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Homotopies for resolutions of skew-hook shapes

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Abstract

We present characteristic-free resolutions and splitting homotopies for the Weyl modules associated to skew-hook shapes.

Résumé

Nous présentons des résolutions en caractéristique-libre, et des homotopies associées aux formes du type « skew-hook ».

1. Introduction

The consideration of the Grothendieck ring in representation theory prompted A. Lascoux [L77] to translate the classical Giambelli determinantal expression of the Schur functions into a resolution, in characteristic zero, of the Schur modules of which these functions were the formal characters. The main idea in this translation is to determine in which dimension of the resolution the terms of the determinantal expansion are to go. Each term in the expansion corresponds to a permutation, and the length of that permutation gives the dimension of that term. Although in [L77] Lascoux dealt only with partitions, it was clear how the extension to skew shapes should be made. The precise definitions of the maps in these resolutions were given later and can be found in [A88,A92,Z87].

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When λ is a partition and μ is a partition such that $\mu_i \leqslant \lambda_i$ for all i, then the Giambelli formula gives the skew Schur function $s_{\lambda/\mu}$ as the determinant $|e_{\lambda'_i-\mu'_j+j-i}|$ where λ', μ' are the partitions conjugate to λ, μ, the e_k are the kth elementary symmetric functions, and $e_{-i}=0$ for $-i<0$. For example, if λ/μ is the skew shape $(3,3,3,2,1)\,/\,(1,0,0)$, then

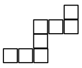
$$s_{\lambda/\mu} = \begin{vmatrix} e_4 & e_6 & e_7 \\ e_2 & e_4 & e_5 \\ e_0 & e_2 & e_3 \end{vmatrix}.$$

Noting that e_k is the formal character of the kth exterior power Λ^k, and replacing the product of polynomials by the tensor product of the corresponding representation modules, the characteristic 0 resolution of the preceding skew shape was given as

$$0 \to \Lambda^7 \otimes \Lambda^4 \to \begin{array}{c} \Lambda^6 \otimes \Lambda^5 \\ \oplus \\ \Lambda^7 \otimes \Lambda^2 \otimes \Lambda^2 \end{array} \to \begin{array}{c} \Lambda^6 \otimes \Lambda^2 \otimes \Lambda^3 \\ \oplus \\ \Lambda^4 \otimes \Lambda^5 \otimes \Lambda^2 \end{array} \to \Lambda^4 \otimes \Lambda^4 \otimes \Lambda^3. \tag{1}$$

In treating Weyl modules, a completely analogous treatment can be given to the Jacobi–Trudi determinantal expansion $s_{\lambda/\mu} = |h_{\lambda_i-\mu_j+j-i}|$. In this case, the entries of the matrix are the complete symmetric polynomials h_i (which are the characters of the ith symmetric power in characteristic zero, and of the ith divided power in arbitrary characteristic). Thus the terms of the resolution would translate in characteristic 0 as the sums of tensor products of symmetric powers. For example, a resolution of the Weyl module $K_{(5,4,3)/(1,0,0)}$ is given by replacing each exterior power in (1) with the corresponding symmetric power. In a characteristic free setting, we want to replace the exterior powers with divided powers. This naive procedure will succeed for the skew-hook shapes treated in this paper, but for more general shapes one must also add terms to the resolution that are not represented in the Jacobi–Trudi expansion.

A number of reasons motivated the study of resolutions of Weyl modules in the characteristic-free context. From a universal algebra point of view, the modules of syzygies are of some considerable importance. But interest also was generated by questions related to $\mathbb{Z}$-forms of rational representations and to intertwining numbers (see, e.g., [AB86]). From the point of view of obtaining an explicit description of the syzygies, the construction of a splitting homotopy seems to be a fruitful way to proceed with these resolutions; the non-zero images of the basis elements under the homotopy usually form a basis for the syzygy modules. In [BR94,B98], examples of such homotopies can be found.

