_3} \right) \text{Shuf} \left( Z_{32}^{(u_1)} y ; Z_{32}^{(u_2)} y \right) y Z_{32}^{(u_3)} \bigotimes \partial_{31}^{(|a| - |u|)} \partial_{21}^{(|c| + |u|)} (m),$ 

using the usual tricks of adding and subtracting terms that we need in order to make usual identities on binomial coefficients work out, shows us that we may define 

$s_{a_1, a_2}^2 \left( Z_{21}^{(c_1)} y Z_{21}^{(c_2)} x \otimes m \right) = $ 

$\sum_{u_i > 0} (-1)^{|u|} \left( \frac{|c| + u_3}{c_2 - u_1} \right) \left( \frac{|a| - |u|}{a_2 - u_3} \right) Z_{32}^{(u_1)} y Z_{32}^{(u_2)} y Z_{32}^{(u_3)} \bigotimes \partial_{31}^{(|a| - |u|)} \partial_{21}^{(|c| + |u|)} (m) +$ 

(6.27) 

$\sum_{u_i > 0} (-1)^{|u|} \left( \frac{|c| + u_3}{c_2 - u_1} \right) \left( \frac{|a| - |u|}{a_2 - u_3} \right) \text{Shuf} \left( Z_{32}^{(u_1)} y ; Z_{32}^{(u_2)} y \right) y Z_{32}^{(u_3)} \bigotimes \partial_{31}^{(|a| - |u|)} \partial_{21}^{(|c| + |u|)} (m) -$ 

(6.27) 

$\sum_{u_i > 0} (-1)^{|u|} \left( \frac{|c|}{c_2 - u_1} \right) \left( \frac{|a| - |u|}{a_2 - u_1 - u_3} \right) \text{Shuf} \left( Z_{32}^{(u_1)} y ; Z_{32}^{(u_2)} y \right) y Z_{32}^{(u_3)} \bigotimes \partial_{31}^{(|a| - |u|)} \partial_{21}^{(|c| + |u|)} (m).$

There is one very important observation that one must make in order to make the statement about straightforward computation true. Namely, we have to put the term 

$(-1)^{a_2} \sum_{k, u_1, u_2} \left( \frac{a_2 + |c| - k}{c_2 - k - u_1} \right) Z_{32}^{(u_1)} y Z_{32}^{(u_2)} y \bigotimes \partial_{31}^{(a_1 - |u|)} \partial_{21}^{(a_2 + |c| + |u| - k)} \partial_{32}^{(a_2 - k)} \partial_{31}^{(k)} (m),$ 

into a form that is more recognizable as part of the boundary of $s_{a_1, a_2}^2 \left( Z_{21}^{(c_1)} y Z_{21}^{(c_2)} x \otimes m \right).$ To do this, we first use the Capelli identity, 2.2, to see that 

$\sum_{\alpha \geq 0} (-1)^{a_2 - k - \alpha} \left( \begin{array}{c} a_2 - \alpha \\ k \end{array} \right) \partial_{31}^{(a_1 - |u|)} \partial_{21}^{(a_2 + |c| + |u| - k)} \partial_{32}^{(a_2 - k)} \partial_{31}^{(k)} =$ 

$\sum_{\alpha \geq 0} (-1)^{a_2 - k - \alpha} \left( \begin{array}{c} a_2 - \alpha \\ k \end{array} \right) \partial_{31}^{(a_1 - |u|)} \partial_{21}^{(a_2)} \partial_{32}^{(a_2 - k)} \partial_{31}^{(k)} \partial_{21}^{(|c| + |u| + \alpha)} \partial_{31}^{(a_2 - \alpha)}.$ 

We then use the fact that 

$\sum_{k} (-1)^{a_2 - k - \alpha} \left( \begin{array}{c} a_2 + |c| - k \\ c_2 - k - u_1 \end{array} \right) \left( \begin{array}{c} a_2 - \alpha \\ k \end{array} \right) = (-1)^{a_2 + \alpha} \left( \begin{array}{c} |c| + \alpha \\ c_2 - u_1 \end{array} \right).$
This is seen as follows:

\[
\sum_k (-1)^{a_2-k-\alpha} \binom{a_2 + |c| - k}{c_2 - k - u_1} \binom{a_2 - \alpha}{k} = \\
(-1)^{a_2 + c_2 + u_1 + \alpha} \sum_k \binom{-a_2 - c_1 - u_1 + 1}{c_2 - k - u_1} \binom{a_2 - \alpha}{k} = \\
(-1)^{a_2 + c_2 + u_1 + \alpha} \binom{-c_1 - u_1 - \alpha + 1}{c_2 - u_1} = (-1)^{a_2 + \alpha} \binom{|c| + \alpha}{c_2 - u_1}.
\]

It is this observation that suggested the introduction of the term 6.26.

6.3. **Discussion.** In this subsection we want to make a few remarks of a purely informal nature.

1. We have not proceeded further than the definition of \(s_{a_1,a_2}^2\) above because the form of the terms was not one that lent itself immediately to generalization. It seemed that the term,

\[
\sum_{u_1 > 0} (-1)^{|u|} \binom{|c|}{c_2 - |u|} Z_{31}^{[a_1 u_1]} y Z_{32}^{[a_2 u_2]} y Z_{31}^{(a_1 - u_1)} Z_{31}^{(a_2 - u_2)} Z_{21}^{(|c| + |u|)} x \otimes m,
\]

would have been more like the term we used in dimension one, but this led to more baroque computations and didn’t seem to suggest any more satisfying form for the terms of \(s_{a_1,a_2}^2\) \(\binom{Z_{21}^{(c_1)} y Z_{21}^{(c_2)} x \otimes m}{}\).

Of course, since we know that \(Z_{31}^{(a_1 - u_1)} Z_{31}^{(a_2 - u_2)} Z_{21}^{(|c| + |u|)} x \otimes m\) and \(Z_{31}^{(|u| - |u_1|)} Z_{21}^{(|c| + |u|)} x \otimes m\) differ by \(\partial s_{a_1-u_1,a_2-u_2}^{(c_1)} Z_{21}^{(|c| + |u|)} x \otimes m\), we could replace the one term by the other and compensate with the boundary term. But so far, that hasn’t led us much further.

2. Another observation that we have not been able to exploit is that we could replace the operation of \(Z_{31}^{(a)}\) on elements of \(D \otimes D \otimes D\) by \(\pm \partial_{31}^{(a)} \partial_{32}^{(a)}\) instead of \(\partial_{31}^{(a)}\) since the difference is a boundary in Stem. In the calculation of some examples, this replacement of the one for the other seemed more natural from a letter-place point of view, but again we were unable to use this fact in any conceptual way.

3. It does seem clear that one need only define the homotopy on the terms of Stem that just involve \(Z_{21}\), since one can extend to the mixed terms in much the same way that we did in the one-dimensional case.

4. In looking for the terms that appear in the definition of the homotopy, we have restricted ourselves to terms in Stem, and the action of \(Z_{31}\) on Stem. Although it is true that, by induction, one can prove that

\[
Z_{31}^{(a_1)} Z_{31}^{(a_2)} V - \binom{a_1 + a_2}{a_2} Z_{31}^{(a_1 + a_2)} V - s_{a_1,a_2}^{u-1} \partial (V)
\]

is a cycle, one cannot really say that it is ‘therefore’ a boundary, since Stem is not necessarily acyclic. Of course, we feel that it is a boundary in Stem, and thus far that is the case. However, the example that gave
rise to our search for the homotopy arose from the very special situation in which \(a_2 = 1\). But when \(a_2 = 1\), the homotopy is much easier to deal with. In this case, we were mapping into an acyclic complex with only the one \(Z_{31}\) term, namely \(Z_{31}^{a+1}\). When \(a_2 > 1\), we are in a situation where we are mapping into an acyclic complex which has far more \(Z_{31}\) terms in it; conceivably the homotopy in general could involve some of these terms (of lower degree).

7. Heisenberg-Weyl and Capelli

In this section, \(^1\) we make a connection between the algebra of place polarizations that we have been discussing, and the Heisenberg-Weyl algebra. The reader has by this time observed that we have had to use a number of off-beat ‘Capelli identities’, and may have wondered how some of them may have been guessed at or even proved initially. In some cases, using the connection with the Heisenberg-Weyl algebra made a number of these identities more tractable.

The Heisenberg-Weyl algebra (H-W algebra, or simply H-W, from now on) is linearly the tensor product of the symmetric algebra in the alphabet \(A = \{a, b, c, \ldots\}\) and the divided power algebra in the alphabet \(\partial A = \{\partial_a, \partial_b, \partial_c, \ldots\}\), subject to the conditions

\[
\partial_a a - a \partial_a = 1; \\
b \partial_a = \partial_b b \text{ if } a \neq b.
\]

A consequence of these relations is Leibnitz’ Rule:

**Proposition 7.1. (Leibnitz Rule)**

1. For all integers \(k, l \geq 0\), we have

\[
\partial_b^{(k)} b^l = \sum_{j \geq 0} \binom{k}{j} b^{l-j} \partial_b^{(k-j)}; \\
\partial_b^{k} b^{(l)} = \sum_{j \geq 0} \binom{k}{j} b^{(l-j)} \partial_b^{k-j}.
\]

2. We also have the ‘signed’ Leibnitz Rule:

\[
b^l \partial_b^{(k)} = \sum_{j \geq 0} (-1)^j \binom{l}{j} \partial_b^{(k-j)} b^{l-j}; \\
b^{(l)} \partial_b^{k} = \sum_{j \geq 0} (-1)^j \binom{k}{j} \partial_b^{k-j} b^{(l-j)}.
\]

\(^1\)We believe that the terminology in this area is not written in stone; to some readers what we will be describing is called the Weyl algebra, while the homogeneous form of this would be the Heisenberg algebra. We are taking the path of least resistance and hyphenating.
Proof. We’ll simply indicate the proof of the first identity. For \( k = 0 \), the result is clearly true for all \( l \). For \( k = 1 \), we use induction on \( l \) (the case \( l = 1 \) being the fundamental identity 7.1. For \( l > 1 \), we have
\[
\partial_b b^{l+1} = \partial_b b^l b = b' \partial_b b + lb^{l-1} b = b'(b \partial_b + 1) + lb^l = b^{l+1} \partial_b + (l + 1)b^l.
\]
Now assume it true for \( k \) (and all \( l \)) and prove it for \( k + 1 \). We have
\[
(k + 1) \partial_b^{(k+1)} b^l = \partial_b \partial_b^{(k)} b^l = \partial_b \sum_{j \geq 0} \binom{l}{j} b^{l-j} \partial_b^{(k-j)} = \sum_{j \geq 0} \binom{l}{j} (b^{l-j} \partial_b + (l - j)b^{l-j-1}) \partial_b^{(k-j)} = \sum_{j \geq 0} \binom{l}{j} (k - j + 1)b^{l-j} \partial_b^{(k-j+1)} + \sum_{j \geq 0} \binom{l}{j} (l - j)b^{l-j-1} \partial_b^{(k-j)}.
\]
But this last is equal to
\[
(k + 1) \sum_{j \geq 0} \binom{l}{j} b^{l-j} \partial_b^{(k+1-j)} - \sum_{j \geq 0} \binom{l}{j} j b^{l-j} \partial_b^{(k-j+1)} + \sum_{j \geq 0} \binom{l}{j} (l - j + 1)b^{l-j} \partial_b^{(k+1-j)} = 0.
\]
Since
\[
\sum_{j \geq 1} \binom{l}{j} (l - j + 1)b^{l-j} \partial_b^{(k+1-j)} - \sum_{j \geq 0} \binom{l}{j} j b^{l-j} \partial_b^{(k-j+1)} = 0,
\]
we get the result. □

From the Leibnitz Rules, we can also obtain the identity:
\[
(-1)^l \binom{l + k}{l} b^k = \sum_{u \geq 0} (-1)^u \partial_b^{(u)} b^{l+k} \partial_b^{(l-u)}.
\]
This last identity is useful in proving, for example, that
\[
\partial_b^{(l)} \partial_b^{(k)} = (-1)^l \partial_b^{(l+k)} \partial_b^{(l)} + \sum_{u \geq 0} (-1)^{l-u} \partial_b^{(u)} \partial_b^{(l+k)} \partial_b^{(l-u)}.
\]
The reader will recall that this is our identity 5.12. Another identity that is basic is
\[
(7.4) \quad \partial_b^{(l)} = \sum_{\alpha \geq 0} (-1)^{\alpha - 1} \binom{k}{\alpha} \partial_b^{(l+\alpha)} b^\alpha + (-1)^k b^k \partial_b^{(l+k)}.
\]
This one is useful in proving the identity that we used in defining the relations 5.54, namely:
\[
\partial_b^{(l)} \partial_b^{(k)} = \sum_{\alpha \geq 0} (-1)^{\alpha - 1} \partial_b^{(l+\alpha)} \partial_b^{(k-\alpha)} b^\alpha + (-1)^k b^k \partial_b^{(l+k)} \partial_b^{(l+k)}.
\]
To prove 7.4, we make use of the Leibnitz Rule and set
\[
\sum_{\alpha \geq 0} (-1)^{\alpha - 1} \binom{k}{\alpha} b^\alpha = \sum_{\alpha \geq 0} (-1)^{\alpha - 1} \binom{k}{\alpha} \sum_{j \geq 0} \binom{\alpha}{j} b^{\alpha - j} \partial_b^{(l+\alpha - j)}.
\]
The coefficient of \( \partial_b^{(l)} \) is \( \sum_{\alpha > 0} (-1)^{\alpha-1} \binom{k}{\alpha} = 1 \). For \( \alpha - j \) strictly between 0 and \( k \), the coefficient of \( b^{\alpha-j} \partial_b^{(l+\alpha-j)} \) is 0. For \( \alpha - j = k \), the coefficient of \( b^k \partial_b^{(l+k)} \) is \( (-1)^{k-1} \). This proves
\[
\partial_b^{(l)} = \sum_{\alpha > 0} (-1)^{\alpha-1} \binom{k}{\alpha} b^{(l+\alpha)} + (-1)^k b^k \partial_b^{(l+k)}.
\]

We imbed the polarization algebra of \( A \) into H-W by sending
\[
D_{ba} \mapsto b \partial_a.
\]

By the polarization algebra of \( A \) we mean the algebra generated by all the place polarizations of the letter-place algebra with a positive letter alphabet and whose positive place alphabet is the alphabet, \( A \).