In this paper, we construct characteristic-free resolutions of the Weyl modules associated to skew hook shapes such as . Each term of the resolution is a direct sum of tensor products of divided powers algebras. This result generalizes the characteristic-free resolutions of Weyl modules associated to hook shapes found in [B98].

Exactness is established by constructing a characteristic-free splitting homotopy; here splitting occurs over the integers. Few explicit homotopies are known for Weyl modules, and this homotopy is new even in characteristic 0. The construction of the homotopy involves a matching between basis elements in modules at adjacent stages of the resolution.

Viewed combinatorially, this matching turns out to be the sign-reversing involution appearing in the Gessel–Viennot proof [GV85] of the relevant Jacobi–Trudi identity.

In Section 2 we recall some constructions from characteristic-free multilinear algebra including the Weyl (or co-Schur) modules from [ABW82] and the letter-place algebras of [DRS76,GRS87]. Section 3 outlines the letter-place construction of the resolution of the nth exterior powers of a free module as described in [B98]. This material provides a vital connection to the general program [BR01] for applying letter-place methods to the resolution of Weyl modules. In Section 4, we introduce a new basis that simplifies the construction in Section 3. The construction we provide produces the resolution of a tensor product of Weyl modules having skew-hook shape as a subquotient complex (indexed by intervals in a Boolean algebra) of the resolution of an exterior power representation. Finally, in Section 5 we construct the desired homotopy.

The resolutions

In this section, we describe how to find characteristic-free resolutions for the Weyl modules associated to the skew-hooks as a subquotient complexes of the Akin–Buchsbaum resolution for the exterior powers. The proofs of these results and are deferred to later sections, as are explicit constructions of both the Weyl modules and the resolutions of the exterior powers.

Let F be a free $\mathbb{Z}$-module, We recall the structure of the characteristic-free resolution [AB86], $\mathbb{X}^n$, of an exterior as a direct sum of tensor products of divided powers. All of these objects are defined formally in Section 2, but the reader accustomed to working in characteristic 0 may temporarily wish to consider the following constructions over the rationals, in which case the ith divided power module may safely be replaced with the ith symmetric power.

Proposition 1.1. *Let F be a free $\mathbb{Z}$-module, $\Lambda^n(F)$ the nth exterior power of F, and $\mathcal{D}_i(F)$ the module of ith divided powers of F.*

Define modules $X_{\{i_1<\cdots<i_{k-1}\}}$ by $X_{\{i_1<\cdots<i_{k-1}\}} \simeq \mathcal{D}_{i_1-0}(L) \otimes \cdots \otimes \mathcal{D}_{n-i_{k-1}}$ where the isomorphism holds over the universal enveloping algebra of gl_F. Then there is a resolution, $\mathbb{X}^n$, of $\Lambda^n(F)$ over the universal enveloping algebra of gl_F such that

$$\mathbb{X}^n \simeq \bigoplus_{E\subseteq\{1,\dots,n-1\}} X_E.$$

Constructions for this resolution are given in Sections 3 and 4. A new proof of exactness is given via the splitting homotopy of Section 5.

Definition 1.1. Suppose $E_0 \subseteq E_1 \subseteq \{1,\dots,n-1\}$. We define $\mathbb{X}(E_0, E_1)$ to be the subquotient complex of $\mathbb{X}^n$ formed by taking all X_E with $E_0 \subseteq E$ in the quotient complex $\mathbb{X}/\mathbb{X}_{\not\subseteq E_1}$ of $\mathbb{X}^n$ by the subcomplex $\mathbb{X}_{\not\subseteq E_1}$ consisting of all X_E with $E \not\subseteq E_1$.

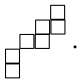
The image of X_E in $\mathbb{X}^n/\mathbb{X}_{\not\subseteq E_1}$ is naturally isomorphic to X_E; we will continue to call it X_E.

With this definition, the complex $\mathbb{X}(\emptyset, \{1, 2, \ldots, n-1\})$ is just $\mathbb{X}^n$ and the complex $\mathbb{X}(E, E)$ is just the single module X_E.

Theorem 1.2. *Let λ/μ be an n-celled shape satisfying $\lambda_{j+1} - \mu_j \in \{0, 1\}$, i.e., $K_{\lambda/\mu}$ is a tensor product of Weyl modules associated to skew shapes. Let $E_1 = \{\sum_{l=1}^{j} \lambda_l - \mu_l \mid j \in \mathbb{Z}^+\}\backslash\{n\}$ and let $E_0 = \{\sum_{l=1}^{j} \lambda_l - \mu_l \mid j \in \mathbb{Z}^+,\ \lambda_{j+1} - \mu_j = 0\}\backslash\{n\}$. The complex $\mathbb{X}(E_0, E_1)$ is exact except at X_{E_1} where its homology is $K_{\lambda/\mu}$.*

This generalizes the resolution [B98] for the hook shapes. The proof, in Section 5, is again by homotopy.

Example. The subquotient complex $\mathbb{X}(\{5\}, \{1, 3, 5, 6\})$ of the resolution, $\mathbb{X}^7$ of Λ^7 forms a characteristic-free resolution for the Weyl module, $K_{(4,4,3,1,1)/(3,2,1)}$, corresponding to shape



Assuming Theorem 1.2, we can interpret the resolution $\mathbb{X}(E_0, E_1)$ as a mapping cone.

Theorem 1.3. *Given $E_0 \subset E_1 \subseteq \{1, \ldots, n-1\}$ and $v \in E_1\backslash E_0$, we have a short exact sequence of complexes $0 \to \mathbb{X}(E_0 \cup \{v\}, E_1) \to \mathbb{X}(E_0, E_1) \to \mathbb{X}(E_0, E_1\backslash\{v\}) \to 0$. Further, $\mathbb{X}(E_0, E_1)$ is the mapping cone of a map $\mathbb{X}(E_0, E_1\backslash\{v\}) \to \mathbb{X}(E_0 \cup v, E_1)$.*

Recognizing the injection in the above theorem as subset inclusion proves exactness. A proof that $\mathbb{X}(E_0, E_1)$ is the claimed mapping cone is given at the end of Section 4.

2. Constructions for characteristic-free representation theory

We review some constructions used in the sequel. The overwhelming majority of these techniques can be found either explicitly or implicitly in [ABW82,DRS76,GRS87], although the outline of the current presentation differs somewhat.

Let M be a rank m free $\mathbb{Z}$-module. The divided powers algebra $\mathcal{D}(M)$ of M is defined to be the graded Hopf algebra dual of the symmetric algebra $Sym(M^*)$ on the linear dual of M. If we fix a basis $b_1, \ldots, b_m$ for M, then $\mathcal{D}(M)$ is isomorphic to the free commutative $\mathbb{Z}$-algebra generated by the symbols $b_i^{(k)}$ for all $1 \leqslant i \leqslant m$ and all $k \geqslant 0$, modulo the ideal generated by the relations

$$b_i^{(j)} b_i^{(k)} = \binom{j+k}{j} b_i^{(j+k)} \quad \text{and} \quad b_i^{(0)} = 1. \tag{2}$$

The module M appears naturally as a submodule of $\mathcal{D}(M)$ and we let b_i denote $b_i^{(1)}$.

Letting each element of M have degree 1, $\mathcal{D}(M)$ is a graded algebra and its ith graded component $\mathcal{D}_i$ is a free $\mathbb{Z}$-submodule with basis $\{b_1^{(j_1)} \cdots b_m^{(j_m)} \mid j_1 + \cdots + j_m = i\}$.