The first thing we have to show is that the fundamental Capelli identity
\[
D_{cb}D_{ba} - D_{ba}D_{cb} - D_{ca}
\]
is carried into zero. That is, we want to show:

**Proposition 7.2.** For all \( a, b \) and \( c \) in \( A \), we have
\[
(c \partial_b)(b \partial_a) = (b \partial_a)(c \partial_b) + c \partial_a.
\]

**Proof.** Applying 7.1,
\[
(c \partial_b)(b \partial_a) = c(b \partial_b + 1) \partial_a =
\]
\[
\partial_b \partial_a + c \partial_a = (b \partial_a)(c \partial_b) + c \partial_a.
\]

Earlier in this paper, we spoke about place polarizations \( D_{ab} \) with \( a \ne b \), and indicated that later we would discuss the case when \( a = b \). This is the point at which we can do this, because in the H-W algebra the product \( a \partial_a \) is perfectly well defined. In fact, interpreting \( \partial_a \) as the differential with respect to the variable \( a \), we see that \( \partial_a(a^n) = na^{n-1} \). Thus \( \partial_a(a^{(n)}) = a^{(n-1)} \) and \( a \partial_a(a^{(n)}) = aa^{(n-1)} = na^{(n)} \). Thus \( D_{aa} \) would mean the place polarization that takes \( a \) to \( a \) and does what we would expect: it takes \( a^{(n)} \) to \( n \) times itself, namely \( a^{(n)} \mapsto a^{(n-1)}a \). Although we could have seen this earlier, it would have seemed contrived; in the context of H-W it is perfectly natural. For these ‘self-polarizations’, we have the following interesting identities:

**Proposition 7.3.** We have
\[
D_{ab}D_{ba} = D_{ba}D_{ab} + D_{aa} - D_{bb}.
\]

and
\[
D_{ab}D_{ba} = D_{aa} + D_{bb}D_{aa}.
\]

**Proof.** The first is easily seen by
\[
(a \partial_b b \partial_a = a(b \partial_b + 1) \partial_a =
\]
\[
b \partial_a \partial_a + a \partial_a = b \partial_a(\partial_a a - 1) + a \partial_a =
\]
\[
b \partial_a \partial_a - b \partial_b + a \partial_a.
\]

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(c \partial_b)(b \partial_a) = c(b \partial_b + 1) \partial_a =
\]
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**Proof.** The first is easily seen by
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\]
\[
b \partial_a \partial_a + a \partial_a = b \partial_a(\partial_a a - 1) + a \partial_a =
\]
\[
b \partial_a \partial_a - b \partial_b + a \partial_a.
\]
The second we see by
\[ a\partial b\partial a = (b\partial + 1)a\partial a = a\partial a + b\partial a\partial a. \]

Finally,

**Proposition 7.4.** We have our basic "power" Capelli identity:
\[ D_{cb}^{(k)}D_{ba}^{(l)} = \sum D_{ba}^{(l-u)}D_{ca}^{(u)}D_{cb}^{(k-u)}. \]

**Proof.** This writes itself out, using Leibnitz’ Rule, 7.1, as:
\[
D_{cb}^{(k)}(c\partial b)^l = c^k \left( \sum_{l} b^{l-j} \partial_{b}^{(k-j)} \partial_{a}^{(l-j)} \partial_{a}^{(j)} \right) = \sum_{l} b^{l-j} \partial_{a}^{(l-j)} c \partial_{a}^{(j)} c^{k-j} \partial_{b}^{(k-j)} = \sum (b\partial a)^{(l-j)}(c\partial a)^{(j)}(c\partial b)^{(k-j)}. 
\]

If we want to take the superalgebra form of all of the foregoing, we proceed as follows: we take a signed alphabet: \( A = A^+ \cup A^- \). In H-W, we add the five additional relations:

\[
\partial_x x + x\partial_x = 1 \text{ if } x \in A^-; \\
x\partial_y + \partial_y x = 0 \text{ if } x \in A^-, y \in A^-; \\
a\partial_x - \partial_x a = 0 \text{ if } x \in A^-, a \in A^+; \\
x\partial_a - \partial_a x = 0 \text{ if } x \in A^-, a \in A^+; \\
\partial_x^2 = 0 \text{ if } x \in A^-.
\]

Linearly, this algebra is the tensor product:
\[ S(A^+) \otimes \Lambda(A^-) \otimes D(\partial A^+) \otimes \Lambda(\partial A^-). \]

8. THE TERMS OF THE RESOLUTIONS OF WEYL MODULES

In this section we will generalize the results of Theorem 6.1 to arbitrary \( n \)-rowed shapes (skew- and almost skew-). Instead of using Stems, however, we will express the resolutions of our \( n \)-rowed shapes in terms of resolutions of \((n - 1)\)-rowed shapes.

As before, we denote an \( n \)-rowed shape by \([p_1, \ldots, p_n; t_1, \ldots, t_{n-1}]\), where, if \( t_{n-1} \geq 0 \) we have a skew-shape, and otherwise an almost skew-shape. Recall that if \( t_{n-1} + \cdots + t_1 \geq 0 \) we say it is of type \( i \) if \( t_{n-1} + \cdots + t_{n-i} < 0 \) and \( t_{n-1} + \cdots + t_{n-i-1} \geq 0 \); if \( t_{n-1} + \cdots + t_1 < 0 \), we say it is of type \( n - 1 \). Thus, it is of type 0 if it is a skew-shape; it is of type 1 if the bottom row doesn’t extend as far to the left as the penultimate row, but does extend beyond (in the weak sense) row \( n - 2 \), etc.
The reader will recall that in section 5.1, we introduced the notation

$$Z^n_{r,m} \otimes Z^{(l)}_{n,m}$$

to stand for the homogeneous strand of the bar complex of total degree $\tau + l$
with initial term of degree $\geq \tau$.

For our final bit of notation, we let $\text{Res}(p_1, \ldots, p_m; t_1, \ldots, t_{m-1})$ stand for
the resolution of the skew-shape $[p_1, \ldots, p_m; t_1, \ldots, t_{m-1}]$. The term $\text{Res}(p_1, \ldots, p_m; \tau_1, \ldots, \tau_{m-1}) \otimes (r)$ stands for the resolution tensored with $D_r$.

We’ll now assume that we know $\text{Res}(p_1, \ldots, p_m; t_1, \ldots, t_{m-1})$ for $m < n$,
and we describe $\text{Res}(p_1, \ldots, p_n; t_1, \ldots, t_{n-1})$ by induction on $n$.

**Theorem 8.1.** 1. For a skew-shape $(p_1, \ldots, p_n; t_1, \ldots, t_{n-1})$, the resolution looks like this:

$$\text{Res}(p_1, \ldots, p_n; t_1, \ldots, t_{n-1}) =$$

\[
\begin{align*}
\text{Res}(p_1, \ldots, p_{n-1}; & t_1, \ldots, t_{n-2}) \otimes (p_n) \oplus \\
\sum_{l_{n-1} \geq 0} & Z^{l_{n-1}+1}_{n,n-1} \otimes Z^{(l_{n-1})}_{n,n-1} \otimes \text{Res}(p_1, \ldots, p_{n-2}, p_{n-1} + \tilde{t}_{n-1}; t_1, \\
& \ldots t_{n-3}, t_{n-2} + \tilde{t}_{n-1}) \otimes (p_n - \tilde{t}_{n-1}) \oplus \\
\sum_{l_{n-j} \geq 0} & (Z^{l_{n-1}+1}_{n,n-1} \otimes Z^{(l_{n-1})}_{n,n-1}) \otimes (Z^{l_{n-2}+1}_{n,n-2} \otimes Z^{(l_{n-2})}_{n,n-2}) \otimes \\
& \text{Res}(p_1, \ldots, p_{n-2} + \tilde{t}_{n-2}, p_{n-1} + \tilde{t}_{n-1}; t_1, \\
& \ldots t_{n-3} + \tilde{t}_{n-2}, t_{n-2} + \tilde{t}_{n-1} - \tilde{t}_{n-2}) \otimes (p_n - \tilde{t}_{n-1} - \tilde{t}_{n-2}) \oplus \\
& \vdots \\
\sum_{l_{n-j} \geq 0} & (Z^{l_{n-1}+1}_{n,n-1} \otimes Z^{(l_{n-1})}_{n,n-1}) \otimes (Z^{l_{n-2}+1}_{n,n-2} \otimes Z^{(l_{n-2})}_{n,n-2}) \otimes \cdots \otimes (Z^{l_1+1}_{n,1} \otimes Z^{(l_1)}_{n,1}) \otimes \\
& \text{Res}(p_1 + \tilde{t}_1, \ldots, p_{n-1} + \tilde{t}_{n-1}; t_1 + \tilde{t}_2 - \tilde{t}_1, \\
& \ldots, t_{n-2} + \tilde{t}_{n-1} - \tilde{t}_{n-2}) \otimes (p_n - \sum_{1 \leq j \leq n-1} \tilde{t}_j),
\end{align*}
\]

where $\tilde{t}_j = t_j + l_j + 1$ for $j = 1, \ldots, n - 1$.

2. For an almost skew-shape $(p_1, \ldots, p_n; t_1, \ldots, t_{n-1})$ of type $k$, let’s denote by $s$ the positive integer $-(t_{n-1} + \cdots + t_{n-k})$, and by $\tau$, the integer
$(t_{n-1} + \cdots + t_{n-k})$. With this notation, the terms of the resolution are

$$\text{Res}(p_1, \ldots, p_n; t_1, \ldots, t_{n-1}) =$$
\[ \text{Res}(p_1, \ldots, p_{n-1}; t_1, \ldots, t_{n-2}) \otimes (p_n) \oplus \]
\[ \sum_{l_{n-1}>0} Z_{n,n-1}^{(l_{n-1})} \otimes \text{Res}(p_1, \ldots, p_{n-2}, p_{n-1} + l_{n-1}; t_1, \ldots, t_{n-3}, t_{n-2} + l_{n-1}) \otimes (p_n - l_{n-1}) \oplus \]
\[ \sum_{l_{n-2}>0, l_{n-1} \geq 0} (Z_{n,n-2}^{(l_{n-2})} \otimes (Z_{n,n-1}^{(l_{n-1})} \otimes \text{Res}(p_1, \ldots, p_{n-2} + l_{n-2}, p_{n-1} + l_{n-1}; t_1, \ldots, t_{n-3} + l_{n-2}, t_{n-2} + l_{n-1} - l_{n-2}) \otimes (p_n - l_{n-1} - l_{n-2}) \oplus \]
\[ \vdots \]
\[ \sum_{l_{n-j} \geq 0} (Z_{n,n-k-2}^{(l_{n-k-2}+1)} \otimes Z_{n,n-k-2}^{(l_{n-k-2})} \otimes (Z_{n,n-k-1}^{(l_{n-k-1})} \otimes \text{Res}(p_1, \ldots, p_{n-2} + l_{n-k-2}, p_{n-1} + \tau + 1 + l_{n-k-1}, p_{n-k} + l_{n-k}, \ldots, p_{n-1} + l_{n-1}; t_1, \ldots, t_{n-k-3} + l_{n-k-2}, t_{n-k-2} + \tau + 1 + l_{n-k-1} - l_{n-k-2}, \]
\[ \ldots, t_{n-k-2} + l_{n-1} - l_{n-2}) \otimes (p_n - \sum_{i=n-k-1}^{n-1} l_{n-i} - \sum_{j=n-k-1}^{n-1} t_j - 2) \oplus \]
\[ \vdots \]
\[ \sum_{l_{n-j} \geq 0} (Z_{n,n-1}^{(l_{n-1}+1)} \otimes Z_{n,n-1}^{(l_{n-1})} \otimes \ldots \otimes (Z_{n,n-k-2}^{(l_{n-k-2}+1)} \otimes Z_{n,n-k-2}^{(l_{n-k-2})} \otimes (Z_{n,n-k-1}^{(l_{n-k-1})} \otimes \text{Res}(p_1 + \bar{\ell}_1, \ldots, p_{n-k-2} + \bar{\ell}_{n-k-2}, p_{n-k-1} + \tau + 1 + l_{n-k-1}, p_{n-k} + l_{n-k}, \ldots, p_{n-1} + l_{n-1}; t_1 + \bar{\ell}_2 - \bar{\ell}_1, \ldots, t_{n-k-3} + \bar{\ell}_{n-k-2} - \bar{\ell}_{n-k-3}, \]
\[ t_{n-k-2} + \tau + 1 + l_{n-k-1} - \bar{\ell}_{n-k-2}, \ldots, t_{n-2} + l_{n-1} - l_{n-2}) \otimes (p_n - \sum_{j=1}^{n-1} \bar{\ell}_j - k), \]

where \( \bar{\ell}_j \) is defined as above.