Given two free modules, L and P^+, one can construct (see [GRS87]) a bilinear pairing $(|)$ of the divided powers algebras $\mathcal{D}(L)$ and $\mathcal{D}(P^+)$ into $\mathcal{D}(L \otimes P^+)$. We have decorated P^+ with a sign following the definition in [GRS87] which properly applies to a direct sum of free modules $P = P^+ \oplus P^-$. Elements of P^+ are said to be *positively signed* and elements of P^- are *negatively signed*. We have specialized above to the case $P^- = 0$.

In terms of bases, this pairing generalizes the permanent. In particular, suppose L and P^+ have bases $\mathcal{L}$ and $\mathcal{P}^+$, respectively. We identify the basis $\{l \otimes p \mid l \in \mathcal{L}, p \in \mathcal{P}\}$ of $L \otimes P^+$ with the set $\{(l|p) \mid l \in \mathcal{L}, p \in \mathcal{P}^+\}$ of "letter-places." The algebra $\mathcal{D}(L \otimes P^+)$ can now be identified with the commutative associative algebra $\mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ generated by all $(l|p)$ and satisfying the relations (2) for all $b = (l|p)$. For $l_1, \ldots, l_k \in \mathcal{L}$ and $p_1, \ldots, p_{k'} \in \mathcal{P}^+$, we have

$$(l_1 l_2 \cdots l_k | p_1 p_2 \cdots p_{k'}) = \begin{cases} \sum_{\sigma \in S_k} (l_{\sigma(1)}|p_1) \cdots (l_{\sigma(k)}|p_k) & \text{when } k = k', \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Turning to the case $P^+ = 0$, we have a bilinear map $(|) : \mathcal{D}(L) \otimes \Lambda(P^-) \to \Lambda(L \otimes P^-)$. If we are given a basis $\mathcal{P}^-$ for P^-, then identifying $\Lambda(L \otimes P^-)$ with the free exterior algebra $\Lambda([\mathcal{L}|\mathcal{P}^-])$ on generators $\{(l|p) \mid l \in \mathcal{L}, p \in \mathcal{P}^-\}$, the biproduct $(|)$ still satisfies Eqs. (3).

We note that for both $P = P^-$ and $P = P^+$, the relations (3) determine $(|)$ over $\mathbb{Z}$.

Example. Suppose we have bases $\mathcal{L} = \{a, b, c\}$, $\mathcal{P}^+ = \{\mathbf{1}, \mathbf{2}\}$, and $\mathcal{P}^- = \{1^-, 2^-\}$ for L, P^+, and P^-, respectively. We have

$$(aabc|\mathbf{1222}) = 12(a|\mathbf{1})(a|\mathbf{2})(b|\mathbf{2})(c|\mathbf{2}) + 6(b|\mathbf{1})(a|\mathbf{2})(a|\mathbf{2})(c|\mathbf{2}) + 6(c|\mathbf{1})(a|\mathbf{2})(a|\mathbf{2})(b|\mathbf{2}),$$

so bilinearity (over $\mathbb{Z}$) implies that

$$\left(a^{(2)}bc|\mathbf{12}^{(3)}\right) = (a|\mathbf{1})(a|\mathbf{2})(b|\mathbf{2})(c|\mathbf{2}) + (b|\mathbf{1})(a|\mathbf{2})^{(2)}(c|\mathbf{2}) + (c|\mathbf{1})(a|\mathbf{2})^{(2)}(b|\mathbf{2}).$$

On the other hand, since $2^-2^- = 0$ in $\Lambda(P)$, we have $(aabc|1^-2^-2^-2^-) = 0$.

For the remainder of this paper, we fix two positively signed, linearly ordered sets $\mathcal{L}$, of cardinality m, and $\mathcal{P}^+ = \{\mathbf{1} < \mathbf{2} < \cdots < \boldsymbol{n}\}$ whose elements will be the basis of free modules L and P^+, respectively. Likewise, we fix a linearly ordered, negatively signed set $\mathcal{P}^- = \{1^- < \cdots < n^-\}$ whose elements are the basis of a rank-n free $\mathbb{Z}$-module P^-.