It must be understood that if the indicated shape isn’t legitimate, then that term vanishes. This puts some constraints on the parameters over which we’re summing.

Proof. The proof of this proposition is much the same as the proof for the three-rowed shapes, except that it proceeds by induction on the total number, \( \omega \), of overlaps of the last row with all the other rows of the diagram. Thus, if \( \omega = 1 \), the shape must be a skew-shape; if the shape is an almost
skew-shape of type $k$, then $\omega \geq k + 1$ (assuming the bottom row is not empty). It is clear from the fundamental exact sequence that if one starts with a shape with $\omega = l$, then the other two terms of the exact sequence have $\omega = l - 1$. Therefore, starting with $\omega = 1$, we see that the fundamental exact sequence gives us two additional shapes which are an $(n - 1)$-rowed shape tensored with $D_{p_n}$ and an $(n - 1)$-rowed skew-shape as kernel. This gets the induction started. The rest proceeds as before. Again, one encounters the ‘boundary’ case of a shape of type $k$ whose indentation, when reduced by one, causes it to become a shape of type $k - 1$. The solution of the problem is the one prescribed at the end of the proof of Theorem 6.1.

If one takes a close look at the resolution of a skew-shape, one sees that the characteristic zero ‘skeleton’ described by Lascoux ([14]) is in there. Recall that Lascoux has described the terms and their placement in the resolutions of Weyl modules (of partitions) in terms of the lengths of the permutations corresponding to the determinantal expansion of the Jacobi-Trudi matrix for that Weyl module. If we let $S_n$ denote the symmetric group on $n$ letters, and $\sigma \in S_n$ a permutation such that $\sigma(n) = i$, then $\sigma$ can be written uniquely as a product:
\[
\sigma = (n, n - 1)(n, n - 2) \cdots (n, i)\sigma',
\]
where $(n, j)$ stands for the transposition on $n$ and $j$, and $\sigma' \in S_{n-1}$. (Notice that the length of $\sigma$ is $n - i + \text{length}(\sigma')$. ) This is the way of recovering the terms of the Lascoux resolution within the resolution described above.

9. Addenda and Alternative Approaches

In this section, we will indicate a few ideas and technical details that arose during the investigation of our problem. Some are simply observations that occasionally proved useful in handling some of the computations that we had to deal with. Others involve a total change in the way we build up the resolution going from the two-rowed to the three-rowed case. We suggest some alternative ways to define the basic identity underlying the operation of $Z_{31}$ on the stems. And finally, we study the first step or two in defining a homotopy for the resolution of a three-rowed partition. This latter is motivated by the fact that, in the two-rowed case, the description of the homotopy gives another proof, independent of mapping cone arguments and fundamental exact sequences, of the exactness of the resolution that we write down.

9.1. Additional, Sometimes Useful, Identities. In the course of working on the problem of homotopy-transitivity, we found that a number of identities relating the operations of $Z_{32}$ and $Z_{31}$ were useful. We will list them here, along with some indication of their proofs.

**Proposition 9.1.** The difference
\[
Z_{32}^{(b)} Z_{31}^{(a)} \left( Z_{21}^{(c)} x v \right) - Z_{31}^{(a)} \left( Z_{32}^{(b)} Z_{21}^{(c)} x v \right)
\]
We have Proposition 9.2.

\[ \partial \left( \sum_{u>0} (-1)^u Z_{32}^{(a)} y Z_{32}^{(b)} y \partial_{31}^{(a-u)} \partial_{21}^{(c+u)}(v) \right), \]

i.e.,

\[ Z_{32}^{(b)} Z_{31}^{(a)} (Z_{21}^{(c)} x v) - Z_{31}^{(a)} (Z_{32}^{(b)} Z_{21}^{(c)} x v) = \partial \left( \sum_{u>0} (-1)^u Z_{32}^{(a)} y Z_{32}^{(b)} y \partial_{31}^{(a-u)} \partial_{21}^{(c+u)}(v) \right). \]

Proof. We have

\[ Z_{32}^{(b)} Z_{31}^{(a)} (Z_{21}^{(c)} x v) = Z_{32}^{(b)} \left( (-1)^{a+c} Z_{21}^{(a+c)} x \partial_{32}^{(a)}(v) + \sum_{u>0} (-1)^u Z_{32}^{(a)} y \partial_{31}^{(a-u)} \partial_{21}^{(c+u)}(v) \right) = \]

\[ \sum_{\alpha \geq 0} (-1)^\alpha Z_{21}^{(a+c-\alpha)} x \partial_{32}^{(b-\alpha)} \partial_{31}^{(a)}(v) + \sum_{u>0} (-1)^u \left( b + 1 \right) Z_{32}^{(b+u)} y \partial_{31}^{(a-u)} \partial_{21}^{(c+u)}(v); \]

and

\[ Z_{31}^{(a)} (Z_{32}^{(b)} Z_{21}^{(c)} x v) = Z_{31}^{(a)} \left( \sum_{\alpha \geq 0} Z_{21}^{(c-\alpha)} x \partial_{32}^{(b-\alpha)} \partial_{31}^{(a)}(v) \right) = \]

\[ \sum_{\alpha \geq 0} (-1)^\alpha Z_{21}^{(a+c-\alpha)} x \partial_{32}^{(b-\alpha)} \partial_{31}^{(a)}(v) + \sum_{u>0} (-1)^u Z_{32}^{(a-u)} y \partial_{31}^{(c-a+u)} \partial_{21}^{(b-\alpha)} \partial_{31}^{(a)}(v). \]

The difference between the two terms is easily seen to be

\[ \partial \left( \sum_{u>0} (-1)^u Z_{32}^{(a)} y Z_{32}^{(b)} y \partial_{31}^{(a-u)} \partial_{21}^{(c+u)}(v) \right). \]

The next thing that we record has to do with commuting \( Z_{32} \) past the product of two \( Z_{31} \) actions:

**Proposition 9.2.** We have

\[ Z_{32}^{(b)} Z_{31}^{(a_1)} Z_{31}^{(a_2)} (Z_{21}^{(c)} x v) - Z_{31}^{(a_1)} Z_{31}^{(a_2)} Z_{32}^{(b)} (Z_{21}^{(c)} x v) = \partial \left\{ \sum_{u>0} (-1)^u Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a_1)} \partial_{31}^{(a_2-u)} \partial_{21}^{(c+u)}(v) - \sum_{u>0} (-1)^u Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a_1-u)} \partial_{21}^{(a_2+c+u)} \partial_{32}^{(a_2)}(v) \right\}. \]
Proof.

\[ Z_{32}^{(b)} Z_{31}^{(a_1)} Z_{31}^{(a_2)} \left( Z_{21}^{(c)} x v \right) = \left( \frac{a_1 + a_2}{a_2} \right) Z_{32}^{(b)} Z_{31}^{(a_1+a_2)} \left( Z_{21}^{(c)} x v \right) - \]

\[ Z_{32}^{(b)} \partial \left\{ \sum_{u > 0} (-1)^{u_1 + u_2} Z_{32}^{(u_1)} y Z_{32}^{(u_2)} y \partial_{31}^{(a_1-u_1)} \partial_{21}^{(a_2-u_2)} \partial_{21}^{(c+u_1+u_2)} (v) \right\} \]

\[ = \left( \frac{a_1 + a_2}{a_2} \right) Z_{31}^{(a_1+a_2)} Z_{32}^{(b)} \left( Z_{21}^{(c)} x v \right) + \]

\[ \partial \left\{ \left( \frac{a_1 + a_2}{a_2} \right) \sum_{u > 0} (-1)^{u} Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a_1+a_2-u)} \partial_{21}^{(c+u)} (v) \right\} - \]

\[ \partial \left\{ \sum_{u > 0} (-1)^{u_1 + u_2} \left( \frac{u + b}{u_1} \right) Z_{32}^{(u_1+b)} y Z_{32}^{(u_2)} y \partial_{31}^{(a_1-u_1)} \partial_{21}^{(a_2-u_2)} \partial_{21}^{(c+u_1+u_2)} (v) \right\} \]

\[ = Z_{31}^{(a_1)} Z_{31}^{(a_2)} Z_{32}^{(b)} \left( Z_{21}^{(c)} x v \right) + \]

\[ \partial \left\{ \sum_{u > 0} \sum_{\alpha \geq 0} (-1)^{u_1 + u_2} Z_{32}^{(u_1)} y Z_{32}^{(u_2)} y \partial_{31}^{(a_1-u_1)} \partial_{21}^{(a_2-u_2)} \partial_{21}^{(c-u_1+u_2)} \right\} + \]

\[ \partial \left\{ \left( \frac{a_1 + a_2}{a_2} \right) \sum_{u > 0} (-1)^{u} Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a_1+a_2-u)} \partial_{21}^{(c+u)} (v) \right\} - \]

\[ \partial \left\{ \sum_{u > 0} (-1)^{u_1 + u_2} \left( \frac{u + b}{u_1} \right) Z_{32}^{(u_1+b)} y Z_{32}^{(u_2)} y \partial_{31}^{(a_1-u_1)} \partial_{21}^{(a_2-u_2)} \partial_{21}^{(c+u_1+u_2)} (v) \right\} \].

So,

\[ Z_{32}^{(b)} Z_{31}^{(a_1)} Z_{31}^{(a_2)} \left( Z_{21}^{(c)} x v \right) - Z_{31}^{(a_1)} Z_{31}^{(a_2)} Z_{32}^{(b)} \left( Z_{21}^{(c)} x v \right) = \]

\[ \partial \left\{ \sum_{u > 0} (-1)^{u_1 + u_2} Z_{32}^{(u_1)} y Z_{32}^{(u_2)} y \partial_{31}^{(a_1-u_1)} \partial_{21}^{(a_2-u_2)} \partial_{21}^{(c+u_1+u_2)} (v) + \right\} \]

\[ \left\{ \sum_{u > 0} (-1)^{u_1 + u_2} \left( \frac{u + b}{u_1} \right) Z_{32}^{(u_1+b)} y Z_{32}^{(u_2)} y \partial_{31}^{(a_1-u_1)} \partial_{21}^{(a_2-u_2)} \partial_{21}^{(c+u_1+u_2)} (v) \right\}. \]

Let’s call the term in the braces above, X. Then it’s fairly easy to see that

\[ \partial \left\{ \sum_{u > 0} (-1)^{u_1 + u_2} Z_{32}^{(u_1)} y Z_{32}^{(u_2)} y \partial_{31}^{(a_1-u_1)} \partial_{21}^{(a_2-u_2)} \partial_{21}^{(c+u_1+u_2)} (v) - \right\} \]

\[ \sum_{u > 0} (-1)^{u} Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a_1-u)} \partial_{21}^{(a_2+c+u)} \partial_{21}^{(a_2)} (v) - \]

\[ \sum_{u > 0} (-1)^{u} Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a_1-u)} \partial_{21}^{(a_2+c+u)} \partial_{21}^{(a_2)} (v). \]
Therefore
\[
\partial(X) = \partial \left\{ \sum_{u>0} (-1)^{u-1} Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a_1)} \partial_{31}^{(a_2-u)} \partial_{21}^{(c+u)} (v) - \sum_{u>0} (-1)^{u-1} Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{31}^{(a_1-u)} \partial_{21}^{(a_2+c+u)} \partial_{32}^{(a_2)} (v) \right\}
\]
and we have the result. ■

Finally, we record this next identity:

**Proposition 9.3.**

\[
Z_{31}^{(a)} \left( Z_{32}^{(b)} y Z_{31}^{(c)} \left( Z_{21}^{(x)} y \right) \right) - Z_{32}^{(b)} y Z_{31}^{(a)} Z_{31}^{(a_2)} \left( Z_{21}^{(c)} y \right) = (-1)^{a_2} \sum_{u>0} (-1)^{u} Z_{32}^{(u)} y Z_{32}^{(b)} y \partial_{21}^{(a_1-u)} \partial_{21}^{(a_2+c+u)} \partial_{32}^{(a_2)} (v).
\]

### 9.2. Some Alternative Actions.

The identity

\[
Z_{31}^{(a)} x \otimes m = (-1)^{a} Z_{21}^{(a+c)} x \otimes \partial_{32}^{(a)} (m) + \sum_{u>0} (-1)^{u} Z_{32}^{(u)} y \otimes \partial_{31}^{(a-u)} \partial_{21}^{(c+u)} (m)
\]

was motivated by one of our Capelli identities. But the underlying assumption was that we had to cover the map \(\partial_{31}^{(a)}\) on the generators of an almost skew-shape. This assumption was made because the map of the corresponding Weyl modules was easily seen to be induced by \(\partial_{31}^{(a)}\). However, since

\[
\partial_{21}^{(c)} \partial_{32}^{(b)} = \sum_{v \geq 0} (-1)^{v} \partial_{32}^{(b-v)} \partial_{21}^{(c-v)} \partial_{31}^{(v)}
\]

we see, setting \(c = b = a\), that

\[
\partial_{31}^{(a)} = (-1)^{a} \partial_{21}^{(a)} \partial_{32}^{(a)} - \sum_{u>0} (-1)^{u} \partial_{32}^{(a)} \partial_{21}^{(a+u)} \partial_{31}^{(a-u)}.
\]

Therefore we have that

\[
\partial_{31}^{(a)} (m) - (-1)^{a} \partial_{21}^{(a)} \partial_{32}^{(a)} (m) = \partial \left( \sum_{u>0} (-1)^{u} Z_{32}^{(u)} y \otimes \partial_{21}^{(a-u)} \partial_{31}^{(a-u)} (m) \right).
\]

This means that we can replace the action of \(Z_{31}^{(a)}\) on \(m\) by that of \((-1)^{a} \partial_{21}^{(a)} \partial_{32}^{(a)}\).

Doing this will then entail defining a completely new set of actions of \(Z_{31}^{(a)}\) on the Stems; we haven’t pursued this approach.

We have also looked at the following approach which may afford some simplification. From 9.1, we had deduced the identity

\[
\partial_{31}^{(a)} \partial_{21}^{(c)} = (-1)^{a} \partial_{21}^{(a+c)} \partial_{32}^{(a)} + \sum_{u>0} (-1)^{u} \partial_{32}^{(a)} \partial_{31}^{(a-u)} \partial_{21}^{(c+u)}.
\]

This has an apparent disadvantage in that it involves variable powers of \(\partial_{21}\), and lower powers of \(\partial_{31}\). It may be convenient to fix this up in the following way.
Proposition 9.4. We may replace the identity 9.2 by the following:

\[(a + c) \sum_{u \geq 0} (-1)^{a-u} \partial_3^{(a)} \partial_2^{(u)} \partial_1^{(a+c)} \partial_3^{(a-u)} = 3^a \partial_1^{(a)} \partial_2^{(c)} \partial_1^{(c)}.\]

Proof. We will prove this using the identity 7.2. First, we translate the above identity into H-W terms. We want to show that

\[3^a \partial_1^{(a)} \partial_2^{(c)} \partial_1^{(c)} = \sum_{u \geq 0} (-1)^{a-u} 3^u \partial_2^{(u)} 2^{a+c} \partial_1^{(a+c)} 3^{a-u} \partial_2^{(a-u)} .\]

(We're using the numerals to represent places, not numbers. Hence \(3^2\), for example, does not mean \(92\).) We see that this reduces to showing that

\[(a + c) \sum_{u \geq 0} (-1)^{a-u} \partial_2^{(u)} 2^{a+c} \partial_2^{(a-u)} = 3^a \partial_1^{(a)} \partial_2^{(c)} \partial_1^{(c)} \sum_{u \geq 0} (-1)^{a-u} \partial_2^{(u)} 2^{a+c} \partial_2^{(a-u)}.
\]

or, more strongly,

\[(a + c) 2^c = \sum_{u \geq 0} (-1)^{a-u} \partial_2^{(u)} 2^{a+c} \partial_2^{(a-u)} .\]

By 7.2 we know that

\[\partial_2^{(u)} 2^{a+c} = \sum_{j} \left( \begin{array}{c} a + c \\ j \end{array} \right) 2^{a+c-j} \partial_2^{(u-j)} .\]

Thus,

\[\sum_{u \geq 0} (-1)^{a-u} \partial_2^{(u)} 2^{a+c} \partial_2^{(a-u)} = \sum_{u \geq 0} (-1)^{a-u} \left( \sum_{j} \left( \begin{array}{c} a + c \\ j \end{array} \right) \left( \begin{array}{c} a - j \\ u - j \end{array} \right) 2^{a+c-j} \partial_2^{(a-j)} \right) .\]

Since

\[\sum_{u \geq 0} (-1)^{a-u} \left( \begin{array}{c} a - j \\ u - j \end{array} \right) = \left\{ \begin{array}{ll} 1 & \text{for } j = a \\ 0 & \text{otherwise} \end{array} \right. ,\]

we see that the sum

\[\sum_{u \geq 0} (-1)^{a-u} \left( \sum_{j} \left( \begin{array}{c} a + c \\ j \end{array} \right) \left( \begin{array}{c} a - j \\ u - j \end{array} \right) 2^{a+c-j} \partial_2^{(a-j)} \right) = \left( \begin{array}{c} a + c \\ j \end{array} \right) 2^c .\]

as desired. \(\blacksquare\)

Using 9.3, we may now define

\[Z_3^{(a)} Z_2^{(c)} x \otimes m = (-1)^a Z_2^{(a+c)} x \otimes \partial_3^{(a)} (m) + \sum_{u \geq 0} (-1)^{a-u} Z_3^{(u)} y \otimes \partial_2^{(a+c)} \partial_3^{(a-u)} (m) .\]

If we use this definition of the action, we still do not get transitivity, but the first step of the homotopy for the homotopy-transitivity is

\[s_{a_1, a_2}^1 (Z_2^{(c)} x \otimes m) = \sum (-1)^{|u|} Z_3^{(u_1)} y Z_3^{(u_2)} y \otimes \partial_2^{(|a|+c)} \partial_3^{(a_1-u_2)} \partial_3^{(a_2-u_1)} .\]

As of this point, we have not explored this alternative further.
9.3. A Very Different Approach. In this section, we’re going to look at the two-rowed skew-shape, and write its resolution in a slightly different way. We will refer to the following diagram:

\[
\begin{array}{ccccccc}
t & \ldots & \sigma & p & \vdots & \sigma & q \\
\end{array}
\]

(9.4)

To write down its resolution, we make use of the fact that this is still a skew-shape after we rotate 180 degrees. Taking account of that, we see that we get the resolution

\[
0 \to P_{q-t} \to \cdots \to P_1 \to P_0,
\]

where \( P_0 = D_p \otimes D_q \), and \( P_k = \sum Z_{12}^{(\sigma+u_k)} Z_{12}^{(u_k)} x Z_{12}^{(u_k)} x \cdots x Z_{12}^{(u_k)} x \otimes m \), where \( u_i > 0 \), and \( m \in D_{p-\sigma-|u|} \otimes D_{q+\sigma+|u|} \). The boundary map is that of the Bar Complex.

Since this resolution and the usual one resolve the same Weyl module, there should be an isomorphism between them. (Actually, we’re only able to say that there should be maps from one to the other such that their compositions are homotopic to the identity, but since the complexes are so similar term by term, it is tempting to see if there is an isomorphism between them that can be made explicit.) To get an isomorphism between the usual resolution and this one, we want to map the term \( Z_{21}^{(t+l)} x \otimes m \) to a term in \( P_1 \). Before doing this, let’s consider a term \( Z_{21}^{(t)} x \otimes m \), with \( t \) a positive integer, and \( m = a^{(p)} \otimes b^{(q)} \), where \( p \) and \( q \) are arbitrary non-negative integers. Its image under the usual boundary map is covered by

\[
\sum_k (-1)^{p-k-l} Z_{12}^{(k)} x \otimes (b^{(p-l-k)} \otimes a^{(p)} b^{(q+l+k-p)}),
\]

since

\[
\sum_k (-1)^{p-k-l} \sigma_{12}^{(k)} (b^{(p-l-k)} \otimes a^{(p)} b^{(q+l+k-p)}) = \\
\sum_{k,\beta} (-1)^{p-k-l} \left( \frac{p-l-\beta}{k-\beta} \right) a^{(\beta)} (b^{(p-l-\beta)} \otimes a^{(p-\beta)} b^{(q+l+\beta-p)}).
\]

For a fixed \( \beta \), the coefficient of the corresponding term is

\[
\sum_k (-1)^{p-k-l} \left( \frac{p-l-\beta}{k-\beta} \right) = \sum_k \left( \frac{-1}{p-l-k} \right) \left( \frac{p-l-\beta}{k-\beta} \right) = \left( \frac{p-l-\beta-1}{p-l-\beta} \right) = \left\{ \begin{array}{ll} 1 & \text{if } \beta = p-l \\ 0 & \text{otherwise} \end{array} \right. .
\]

Notice that if we look at what this means in the case of our resolutions, we may assume that \( l = t + l', p = p' + t + l', q = q' - t - l' \). Then \( q + l + k - p = -\sigma + k - l' \geq 0 \) implies that \( k \geq \sigma + l' \), which puts us exactly into the range we want for \( k \). That is, the term

\[
\sum_k (-1)^{p-k-l} Z_{12}^{(k)} x \otimes (b^{(p-l-k)} \otimes a^{(p)} b^{(q+l+k-p)})
\]

is in the complex we want it to be in (having changed, of course, the \( p \) and \( q \) of our original setup to \( p' \) and \( q' \)). In terms of place polarizations, this means
that we map the element \( Z^{(l)}_{21} x \otimes m \) to the element \( \sum k(-1)^{p-k-l} Z^{(k)}_{12} x \otimes \partial^{(p-l-k)}_{13} \partial^{(p)}_{21} \partial^{(p-l-k)}_{32}(m) \).

The next step of the map between complexes is given (at least, up to sign) by:

\[
Z^{(l_1)}_{21} x Z^{(l_2)}_{21} x \otimes (a^{(p)} \otimes b^{(q)}) \to \\
\sum (-1)^{p-|l|-|k|} \binom{p-|k|}{l_2 - k_2} Z^{(k_1)}_{12} x Z^{(k_2)}_{12} x \otimes (b^{p-|l|-|k|} \otimes a^{(p)} b^{-(p-|l|-|k|)}).
\]

We would guess that the general situation is:

\[
Z^{(l_1)}_{21} x Z^{(l_2)}_{21} x \cdots x Z^{(l_n)}_{21} x \otimes (a^{(p)} \otimes b^{(q)}) \to \\
\sum (-1)^{p-|l|-|k|} \binom{p-|k|}{l_n - k_n} Z^{(k_1)}_{12} x Z^{(k_2)}_{12} x \cdots x Z^{(k_n)}_{12} x \otimes (b^{p-|l|-|k|} \otimes a^{(p)} b^{-(p-|l|-|k|)}).
\]

9.3.1. **Three-rowed case; at most one triple overlap.** If we use this alternative description of the two-rowed resolution, we can alter the boundary map in the ‘uncomplicated’ three-rowed resolution, namely in the case of the three-rowed skew-shape with at most one triple overlap. Since the place polarization from \( b \) to \( c \) commutes with that of \( b \) to \( a \), the boundary map in the three-rowed resolution can simply use the fact that

\[
Z^{(k)}_{32} Z^{(l)}_{12} x = Z^{(l)}_{12} x Z^{(k)}_{32}.
\]

Thus, for example, the image of

\[
Z^{(k)}_{32} y Z^{(l)}_{12} x \otimes m
\]

is

\[
Z^{(k)}_{32} y \otimes \partial^{(l)}_{12}(m) - Z^{(l)}_{12} x \otimes \partial^{(k)}_{32}(v).
\]

In this section and in future sections, we’ll be dealing with the skew-shape

\[
\begin{array}{c}
| & | & | \\
\sigma & & \\
| & \big| & \big| \\
t_2 & t_1 & \sigma_p \\
| & \big| & \big| \\
p_1 & p_2 & \sigma \\
| & \big| & \big| \\
p_3 & & \sigma \\
\end{array}
\]