The preceding example suggests the following.

Notation. Let $\underline{u} = u_1, \ldots, u_i$ be a sequence of elements taken from some set S. For $s \in S$, define $c_s(\underline{u}) = |\{j \mid u_j = s\}|$ Further, for sequences $\underline{w}$ and $\underline{v}$ of elements in $\mathcal{L}$ and $\mathcal{P}^+$, respectively, define

$$(w_1, w_2, \ldots, w_k | v_1, v_2, \ldots, v_k) = \left(\prod_{l \in \mathcal{L}} l^{(c_l(\underline{w}))} \middle| \prod_{p \in \mathcal{P}^+} p^{(c_p(\underline{v}))} \right).$$

Likewise, for sequences $\underline{w}$ and $\underline{v}$ of elements in $\mathcal{L}$ and $\mathcal{P}^-$, respectively, define

$$(w_1, w_2, \ldots, w_k | v_1, v_2, \ldots, v_k) = \left(\prod_{l \in \mathcal{L}} l^{(c_l(\underline{w}))} \middle| v_1 v_2 \cdots v_k \right).$$

We recall the following expansion property of the biproduct.

Proposition 2.1. *Given a sequence $\underline{w}$ of elements in $\mathcal{L}$ and $v, v' \in \mathcal{D}(P^+)$, we have*

$$\left(\prod_{l \in \mathcal{L}} l^{(c_l(\underline{w}))} \middle| v v' \right) = \sum_{\underline{w_1}, \underline{w_2}} \left(\prod_{l \in \mathcal{L}} l^{(c_l(\underline{w}_1))} \middle| v \right) \left(\prod_{l \in \mathcal{L}} l^{(c_l(\underline{w}_2))} \middle| v' \right),$$

where the sum is taken over all weakly increasing sequences $\underline{w}_1, \underline{w}_2$ such that $\underline{w} = \underline{w}_1 + \underline{w}_2$ as multisets.

In Hopf algebras notation, one could instead write $(w|vv') = \sum_{(w)} ((w)_1|v)((w)_2|v')$ where $\sum_{(w)} (w)_1 \otimes (w)_2$ is the Sweedler notation for $\Delta(w)$, the coproduct applied to w. Indeed, this and a few closely related properties are commonly taken (see [GRS87]) as the definition of the biproduct over an arbitrary commutative ring.

Certainly, the $\mathbb{Z}$-linear span of all $(w_1, \ldots, w_k|\mathbf{1}, \ldots, \mathbf{1})$ with $w_i \in \mathcal{L}$ has as basis all biproducts $(w_1, \ldots, w_k|\mathbf{1}, \ldots, \mathbf{1})$ with $w_1 \leqslant \cdots \leqslant w_k$. Further, this $\mathbb{Z}$-span is isomorphic to $\mathcal{D}(P)$ as a gl_n-representation. To make this formal over $\mathbb{Z}$, we let $U(gl_{\mathcal{L}})$ be the Kostant $\mathbb{Z}$-form of the universal enveloping algebra of the general linear group generated by all the divided powers of all $e_{a,b}$ with $a \neq b$ and by all elements $\binom{e_{a,a}}{k}$, where $a, b \in \mathcal{L}$ and $k \in \mathbb{Z}^+$. We omit the basis-free construction of this $\mathbb{Z}$-form. The element $e_{a,b}$ acts on $\mathcal{D}(L)$ as the letter polarization $a\partial/\partial b$ and on $\mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ as the polarization $\sum_{\boldsymbol{i} \in \mathcal{P}^+} (a|i)\partial/\partial(b|i)$. The action of $e_{a,b}$ on $\Lambda([\mathcal{L}|\mathcal{P}^-])$ may be defined by requiring that $e_{a,b}(x|i) = 0$ if $x \neq b$, that $e_{a,b}(b|i) = (a|i)$, and that $e_{a,b}$ is a derivation, namely $e_{a,b}(pq) = e_{a,b}(p) \cdot q + p \cdot e_{a,b}(q)$.

More generally, one constructs $U(gl_{\mathcal{L}})$-modules as is detailed below.

Definition 2.1. Given positive integers $0 = i_0 < i_1 < \cdots < i_k$, a sequence $w_1, \ldots, w_{i_k}$ in $\mathcal{L}$, and a sequence $v_1, \ldots, v_{i_k}$ in $\mathcal{P}^+$ (respectively $\mathcal{P}^-$), define a generalized *bitableau*

$$\left(\begin{array}{r|r} w_{i_1}, \ldots, w_1 & v_{i_1}, \ldots, v_1 \\ w_{i_2}, \ldots, w_{i_1+1} & v_{i_2}, \ldots, v_{i_1+1} \\ \cdots & \cdots \\ w_{i_k}, \ldots, w_{i_{k-1}+1} & v_{i_k}, \ldots, v_{i_{k-1}+1} \end{array} \right) = \prod_{l=1}^{k} (w_{i_l}, \ldots, w_{i_{l-1}+1} | v_{i_l}, \ldots, v_{i_{l-1}+1})$$

as an element of $\mathcal{D}([\mathcal{L}|\mathcal{P}])$ (respectively $\Lambda([\mathcal{L}|\mathcal{P}])$).

In writing these bitableaux, we may choose to denote a repeated consecutive subsequence $l, \ldots, l$ of i copies of l by $l^{(i)}$.

The last shorthand for subsequences is chosen to be suggestive for identities such as $(a, b, c, d|\mathbf{4}, \mathbf{3}, \mathbf{1}^{(2)}) = (abcd|\mathbf{431}^{(2)})$, the latter also equals $(a, b, c, d|\mathbf{4}, \mathbf{3}, \mathbf{1}, \mathbf{1})$.

The following proposition is a direct consequence of the fact that regardless of the sign of $\mathcal{P}$, $e_{a,b}(\underline{w}|\underline{v}) = (e_{a,b}\underline{w}|\underline{v})$.

Proposition 2.2. *If we fix positive integers $i_1 < \cdots < i_k$ and fix a sequence $v_1, \ldots, v_{i_k}$ in $\mathcal{P}^+$ (respectively $\mathcal{P}^-$) then the $\mathbb{Z}$-linear span, ranging over all sequences $w_1, \ldots, w_{i_k}$ in $\mathcal{L}$, of the bitableaux*

$$\left(\begin{array}{r|r} w_{i_1}, \ldots, w_1 & v_{i_1}, \ldots, v_1 \\ w_{i_2}, \ldots, w_{i_1+1} & v_{i_2}, \ldots, v_{i_1+1} \\ \cdots & \cdots \\ w_{i_k}, \ldots, w_{i_{k-1}+1} & v_{i_k}, \ldots, v_{i_{k-1}+1} \end{array}\right)$$

is a $U(gl_{\mathcal{L}})$-representation.

Note that the order of the elements in each row of the above bitableau makes no difference in the module defined in Proposition 2.2.

Example. If for $1 \leqslant l \leqslant k$ in Proposition 2.2 we let $v_{i_l} = v_{i_l - 1} = \cdots = v_{i_{l-1}+1} = \boldsymbol{i_{l-1} + 1}$ for all $1 \leqslant l \leqslant k$ where again $i_0 = 0$, then the representation constructed in Proposition 2.2 is isomorphic, over $U(gl_{\mathcal{L}})$, to

$$\mathcal{D}_{i_1 - i_0}(L) \otimes \cdots \otimes \mathcal{D}_{i_k - i_{k-1}}(L).$$

If in Proposition 2.2 we instead had let $i_l = l$ and $v_{i_l} = 1^-$ for all $1 \leqslant l \leqslant k$, then the corresponding representation is isomorphic to $\Lambda^k(L)$.