The terms of the resolution are now of the form

\[
\Res(p_1, p_2; \sigma) \otimes D_{p_3} \oplus \\
\sum_{l \geq 0} Z^{(l)}_{32} \otimes Z^{(l)}_{32} \otimes \Res(p_1, p_2 + t_2 + 1 + l; \sigma) \otimes D_{p_3 - (t_2 + 1 + l)}.
\]

This gives a significant simplification to the boundary map; whether it helps in the long run is something that remains to be seen.
9.3.2. New actions. Now that we’re building up the resolutions using the fixed \( \sigma \) of the two-rowed resolutions that occur in the short exact sequences, we have to get a different action of \( Z_{31}^{(l)} \) on the terms of the two-rowed resolution. The Capelli identities come in to play here. In fact,

\[
\partial_{31}^{(l)} \partial_{12}^{(k)} = \partial_{12}^{(k)} \partial_{31}^{(l)} + \sum_{a>0} \partial_{32}^{(a)} \partial_{12}^{(k-a)} \partial_{31}^{(l-a)}.
\]

This suggests that we set

\[
Z_{31}^{(l)} Z_{12}^{(k)} = \partial_{12}^{(k)} \partial_{31}^{(l)} + \sum_{a>0} Z_{32}^{(a)} \partial_{12}^{(k-a)} \partial_{31}^{(l-a)} (m).
\]

Working it out for the next case, we are led to the general formula:

\[
Z_{31}^{(l)} Z_{12}^{(k_1)} x Z_{12}^{(k_2)} x \cdots x Z_{12}^{(k_n)} x \otimes \partial_{31}^{(l)} (m) + \sum_{j=1}^n (-1)^{(j+1)(n-1)} Z_{32}^{(a_j)} y \cdots y Z_{32}^{(a_1)} y Z_{12}^{(k_1)} x \cdots x Z_{12}^{(k_{j-1})} x \otimes \partial_{12}^{(k_j-a_j)} \cdots \partial_{12}^{(k_n-a_n)} \partial_{31}^{(l-\vert a \vert)} (m).
\]

The proof for \( n = 2, 3 \) is straightforward. The proof in general goes like this:

First, some notation. If we let

\[
V = Z_{12}^{(k_1)} x Z_{12}^{(k_2)} x \cdots x Z_{12}^{(k_n)} x \otimes mv,
\]

we set

\[
P_t^{(l)} (V) = \sum Z_{32}^{(a_1)} y \cdots y Z_{32}^{(a_1)} y \otimes \partial_{12}^{(k_1-a_1)} \cdots \partial_{12}^{(k_j-a_j)} \partial_{31}^{(l-\vert a \vert)} (m).
\]

The crucial lemma to prove is:

**Lemma 9.5.**

1) \( \partial P_t^{(l)} (V) - P_{t-1}^{(l)} (\partial V) = \sum \left\{ \left( \begin{array}{c} k_1 + k_2 \\ k_2 \end{array} \right) - \left( \begin{array}{c} k_1 + k_2 - \alpha_2 \\ k_2 - \alpha_2 \end{array} \right) \right\} Z_{32}^{(a_2)} y \cdots y Z_{32}^{(a_1)} y \otimes \partial_{12}^{(k_1+k_2-\alpha_2)} \cdots \partial_{12}^{(k_1-\alpha_1)} \partial_{31}^{(l-\vert a \vert)} (m); \)

2) \( \partial \left( P_{t+1}^{(l)} (Z_{12}^{(k)} \times V) \right) = P_t^{(l)} (Z_{12}^{(k)} \times \partial V) + (-1)^t \sum \left\{ \left( \begin{array}{c} k + k_1 \\ k \end{array} \right) - \left( \begin{array}{c} k + k_1 - \alpha_1 \\ k_1 - \alpha_1 \end{array} \right) \right\} \right. \)

\[
Z_{32}^{(a_1)} y \cdots y Z_{32}^{(a_1)} y \otimes \partial_{12}^{(k_1+k_2-\alpha_1)} \cdots \partial_{12}^{(k_1-\alpha_1)} \partial_{31}^{(l-\vert a \vert)} (m).
\]

**Proof.** The main observation to make here is that when we take the boundary of \( P_t^{(l)} (V) \), we can add and subtract convenient terms. All the \( \alpha_i \) in the summation are supposed to be positive. However, we may let one of the indices be zero and consider the image of such a term under multiplication by the zero index on the right, and on the left (with opposite sign). This is the standard trick we have used often. Since these terms cancel out, we collect them judiciously and get the results mentioned (the “correction” term that
appears at the end is due to the fact that we can’t let the first index be zero since it has nothing on the left to cancel multiplication on the right).

With this lemma, we proceed as follows: Observe that

\[ Z_{31}^{(l)}(Z_{12}^{(k)} x V) = Z_{12}^{(k)} x Z_{31}^{(l)} V + \sum Z_{32}^{(\alpha\alpha)} y \cdots y Z_{32}^{(\alpha_1)} y \otimes \partial_{12}^{(k-\alpha_1)} \cdots \partial_{12}^{(k_1-\alpha_1)} \partial_{31}^{(l-\alpha_1)}(m) \]

\[ = Z_{12}^{(k)} x Z_{31}^{(l)} V + P_{t+1}(Z_{12}^{(k)} x V). \]

(As in a previous case, we define

\[ Z_{12}^{(k)} x Z_{31}^{(l)} y = -Z_{31}^{(l)} y Z_{12}^{(k)} x \]

while we set

\[ Z_{12}^{(k)} Z_{31}^{(l)} y = Z_{31}^{(l)} y Z_{12}^{(k)}. \]

We also observe that

\[ \partial(Z_{12}^{(k)} x V) = Z_{12}^{(k)} x \partial V + (-1)^t Z_{12}^{(k)} V. \]

Now we start to calculate and get

\[ \partial Z_{31}^{(l)}(Z_{12}^{(k)} x V) = \partial \left( Z_{12}^{(k)} x Z_{31}^{(l)} V + P_{t+1}^{(l)}(Z_{12}^{(k)} x V) \right) \]

\[ = Z_{12}^{(k)} x \partial Z_{31}^{(l)} V + (-1)^t Z_{12}^{(k)} Z_{31}^{(l)} V + \partial P_{t+1}^{(l)}(Z_{12}^{(k)} x V). \]

This last is seen to be

\[ Z_{12}^{(k)} x \partial Z_{31}^{(l)} V + (-1)^t Z_{12}^{(k)} Z_{31}^{(l)} V + P_{t+1}^{(l)}(Z_{12}^{(k)} x \partial V) + \]

\[ (-1)^t \sum \left\{ \left( \begin{array}{c} k + k_1 \\ k_1 \end{array} \right) - \left( \begin{array}{c} k + k_1 - \alpha_1 \\ k_1 - \alpha_1 \end{array} \right) \right\} Z_{32}^{(\alpha_1)} y \cdots y Z_{32}^{(\alpha)} y \]

\[ \otimes \partial_{12}^{(k_1+k_2-\alpha_1)} \cdots \partial_{12}^{(k_1-\alpha_1)} \partial_{31}^{(l-\alpha_1)}(m) \]

\[ = Z_{31}^{(l)} \partial(Z_{12}^{(k)} x V) + (-1)^{t+1} Z_{12}^{(k)} Z_{31}^{(l)} V + (-1)^t Z_{12}^{(k)} Z_{31}^{(l)} V + (-1)^t K \]

where

\[ K = \sum \left\{ \left( \begin{array}{c} k + k_1 \\ k_1 \end{array} \right) - \left( \begin{array}{c} k + k_1 - \alpha_1 \\ k_1 - \alpha_1 \end{array} \right) \right\} Z_{32}^{(\alpha_1)} y \cdots y Z_{32}^{(\alpha)} y \otimes \partial_{12}^{(k_1+k_2-\alpha_1)} \cdots \partial_{12}^{(k_1-\alpha_1)} \partial_{31}^{(l-\alpha_1)}(m). \]

But

\[ (-1)^{t+1} Z_{31}^{(l)} Z_{12}^{(k)} V + (-1)^t Z_{12}^{(k)} Z_{31}^{(l)} V = (-1)^{t+1} K \]

This shows that

\[ \partial Z_{31}^{(l)}(Z_{12}^{(k)} x V) = Z_{31}^{(l)} \partial(Z_{12}^{(k)} x V) \]

and we’re done.

The important thing to observe here is that our operation of \( Z_{31}^{(l)} \) on \( Z_{12}^{(k)} x Z_{12}^{(k_1)} x \cdots x Z_{12}^{(k_s)} x \otimes m \) is exactly what we would want it to be. That is, if we set \( V = Z_{12}^{(k_1)} x \cdots x Z_{12}^{(k_s)} x \otimes m \), then we have

\[ Z_{31}^{(l)} \left( Z_{12}^{(k_1)} x \cdots x Z_{12}^{(k_s)} x \otimes m \right) = Z_{12}^{(k_1)} x Z_{12}^{(l)}(V) + P_{t+1}^{(l)} \left( Z_{12}^{(k_1)} x \cdots x Z_{12}^{(k_s)} x \otimes m \right). \]

The next thing one has to do is consider transitivity of the operation. But this forces us to consider how \( Z_{31}^{(l)} \) would act on some \( Z_{32}^{(\alpha)} y \) terms, and this is what leads one to consider the relationship between the two types
of resolutions of two-rowed shapes. If one looks at the first place such an action occurs, it’s in the resolution of the three-rowed skew-shape with two triple overlaps. There we have to define the map of the resolution of

\[
\begin{array}{c|c|c}
 t_1 + t_2 + 1 & p_1 \\
 t_2 + 1 & \sigma \\
 & p_2 + t_2 + 1 \\
 & p_3 - t_2 - 1 \\
\end{array}
\]

into that of

\[
\begin{array}{c|c}
 t_2 + 1 & \sigma \\
 t_1 & p_1 \\
 & p_2 \\
 & p_3 \\
\end{array}
\]

To handle this case, we use the identity

\[
(\ast) \quad \partial_{32}^{(i)} \partial_{31}^{(j)} = \sum_{\alpha > 0} (-1)^{\alpha - 1} \partial_{32}^{(\alpha + i)} \partial_{31}^{(\alpha - j)} (\partial_{21}^{(\alpha)} + (-1)^{k} \partial_{21}^{(k)} \partial_{32}^{(i + k)})
\]

which was used in a previous section.

When \( k = t_1 + 1 \), the term \((-1)^{k} \partial_{21}^{(k)} \partial_{32}^{(i + k)} \) would make sense if we were using the old version of the resolution of two-rowed shapes. Since we’re not, we have to translate that term into ones that make sense for our current form. This is still all very incomplete. In fact, it seems to us that we should use a different form of \((\ast)\), perhaps only involving \(\partial_{12}^{(\alpha)}\) instead of \(\partial_{21}^{(\alpha)}\). In any event, what we have so far is the following:

We should define

\[
\begin{align*}
Z_{32}^{(i)} Z_{31}^{(j)} & \equiv (\cdot(t_1 + t_1 + 1) \otimes b(p_2 + t_2 + 1)) = \\
& \sum_{k > 0} (-1)^{k+1} Z_{32}^{(i)} y \otimes (a(p_1) \otimes a(k) b(p_2 + t_2 + 1) \otimes a(t_1 + 1 - k)) \\
& \sum_{k > 0} (-1)^{i-1} Z_{12}^{(j)} x \otimes (b(p_1 - \sigma - \alpha) \otimes a(p_1 + t_1 + 1) b(\alpha - 1) \otimes b(p_3)).
\end{align*}
\]

We have yet to define the term

\[
Z_{32}^{(i)} Z_{31}^{(j)} z \otimes (a(p) \otimes b(q)) = Z_{12}^{(k)} Z_{12}^{(k)} x \otimes m.
\]

What we really should write above is:

\[
\begin{align*}
Z_{32}^{(m)} Z_{31}^{(n)} z \otimes (a(p) \otimes b(q)) & = \\
& \sum_{k > 0} (-1)^{k+1} Z_{32}^{(m + k)} y \otimes (a(p-n) \otimes a(k) b(q) \otimes a(n-k)) \\
& \sum_{k > 0} (-1)^{p-n} Z_{12}^{(\lambda)} x \otimes (b(p-n-\lambda) \otimes a(p) b(q + \lambda - m - p) \otimes b(m+n)).
\end{align*}
\]

But this only holds when \( m + p - q > 0 \) (which is the case when our general parameters are specialized to the ones immediately above). We see that when our original parameters are those dictated by our mapping cone situation, the resulting parameters put us in the right complex. More generally, we get:

\[
\partial_{32}^{(m)} \partial_{31}^{(n)} (a(p) \otimes b(q)) =
\]
\[
\sum_{k>0} (-1)^{k+1} \partial_{32}^{(m+k)} (a^{(p-n)} \otimes a^{(k)} b^{(q)} \otimes a^{(n-k)}) +
\]

\[
\sum_{\lambda>0} (-1)^{p-\lambda} \partial_{12}^{(\lambda)} (b^{(p-n-\lambda)} \otimes a^{(p)} b^{(q-\lambda-m-p)} \otimes b^{(m+n)}) +
\]

\[
(-1)^{p+1} b^{(p-n)} \otimes a^{(p)} b^{(q-m-p)} \otimes b^{(m+n)}
\]

if we insist that our summation over \( \lambda \) is for positive \( \lambda \) only. If we allow \( \lambda \) to be zero as well, we don’t get the extra term:

\[
(-1)^{p+1} b^{(p-n)} \otimes a^{(p)} b^{(q-m-p)} \otimes b^{(m+n)}.
\]

The above discussion indicates some of the directions one can go in if one makes the simplification of using polarization from place 2 to place 1 instead of the reverse polarization we discussed in the main part of the paper. As the reader can see, there are complications even though some of the actions seem more pleasing aesthetically. The complications compound (at least for us) when we try to keep track, in the general case, of the effects of rotating the top part of an almost skew-shape (which is a skew-shape, after all), and changing the senses of the polarization operators. We offer this variation of the method we have been using in the event that there is someone who can carry it to successful conclusion.