Definition 2.2. If we choose the sequence $v_{i_l}, \ldots, v_{i_{l-1}+1}$ to be

$$\bigl(i_l - (l-1)\bigr)^-, \ldots, \bigl(i_{l-1} + 1 - (l-1)\bigr)^-$$

then the representation given in Proposition 2.2 is by definition the *Weyl module* $K_{\lambda/\mu}$ associated to the *skew-hook* λ/μ with $\lambda_l - \mu_l = i_l - i_{l-1}$ and $\mu_l + 1 = \lambda_{l+1}$.

Notation. When $P = P^-$, it is convenient to denote a bitableau of the form given in Proposition 2.2 by $[T]$ where T is a tableau with cells filled as follows: For $i_r \geqslant i > i_{r-1}$, T has a cell in row r from the top and column v_i from the right; this cell contains the element w_i. Let $D(\underline{v})$ be the set of weak descents of $\underline{v}$, i.e., $D(\underline{v}) = \{1 \leqslant i < i_k \mid v_i \geqslant v_{i+1}\}$. When the weak descents of $\underline{v}$ occur only in positions i_l, then $\underline{w}$ is the usual *row reading-word* of T.

Example. If in Definition 2.2 we start with $i_0, \ldots, i_4 = 0, 1, 4, 5, 7$, then we construct the representation $K_{(4,4,2,2)/(3,1,1,0)}$ as the span over all sequences $w_1, \ldots, w_7$ in $\mathcal{L}$ of the bitableaux

$$\left(\begin{array}{r|r} w_1 & 1^- \\ w_4, w_3, w_2 & 3^-, 2^-, 1^- \\ w_5 & 3^- \\ w_7, w_6 & 4^-, 3^- \end{array}\right) = \left[\begin{array}{cccc} & & & \boxed{w_1} \\ & \boxed{w_4} & \boxed{w_3} & \boxed{w_2} \\ & \boxed{w_5} & & \\ \boxed{w_7} & \boxed{w_6} & & \end{array}\right] \tag{4}$$

and $w_1, w_2, \ldots, w_7$ is the row-reading word of the tableau on the right-hand side of Eq. 4.

The following is a special case of the fact that the semi-standard Young tableaux of skew shape λ/μ index a $\mathbb{Z}$-basis for the Weyl module of shape λ/μ.

Proposition 2.3. *If λ/μ is a skew hook, then, following the notation of Definition* 2.2*, the Weyl module $K_{\lambda/\mu}$ has a $\mathbb{Z}$-basis consisting of all bitableaux*

$$\left(\begin{array}{r|r} w_{i_1}, \ldots, w_1 & i_1^-, \ldots\ldots, 1^- \\ \cdots & \cdots \\ w_{i_k}, \ldots, w_{i_{k-1}+1} & (i_k - (k-1))^-, \ldots, (i_{k-1} + 1 - (k-1))^- \end{array}\right),$$

where the word $w_1, w_2, \ldots, w_{i_k}$ satisfies $w_j \geqslant w_{j+1}$ for $j \notin \{i_1, \ldots, i_{k-1}\}$ and $w_j < w_{j+1}$ when $j \in \{i_1, \ldots, i_{k-1}\}$. In other words, the set of strict ascents *of $w_1, w_2, \ldots, w_{i_k}$ is precisely $\{i_1, \ldots, i_{k-1}\}$.*

3. Letter-place methods for the resolution of $\Lambda^n(L)$

In this section we modify slightly the presentation of the characteristic-free resolution of $\Lambda^n(L)$ given, via the letter-place techniques of [BR94], in [B98].

Let L be a rank-m free $\mathbb{Z}$-module with basis $\mathcal{L}$. We construct a characteristic-free resolution $\mathbb{X}^n$ of $\Lambda^n(L)$ inside a certain *bar complex* [BR94] as follows.