9.4. Homotopy for Partitions. Here we describe, as far as we can, the explicit homotopy for the resolution of a three-rowed partition, \((p, q, r)\).

Our terms of dimension 0, \(X_0\), are generated freely by the standard bitableaux

\[
\begin{bmatrix}
  w_1 & a^{(p)} & b^{(q_1)} & c^{(r_1)} \\
  w_2 & b^{(q_2)} & c^{(r_2)} \\
  w_3 & c^{(r_3)}
\end{bmatrix},
\]

with \(q_1 + q_2 = q\) and \(r_1 + r_2 + r_3 = r\). Our terms in dimension 1, \(X_1\), are of the form

\[
Z_{21}^{(k)} x \otimes m \ 	ext{with} \ m \in D_p \otimes D_{q-k} \otimes D_r \ 	ext{and}
\]

\[
Z_{32}^{(l)} y \otimes m \ 	ext{with} \ m \in D_p \otimes D_{q+l} \otimes D_{r-l}.
\]

Our homotopy \(s_0 : X_0 \to X_1\) is defined by
case, our homotopy on the terms of the resolution which just involve the take the value 0 on these terms. Using our experience with the two-rowed prefixes, will simply be 21

\[
\begin{pmatrix}
0 \\
Z_{21}^{(q_1)} x \otimes \begin{pmatrix}
w_1 & a^{(p+q_1)} & c^{(r_1)} \\
w_2 & b^{(q_2)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{pmatrix} \\
Z_{32}^{(r_1)} y \otimes \begin{pmatrix}
w_1 & a^{(p)} & b^{(r_1)} \\
w_2 & b^{(q)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{pmatrix} \\
\sum_{\alpha>0} (r_2+\alpha) \alpha^{Z_{21}^{(\alpha)} x} \otimes \begin{pmatrix}
w_1 & a^{(p+\alpha)} & c^{(r_1-\alpha)} \\
w_2 & b^{(q-\alpha)} & c^{(r_2+\alpha)} \\
w_3 & c^{(r_3)}
\end{pmatrix}
\end{pmatrix}
\]

if \( q_1 = r_1 = r_2 = 0 \)

if \( q_1 > 0 \)

if \( q_1 = 0 \) and \( r_1 > 0 \)

if \( q_1 = r_1 = 0 \) and \( r_2 > 0 \)

From this we see that a basis for the syzygies in dimension 0 is the free module generated by terms of the form

\[
Z_{21}^{(k)} x \otimes \begin{pmatrix}
w_1 & a^{(p+k)} & c^{(r_1)} \\
w_2 & b^{(p-k)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{pmatrix};
\]

\[
Z_{32}^{(l)} y \otimes \begin{pmatrix}
w_1 & a^{(p)} & b^{(l)} \\
w_2 & b^{(q)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{pmatrix};
\]

\[
Z_{32}^{(l)} y \otimes \begin{pmatrix}
w_1 & a^{(p)} & c^{(r_2-l)} \\
w_2 & b^{(q+l)} & c^{(r_3)}
\end{pmatrix}.
\]

Since we’re trying to build a splitting homotopy, our next homotopy, \( s_1 \), will take the value 0 on these terms. Using our experience with the two-rowed case, our homotopy on the terms of the resolution which just involve the \( Z_{21} \) prefixes, will simply be

\[
\begin{pmatrix}
0 \\
Z_{21}^{(k_1)} x \cdots x Z_{21}^{(k_n)} x \otimes \begin{pmatrix}
w_1 & a^{(p+|k|)} & b^{(q_1)} & c^{(r_1)} \\
w_2 & b^{(q_2)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{pmatrix}
\end{pmatrix} =
\]

\[
\begin{pmatrix}
\sum_{\alpha>0} (r_2+\alpha) \alpha^{Z_{21}^{(\alpha)} x} \otimes \begin{pmatrix}
w_1 & a^{(p+|k|+q_1)} & c^{(r_1)} \\
w_2 & b^{(q_2)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{pmatrix}
\end{pmatrix}
\]

if \( q_1 = 0 \)

if \( q_1 > 0 \).
In particular, then, our homotopy $s_1$ is defined on the terms of the first type, i.e., those with prefix $Z_{21}$. To define it on the terms prefixed with a $Z_{32}$, we have to consider some cases.

We'll first consider a term of the form

$$Z^{(l)}_{3,2} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(l+t)} & c^{(r_1)} \\ w_2 & b^{(q-t)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{bmatrix}$$

with $r_1 + r_2 + r_3 = r - l$, and $t > 0$. Set

$$s_1 \left( Z^{(l)}_{32} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(l+t)} & c^{(r_1)} \\ w_2 & b^{(q-t)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{bmatrix} \right) =$$

$$\sum_{k>0} \left( -l - 1 \right) \binom{l}{k} \cdot \sum_{\gamma>0} \binom{\gamma-1}{\beta} \sum_{\beta>0} \left( c^{(r_1)} \right) \left( \gamma - 1 \right) \left( c^{(r_1)} \right) \left( \beta \right) \left( c^{(r_1)} \right) +$$

$$s_1 \left( \sum_{\gamma>0} \binom{\gamma}{\beta} \sum_{\beta>0} \left( -1 \right)^{\gamma-\beta-1} Z^{(\gamma)}_{21} x \otimes \begin{bmatrix} w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} & c^{(r_1+\beta)} \\ w_2 & b^{(q-t-\beta)} & c^{(r_2+\beta)} \\ w_3 & c^{(r_3)} \end{bmatrix} \right).$$

Now even though the tableaux

$$\begin{bmatrix} w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} & c^{(r_1+\beta)} \\ w_2 & b^{(q-t-\beta)} & c^{(r_2+\beta)} \\ w_3 & c^{(r_3)} \end{bmatrix}$$

are not necessarily standard, the homotopy $s_1$ is defined on all terms with prefix $Z_{21}$, and is linear there. Hence we don’t have to straighten in order to apply the homotopy to these terms. The fact that this homotopy does what it should is a straightforward calculation using the usual identities on binomial coefficients. To illustrate this, we do it in some detail:

$$\partial s_1 \left( Z^{(l)}_{32} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(l+t)} & c^{(r_1)} \\ w_2 & b^{(q-t)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{bmatrix} \right) =$$
\[
\sum_{k>0} \left( \frac{-l-1}{k-1} \right) Z_{32}^{(l)} y \otimes \partial_{21}^{(l+k)} \begin{bmatrix}
w_1 & a^{(p+l+k)} & b^{(t-k)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix} - \\
\sum_{k>0} \left( \frac{-l-1}{k-1} \right) Z_{32}^{(l)} Z_{21}^{(l+k)} x \otimes \begin{bmatrix}w_1 & a^{(p+l+k)} & b^{(t-k)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix} + \\
\sum_{\gamma>0} \sum_{\beta} (-1)^{\gamma-\beta-1} \begin{bmatrix}r_1 + l - \beta \\
r_2 \end{bmatrix} = \\
\partial s_1 \begin{bmatrix}Z_{21}^{(\gamma)} x \otimes \begin{bmatrix}w_1 & a^{(p+\gamma)} & b^{(t-\gamma+\beta)} & c^{(r_1+l-\beta)} \\
w_2 & b^{(q-t-\beta)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix}
\end{bmatrix}.
\]

But
\[
\partial_{21}^{(l+k)} \begin{bmatrix}w_1 & a^{(p+l+k)} & b^{(t-k)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix} = \begin{bmatrix}w_1 & a^{(p+l)} & b^{(t+l)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix},
\]
so that
\[
\sum_{k>0} \left( \frac{-l-1}{k-1} \right) Z_{32}^{(l)} y \otimes \partial_{21}^{(l+k)} \begin{bmatrix}w_1 & a^{(p+l+k)} & b^{(t-k)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix} = \\
\sum_{k>0} \left( \frac{-l-1}{k-1} \right) \begin{bmatrix}t + l \\
t - k\end{bmatrix} Z_{32}^{(l)} y \otimes \begin{bmatrix}w_1 & a^{(p)} & b^{(t+l)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix} = \\
\begin{bmatrix}t - 1 \\
t - 1\end{bmatrix} Z_{32}^{(l)} y \otimes \begin{bmatrix}w_1 & a^{(p)} & b^{(t+l)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix} = \\
Z_{32}^{(l)} y \otimes \begin{bmatrix}w_1 & a^{(p)} & b^{(t+l)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix}.
\]

This takes care of the first term in the sum we’re computing. Now
\[
Z_{32}^{(l)} Z_{2,1}^{(l+k)} x \otimes \begin{bmatrix}w_1 & a^{(p+l+k)} & b^{(t-k)} & c^{(r_1)} \\
w_2 & b^{(q-t)} & c^{(r_2)} \\
w_3 & c^{(r_3)}
\end{bmatrix} = \\
\sum_{\alpha} Z_{21}^{(l+k-\alpha)} x \otimes \begin{bmatrix}r_1 + \alpha \\
\alpha\end{bmatrix} \sum_{\beta} \begin{bmatrix}r_1 + l - \beta \\
r_1 + \alpha \end{bmatrix} = \\
\begin{bmatrix}w_1 & a^{(p+l+k-\alpha)} & b^{(t-k-l+\alpha+\beta)} & c^{(r_1+l-\beta)} \\
w_2 & b^{(q-t-\beta)} & c^{(r_2+\beta)} \\
w_3 & c^{(r_3)}
\end{bmatrix},
\]
so that

\[
\sum_k Z_{32}^{(l)} Z_{21}^{(l+k)} x \otimes \begin{bmatrix}
  w_1 & a^{(p+l+k)} & b^{(l-t)} & c^{(r_1)} \\
  w_2 & a^{(q-t)} & b^{(l-t)} & c^{(r_2)} \\
  w_3 & c^{(r_3)} & c^{(r_3)} & c^{(r_3)}
\end{bmatrix} =
\sum_k \sum_\alpha Z_{21}^{(l+k-\alpha)} x \otimes \left( r_1 + \alpha \right) \sum_\beta \left( r_1 + l - \beta \right) \left( r_2 + \beta \right) \begin{bmatrix}
  w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} & c^{(r_1+l-\beta)} \\
  w_2 & b^{(q-t-\beta)} & b^{(l-\gamma+\beta)} & c^{(r_2+\beta)} \\
  w_3 & c^{(r_3)} & c^{(r_3)} & c^{(r_3)}
\end{bmatrix}.
\]

If we let \( \gamma = l + k - \alpha \), the coefficient of the term

\[
Z_{21}^{(\gamma)} x \otimes \begin{bmatrix}
  w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} & c^{(r_1+l-\beta)} \\
  w_2 & b^{(q-t-\beta)} & b^{(l-\gamma+\beta)} & c^{(r_2+\beta)} \\
  w_3 & c^{(r_3)} & c^{(r_3)} & c^{(r_3)}
\end{bmatrix}
\]

is

\[
\sum_k \sum_\alpha \left( l - 1 \right) \left( r_1 + \alpha \right) \left( r_1 + l - \beta \right) \left( r_2 + \beta \right) =
\sum_k \sum_\alpha \left( l - 1 \right) \left( \gamma + k - \alpha \right) \left( r_1 + l - \beta \right) \left( r_2 + \beta \right) =
\left( -\beta - 1 \right) \left( r_1 + l - \beta \right) \left( r_2 + \beta \right) =
\left( -1 \right) \gamma - \beta - 1 \left( \gamma + 1 \right) \left( r_1 + l - \beta \right) \left( r_2 + \beta \right).
\]

This gives us that

\[
\partial s_1 \left( Z_{32}^{(l)} y \otimes \begin{bmatrix}
  w_1 & a^{(p)} & b^{(l+t)} & c^{(r_1)} \\
  w_2 & b^{(q-t)} & b^{(l-t)} & c^{(r_2)} \\
  w_3 & c^{(r_3)} & c^{(r_3)} & c^{(r_3)}
\end{bmatrix} \right) =
\]
On the other hand,

$Z_{32}^{(t)} y \otimes \left[ \begin{array}{ccc} w_1 & a^{(p)} & b^{(l+t)} \\ w_2 & b^{(q-t)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{array} \right] = \sum_{\gamma > 0} \sum_{\beta} (-1)^{\gamma - \beta - 1} \left( \begin{array}{c} \gamma - 1 \\ \beta \end{array} \right) \left( \begin{array}{cc} r_1 + l - \beta \\ r_1 \end{array} \right) \left( \begin{array}{cc} r_2 + \beta \\ r_2 \end{array} \right).$

$Z_{21}^{(\gamma)} x \otimes \left[ \begin{array}{ccc} w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} \\ w_2 & b^{(q-t-\beta)} & c^{(r_2+\beta)} \\ w_3 & c^{(r_3)} \end{array} \right] + \sum_{\gamma > 0} \sum_{\beta} (-1)^{\gamma - \beta - 1} \left( \begin{array}{c} \gamma - 1 \\ \beta \end{array} \right) \left( \begin{array}{cc} r_1 + l - \beta \\ r_1 \end{array} \right) \left( \begin{array}{cc} r_2 + \beta \\ r_2 \end{array} \right).$

$\partial s_1 \left( Z_{21}^{(\gamma)} x \otimes \left[ \begin{array}{ccc} w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} \\ w_2 & b^{(q-t-\beta)} & c^{(r_2+\beta)} \\ w_3 & c^{(r_3)} \end{array} \right] \right).$

On the other hand,

$s_0 s_0 Z_{32}^{(t)} y \otimes \left[ \begin{array}{ccc} w_1 & a^{(p)} & b^{(l+t)} \\ w_2 & b^{(q-t)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{array} \right] = s_0 \left( \sum_{\beta} \left( \begin{array}{cc} r_1 + l - \beta \\ r_1 \end{array} \right) \left( \begin{array}{cc} r_2 + \beta \\ r_2 \end{array} \right) \right) \left[ \begin{array}{ccc} w_1 & a^{(p)} & b^{(l+t)} \\ w_2 & b^{(q-t)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{array} \right].$

Now let’s use the fact that

$Z_{21}^{(\gamma)} x \otimes \left[ \begin{array}{ccc} w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} \\ w_2 & b^{(q-t-\beta)} & c^{(r_2+\beta)} \\ w_3 & c^{(r_3)} \end{array} \right] = \partial s_1 \left( Z_{21}^{(\gamma)} x \otimes \left[ \begin{array}{ccc} w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} \\ w_2 & b^{(q-t-\beta)} & c^{(r_2+\beta)} \\ w_3 & c^{(r_3)} \end{array} \right] \right) + s_0 \partial \left( Z_{21}^{(\gamma)} x \otimes \left[ \begin{array}{ccc} w_1 & a^{(p+\gamma)} & b^{(l-\gamma+\beta)} \\ w_2 & b^{(q-t-\beta)} & c^{(r_2+\beta)} \\ w_3 & c^{(r_3)} \end{array} \right] \right).$
This tells us that, modulo a boundary (which we can describe),

$$
\sum_{\gamma > 0} \sum_{\beta} (-1)^{\gamma - \beta - 1} \binom{\gamma - 1}{\beta} \binom{r_1 + l - \beta}{r_1} \binom{r_2 + \beta}{r_2} .
$$

$$
Z^{(\gamma)}_{21} x \otimes \left[ \begin{array}{ccc}
  w_1 & a(p+\gamma) & b(t-\gamma+\beta) & c(r_1+1-l-\beta) \\
  w_2 & b(q-t-\beta) & c(r_2+\beta) \\
  w_3 & c(r_3) 
\end{array} \right] = 
$$

$$
\sum_{\gamma > 0} \sum_{\beta} (-1)^{\gamma - \beta - 1} \binom{\gamma - 1}{\beta} \binom{r_1 + l - \beta}{r_1} \binom{r_2 + \beta}{r_2} .
$$

$$
\partial Z^{(\gamma)}_{21} s_0 \left[ \begin{array}{ccc}
  w_1 & a(p+\gamma) & b(t-\gamma+\beta) & c(r_1+1-l-\beta) \\
  w_2 & b(q-t-\beta) & c(r_2+\beta) \\
  w_3 & c(r_3) 
\end{array} \right] = 
$$

$$
\sum_{\gamma > 0} \sum_{\beta} (-1)^{\gamma - \beta - 1} \binom{\gamma - 1}{\beta} \binom{r_1 + l - \beta}{r_1} \binom{r_2 + \beta}{r_2} \binom{t + \beta}{t} .
$$

$$
\partial^{(\gamma)}_{21} s_0 \left[ \begin{array}{ccc}
  w_1 & a(p) & b(t+\beta) & c(r_1+1-l-\beta) \\
  w_2 & b(q-t-\beta) & c(r_2+\beta) \\
  w_3 & c(r_3) 
\end{array} \right] = 
$$

$$
\sum_{\gamma > 0} \sum_{\beta} \binom{\gamma}{\gamma - \beta - 1} \binom{r_1 + l - \beta}{r_1} \binom{r_2 + \beta}{r_2} \binom{t + \beta}{t} .
$$

$$
\partial^{(\gamma)}_{21} s_0 \left[ \begin{array}{ccc}
  w_1 & a(p) & b(t+\beta) & c(r_1+1-l-\beta) \\
  w_2 & b(q-t-\beta) & c(r_2+\beta) \\
  w_3 & c(r_3) 
\end{array} \right] = 
$$

$$
\sum_{\gamma > 0} \sum_{\beta} \binom{\gamma}{\gamma - \beta - 1} \binom{r_1 + l - \beta}{r_1} \binom{r_2 + \beta}{r_2} .
$$

$$
\partial^{(\gamma)}_{21} s_0 \left[ \begin{array}{ccc}
  w_1 & a(p) & b(t+\beta) & c(r_1+1-l-\beta) \\
  w_2 & b(q-t-\beta) & c(r_2+\beta) \\
  w_3 & c(r_3) 
\end{array} \right] = 
$$

So we see that the homotopy works on the terms of the form selected. Now we have to look at terms of the form

$$
Z^{(t)}_{32} y \otimes \left[ \begin{array}{ccc}
  w_1 & a(p) & b(t-t) & c(r_1) \\
  w_2 & b(q+t) & c(r_2) \\
  w_3 & c(r_3) 
\end{array} \right],
$$

where \( t \geq 0 \).
Of the terms $Z^{(l)}_{32} y \otimes \left[ \begin{array}{c|ccc} w_1 & a^{(p)} & b^{l(t)} & c^{(r_1)} \\ w_2 & b^{(q+t)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{array} \right]$, with $t \geq 0$, the two extreme cases: $t = 0$ and $t = l$, when coupled with the conditions $r_1 = 0$ in the first case, and $r_1 = r_2 = 0$ in the second, will be mapped to zero since they are in the image of $s_0$. (E.g.,

$$\partial Z^{(l)}_{32} y \otimes \left[ \begin{array}{c|ccc} w_1 & a^{(p)} & b^{l(t)} & c^{(r_2)} \\ w_2 & b^{(q)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{array} \right] =$$

$$\sum_{\alpha>0} \binom{r_2 + \alpha}{r_2} \left( \begin{array}{c|c|c} w_1 & a^{(p)} & b^{l(\alpha)} \\ w_2 & b^{(q-\alpha)} & c^{(r_2+\alpha)} \\ w_3 & c^{(r_3)} \end{array} \right) + \left( \begin{array}{c|c} a^{(p)} & b^{l(t)} \\ b^{(q)} & c^{(r_2)} \\ c^{(r_3)} \end{array} \right),$$

so that

$$s_0 \partial Z^{(l)}_{32} y \otimes \left[ \begin{array}{c|ccc} w_1 & a^{(p)} & b^{l(t)} & c^{(r_2)} \\ w_2 & b^{(q)} & c^{(r_2)} \\ w_3 & c^{(r_3)} \end{array} \right] =$$

$$\sum_{\alpha>0} \binom{r_2 + \alpha}{r_2} Z^{(\alpha)}_{21} x \otimes \left( \begin{array}{c|cc} w_1 & a^{(p+\alpha)} & b^{l(\alpha)} \\ w_2 & b^{(q-\alpha)} & c^{(r_2+\alpha)} \\ w_3 & c^{(r_3)} \end{array} \right) +$$

$$Z^{(l)}_{32} y \otimes \left( \begin{array}{c|c} a^{(p)} & b^{l(t)} \\ b^{(q)} & c^{(r_2)} \\ c^{(r_3)} \end{array} \right) -$$

$$\sum_{\alpha>0} \binom{r_2 + \alpha}{r_2} Z^{(\alpha)}_{21} x \otimes \left( \begin{array}{c|cc} w_1 & a^{(p+\alpha)} & b^{l(\alpha)} \\ w_2 & b^{(q-\alpha)} & c^{(r_2+\alpha)} \\ w_3 & c^{(r_3)} \end{array} \right)$$

$$= Z^{(l)}_{32} y \otimes \left( \begin{array}{c|c} a^{(p)} & b^{l(t)} \\ b^{(q)} & c^{(r_2)} \\ c^{(r_3)} \end{array} \right).$$

It’s even easier for the term with $t = l$ and $r_1 = r_2 = 0$.)

The next case we should dispose of is the one: $(l - t) + r_1 = 0$ and $r_2 > 0$. Here we have
Thus, if we set

\[ s_1 \left( Z_{32}^{(l)} y \otimes \begin{bmatrix} w_1 & a^{(p)} \\ w_2 & b^{(q+l)} \\ w_3 & c^{(r_3)} \end{bmatrix} \right) = Z_{32}^{(r_2+l)} y \otimes \begin{bmatrix} w_1 & a^{(p)} \\ w_2 & b^{(q+r_2+l)} \\ w_3 & c^{(r_3)} \end{bmatrix}, \]

we see right away that this works. This means that we want to assume that 
\( (l-t) + r_1 > 0 \) and that \( t + r_1 > 0 \) (for we’ve disposed of the case \( t = r_1 = 0 \)).

Let’s look at the case \( t = l, r_1 > 0 \). That is, we’re starting with the term

\[ Y = Z_{32}^{(l)} y \otimes \begin{bmatrix} w_1 & a^{(p)} \\ w_2 & b^{(q+l)} \\ w_3 & c^{(r_3)} \end{bmatrix}, \]

and we assume that \( r_1 < l \). Define

\[ s_1(Y) = \sum_{\alpha > 0} (-1)^{\alpha - 1} \binom{l + \alpha}{\alpha} Z_{32}^{(l)} y \otimes Z_{31}^{(\alpha)} \begin{bmatrix} w_1 & a^{(p+\alpha)} \\ w_2 & b^{(q+l)} \\ w_3 & c^{(r_3)} \end{bmatrix} \]

\[ + \sum_{\beta} (-1)^{r_1 - \beta - 1} Z_{32}^{(r_1)} y \otimes Z_{32}^{(l+\beta-r_1)} y \otimes \begin{bmatrix} w_1 & a^{(p)} \\ w_2 & b^{(q+l)} \\ w_3 & c^{(r_3)} \end{bmatrix}. \]

(In both of the terms above, the hypothesis that \( r_1 < l \) is crucial: in the first term, one must have \( \alpha \leq l \) for the term to make sense, and in the second, \( l + \beta - r_1 \) must be positive. Thus, there is no restriction on \( \beta \) if we assume that \( r_1 < l \).)

We’ll show that

**Proposition 9.6.** \( \partial s_1(Y) + s_0 \partial(Y) = Y \).

**Proof.** The element

\[ Z_{32}^{(l)} y \otimes Z_{31}^{(\alpha)} \begin{bmatrix} w_1 & a^{(p+\alpha)} \\ w_2 & b^{(q+l)} \\ w_3 & c^{(r_3)} \end{bmatrix} \]
goes to
\[
\left( \begin{array}{l}
{r_1} \\
\alpha
\end{array} \right) Z_{32}^{(l)} y \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p)} & a^{(r_1)} \\
w_2 & b^{(q+l)} & c^{(r_2)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right] - \]
\[
(-1)^{\alpha} \left( \begin{array}{l}
{r_2} + l + \alpha \\
{r_2}
\end{array} \right) Z_{21}^{(\alpha)} x \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p+\alpha)} & c^{(r_1-\alpha)} \\
w_2 & b^{(q-\alpha)} & c^{(r_2+l+\alpha)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right] +
\]
\[
\sum_{u>0} (-1)^u \left( \begin{array}{l}
{r_1} - u \\
{r_1} - \alpha
\end{array} \right) Z_{32}^{(l+u)} y \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p)} & b^{(u)} \\
w_2 & b^{(q+l)} & c^{(r_2)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right].
\]

Therefore, the element
\[
\sum_{\alpha>0} (-1)^{\alpha-1} \left( \begin{array}{l}
{l} + \alpha \\
{\alpha}
\end{array} \right) Z_{32}^{(l)} y Z_{31}^{(\alpha)} z \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p+\alpha)} & c^{(r_1-\alpha)} \\
w_2 & b^{(q+l)} & c^{(r_2)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right]
\]
goes to
\[
\sum_{\alpha>0} (-1)^{\alpha-1} \left( \begin{array}{l}
{l} + \alpha \\
{\alpha}
\end{array} \right) \left( \begin{array}{l}
{r_1} \\
{r_2}
\end{array} \right) Z_{32}^{(l)} y \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p)} & c^{(r_1)} \\
w_2 & b^{(q+l)} & c^{(r_2)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right] +
\]
\[
\sum_{\alpha>0} \left( \begin{array}{l}
{l} + \alpha \\
{\alpha}
\end{array} \right) \left( \begin{array}{l}
{r_2} + l + \alpha \\
{r_2}
\end{array} \right) Z_{21}^{(\alpha)} x \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p+\alpha)} & c^{(r_2+l+\alpha)} \\
w_2 & b^{(q-\alpha)} & c^{(r_3)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right] +
\]
\[
\sum_{\alpha>0} (-1)^{\alpha-1} \left( \begin{array}{l}
{l} + \alpha \\
{\alpha}
\end{array} \right) \sum_{u>0} (-1)^u \left( \begin{array}{l}
{r_1} - u \\
{r_1} - \alpha
\end{array} \right) Z_{32}^{(l+u)} y \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p)} & b^{(u)} \\
w_2 & b^{(q+l)} & c^{(r_2)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right]
\]

which, after doing the indicated sums, comes out to be
\[
Y + (-1)^{r_1-1} \left( \begin{array}{l}
{l} \\
{r_1}
\end{array} \right) Y + \sum_{u>0} (-1)^{r_1+u-1} \left( \begin{array}{l}
{l} + u \\
{r_1}
\end{array} \right) Z_{32}^{(l+u)} y \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p)} & b^{(u)} \\
w_2 & b^{(q+l)} & c^{(r_2)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right] +
\]
\[
\sum_{\alpha>0} \left( \begin{array}{l}
{l} + \alpha \\
{\alpha}
\end{array} \right) \left( \begin{array}{l}
{r_2} + l + \alpha \\
{r_2}
\end{array} \right) Z_{21}^{(\alpha)} x \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p+\alpha)} & c^{(r_2+l+\alpha)} \\
w_2 & b^{(q-\alpha)} & c^{(r_3)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right].
\]

We note that
\[
(-1)^{r_1-1} \left( \begin{array}{l}
{l} \\
{r_1}
\end{array} \right) Y + \sum_{u>0} (-1)^{r_1+u-1} \left( \begin{array}{l}
{l} + u \\
{r_1}
\end{array} \right) Z_{32}^{(l+u)} y \otimes \left[ \begin{array}{ccc}
w_1 & a^{(p)} & b^{(u)} \\
w_2 & b^{(q+l)} & c^{(r_2)} \\
w_3 & c^{(r_3)} & c^{(r_3)}
\end{array} \right] =
\]
Next we note that the element
goes to
\[ (-1)^{r_1+\beta-1}Z_{32}^{(r_1)} y Z_{32}^{(l+\beta-r_1)} y \otimes \begin{bmatrix} w_1 & a(p) & b(\beta) & c(r_1-\beta) \\ w_2 & b(q+l) & c(r_2) \\ w_3 & c(r_3) \end{bmatrix}. \]

Hence the element
\[ \sum_{\beta} (-1)^{r_1-\beta-1}Z_{32}^{(r_1)} y Z_{32}^{(l+\beta-r_1)} y \otimes \begin{bmatrix} w_1 & a(p) & b(\beta) & c(r_1-\beta) \\ w_2 & b(q+l) & c(r_2) \\ w_3 & c(r_3) \end{bmatrix} \]
goes to
\[ -\binom{l+r_2}{r_2}Z_{32}^{(r_1)} y \otimes \begin{bmatrix} w_1 & a(p) & b(r_1) \\ w_2 & b(q) & c(l+r_2) \\ w_3 & c(r_3) \end{bmatrix}. \]

(We see that we already have cancellation of this latter term with one of the terms we obtained earlier.) Now we want to calculate \( s_0 \partial(Y) \). We have that
\[ \partial(Y) = \binom{l+r_2}{r_2} \begin{bmatrix} w_1 & a(p) & c(r_1) \\ w_2 & b(q) & c(l+r_2) \\ w_3 & c(r_3) \end{bmatrix}, \]
and this goes, under $s_0$, to

\[
\begin{pmatrix} l + r_2 \\ r_2 \end{pmatrix} Z_{32}^{(r_1)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(r_1)} \\ w_2 & b^{(q)} & c^{(l+r_2)} \\ w_3 & c^{(r_3)} \end{bmatrix} = \\
\sum_{\alpha > 0} \left( l + \alpha \right) \begin{pmatrix} r_2 + l + \alpha \\ r_2 \end{pmatrix} Z^{(\alpha)}_{21} x \otimes \begin{bmatrix} w_1 & a^{(p+\alpha)} & b^{(q-\alpha)} \\ w_2 & c^{(r_2+l+\alpha)} \\ w_3 & c^{(r_3)} \end{bmatrix}.
\]

Now it’s trivial to see that we’ve proven the Proposition.

It is easy to see that these calculations also check out for $r_1 = l$, so we really have to consider only the case $r_1 > l$.

We have looked at two special cases for $r_1 > l$: one is the situation when $r = 2$, and the other when $r = 3$. We record what we found, even though the picture is far from complete.

When $r = 2$, we want to consider the case $l = r_1 = 1$ and hence the element $Y = Z_{32}^{(1)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(q+1)} \\ w_2 & c \end{bmatrix}$.

In this case, we can define

\[
s_1 \left( Z_{32}^{(1)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(q+1)} & c \end{bmatrix} \right) = 2Z_{32}^{(1)} y Z_{31}^{(1)} z \otimes \begin{bmatrix} w_1 & a^{(p+1)} & b^{(q+1)} \end{bmatrix} - Z_{32}^{(1)} y Z_{32}^{(1)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(q+1)} \end{bmatrix}.
\]

To see this, we first calculate $\partial s_1(Y)$:

\[
\partial s_1(Y) = 2Y + 2Z_{21}^{(1)} x \otimes \begin{bmatrix} w_1 & a^{(p+1)} & b^{(q-1)} & c^{(2)} \end{bmatrix} - 2Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(q+1)} \end{bmatrix} - \\
Z_{32}^{(1)} y \otimes \begin{bmatrix} w_1 & a^{(p+1)} & b^{(q+1)} & c \end{bmatrix} - Z_{32}^{(1)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(q)} & c \end{bmatrix} + 2Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(q+1)} \end{bmatrix} = \\
y + 2Z_{21}^{(1)} x \otimes \begin{bmatrix} w_1 & a^{(p+1)} & b^{(q-1)} & c^{(2)} \end{bmatrix} - Z_{32}^{(1)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(q)} & c \end{bmatrix}.
\]

\[
s_0 \partial(Y) = s_0 \left( \begin{bmatrix} w_1 & a^{(p)} & c \\ w_2 & b^{(q)} & c \end{bmatrix} \right) = Z_{32}^{(1)} y \otimes \begin{bmatrix} w_1 & a^{(p)} & b^{(q)} & c \end{bmatrix} - 2Z_{21}^{(1)} x \otimes \begin{bmatrix} w_1 & a^{(p+1)} & b^{(q-1)} & c^{(2)} \end{bmatrix}.
\]

So we end up with $\partial s_1(Y) + s_0 \partial(Y) = Y$. 
The next case to consider is one in which \( l < r_1 \), so we let \( Y = Z_{32}^{(1)} \). Here, we found that we can set

\[
\partial s_1(Y) = 2Y + Z_{21}^{(1)} x \otimes \begin{bmatrix} w_1 & a^{(p+1)} b \otimes c \end{bmatrix} - Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p)} b \otimes c \end{bmatrix} - Z_{21}^{(1)} x \otimes \begin{bmatrix} w_1 & a^{(p+1)} b \otimes c \end{bmatrix} + 3Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p+2)} b \otimes c \end{bmatrix} + 3Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p+2)} b \otimes c \end{bmatrix} - 3Z_{32}^{(3)} y \otimes \begin{bmatrix} w_1 & a^{(p)} b \otimes c \end{bmatrix} -
\]

To see why this works, we calculate:

\[
\partial s_1(Y) = 2Y + Z_{21}^{(1)} x \otimes \begin{bmatrix} w_1 & a^{(p+1)} b \otimes c \end{bmatrix} - Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p)} b \otimes c \end{bmatrix} - Z_{21}^{(1)} x \otimes \begin{bmatrix} w_1 & a^{(p+1)} b \otimes c \end{bmatrix} + 3Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p+2)} b \otimes c \end{bmatrix} + 3Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p+2)} b \otimes c \end{bmatrix} - 3Z_{32}^{(3)} y \otimes \begin{bmatrix} w_1 & a^{(p)} b \otimes c \end{bmatrix} -
\]

On the other hand,

\[
s_0 \partial(Y) = Z_{32}^{(2)} y \otimes \begin{bmatrix} w_1 & a^{(p+2)} b \otimes c \end{bmatrix} - Z_{21}^{(1)} x \otimes \begin{bmatrix} w_1 & a^{(p+1)} b \otimes c \end{bmatrix} -
\]

So here again we see that \( \partial s_1(Y) + s_0 \partial(Y) = Y \).
As the reader can see, all the calculations are of a piece, but as yet the pattern hasn’t emerged. We hope that all of the above will be revealed to be transparent.

10. Appendix/Apologia

The reader will understand that this paper represents the joint work of the authors, but was written after the death of Gian-Carlo Rota. Consequently, it is bound to have the shortcomings that result from a one-sided presentation of a two-sided collaboration. Another major shortcoming is that this work is clearly incomplete; the focus of the project had been to complete the description of the resolutions and work out some applications. In fact, it is this incompleteness that was responsible for our not having written up the results we had obtained as we went along. With the death of Rota, it was felt that a large part of this collaborative work should be made available to those, other than myself, who might be interested in carrying the project forward. It is for this reason that some of the proofs have been elaborated in some detail; this was done where the results and/or methods represented a combination of techniques which might be unfamiliar to the reader, but which seem to form basic tools in approaching this problem.

I of course must also apologize for the fact that those parts of the paper which fall more naturally into the expertise of Rota were probably not presented as faithfully, fully or elegantly as they would have been by Rota himself.

11. List of Notations

In the following very informal list of notation and terminology, we’ve indicated the page on which certain terms that do not necessarily appear in the Table of Contents appear for the first time in the paper.

- $D_l$: the $l^{th}$ divided power (of some underlying free module); 5
- $(v|1^{(v)}2^{(s)})$ is an element of the letter-place algebra; 6
- \[
\begin{pmatrix}
    w \\
    w'
\end{pmatrix}
\begin{pmatrix}
    1^{(p)}2^{(k)} \\
    2^{(q-k)}
\end{pmatrix}
\]
  is a double tableau; 6
- positive and negative places; 7
- $\partial_{ij}$ is the place polarization from place $j$ to place $i$; 9
- The mapping cone of $f$, $M(f)$; 11
- $[p_1, p_2, \ldots, p_n; t_1, \ldots, t_n]$ : a skew-shape or almost skew-shape; 16
- Almost skew-shape of type $i$; 16
- Separators or separator variables; 20
- Bar$(A; S)$ : the Bar Algebra on the algebra $A$ with separators $S$; 21
- $T$-grading of Bar$(A; S)$: 21
Bar(A; S; T, i); 21
Bar(M, A; S): the free bar module of the A-module M, with set of separators S; 21
\(\partial_k\): antiderivation; 22
\(\partial_T = \sum_{x \in T} \partial_k\): the T-boundary operator; 22
(t\(^+\))-graded strand of degree n; 23
Weyl map as composite of place polarizations; 23
\(Z_{2,1}\): “virtual” polarization operator; 23
Cap(1,2); 28
Res([p_1, p_2, p_3; t_1, t_2]); 33
\(Z_{n,m}^* \odot Z_{n,m}^{(d)}\); 35
Cap'(1,2,3); 37
Cap''(1,2,3); 49
Stem([p_1, p_2, p_3; t_1, t_2]); 52
Shuf(\(Z_{32}^{(b)} y\); \(Z_{32}^{(u)} y\)); 65
Shuf(\(\partial (Z_{32}^{(b)} y); Z_{32}^{(u)} y\)); 67
Leibnitz’ Rule; 73
Polarization algebra; 75
Res(p_1, \ldots, p_m; t_1, \ldots, t_{m-1}); 77

References


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