Definition 3.1. Let A be the free algebra with divided powers on $\binom{n}{2}$ generators $\{Z_{j,i} \mid n \geqslant j > i \geqslant 1\}$ modulo the ideal generated by relations

$$Z_{j,i}^{(c)} Z_{l,k}^{(d)} = Z_{l,k}^{(d)} Z_{j,i}^{(c)}, \quad \text{for } i > l. \tag{5}$$

Let $\{x_1, \ldots, x_{n-1}\}$ be a basis for another free $\mathbb{Z}$-module, and let $\Lambda = \Lambda(\{x_1, \ldots, x_{n-1}\})$ be the exterior algebra generated by this module. The free product of A and Λ is known as the *bar algebra* on the algebra A and the separators $\{x_1, \ldots, x_{n-1}\}$, and is denoted by $\operatorname{Bar}(A; \{x_1, \ldots, x_{n-1}\})$.

A boundary map ∂ is defined on $\operatorname{Bar}(A; \{x_1, \ldots, x_{n-1}\})$ by taking the sum of the polarizations of the x_i's to 1. Specifically, ∂ is the skew-derivation defined on $\operatorname{Bar}(A; \{x_1, \ldots, x_{n-1}\})$ by $\partial(A) = 0$, $\partial(x_i) = 1$, and $\partial(pq) = \partial(p)q + (-1)^{|p|} p \partial(q)$ with $|p|$ defined as the number of x_i's, counted with multiplicity, appearing in p.

We define a quotient algebra, B, of $\mathrm{Bar}(A; \{x_1, \dots, x_{n-1}\})$ by imposing the relations

$$Z_{j,i} x_k = x_k Z_{j,i} \quad \text{for } i > k, \tag{6}$$

$$Z_{j,i}^{(u)} Z_{i,k}^{(v)} x_k = \sum_{\alpha \leqslant u,v} Z_{i,k}^{(v-\alpha)} x_k Z_{j,i}^{(u-\alpha)} Z_{j,k}^{\alpha} \quad \text{for } u, v > 0. \tag{7}$$

We can now consider $B \otimes_A \mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ where the action of A on $\mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ is defined by letting the element $Z_{j,i}$ act as the place polarization $D_{j,i} = \sum_l (l|\boldsymbol{j}) \frac{\partial}{\partial \boldsymbol{i}}$. The map ∂ descends from $\mathrm{Bar}(A; \{x_1, \dots, x_{n-1}\})$ to B and thence extends to $B \otimes_A \mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ by $\partial(p \otimes m) = \partial(p)$.

Definition 3.2. Define $\mathbb{X}^n$ to be the subcomplex of the complex $B \otimes_A \mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ generated by all elements $b_{\{i_1,\dots,i_{k-1}\}}(\underline{w})$ for all $0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = n$ and all $w_j \in \mathcal{L}$, where $b_{\{i_1,\dots,i_{k-1}\}}(w_1, \dots, w_n)$ is defined as

$$\mathbf{Y}_{n,i_{k-1}} \mathbf{Y}_{i_{k-1},i_{k-2}} \cdots \mathbf{Y}_{i_2,i_1} \mathbf{Y}_{i_1,0} \left(\begin{array}{r|r} w_{i_1}, \dots, w_1 & \mathbf{1}^{(i_1)} \\ w_{i_2}, \dots, w_{i_1+1} & (\boldsymbol{i}_1 + \mathbf{1})^{(i_2-i_1)} \\ \cdots & \cdots \\ w_{i_k}, \dots, w_{i_{k-1}+1} & (\boldsymbol{i}_{k-1} + \mathbf{1})^{(i_k-i_{k-1})} \end{array} \right) \tag{8}$$

with

$$\mathbf{Y}_{j,i} = Z_{j,j-1} x_{j-1} Z_{j-1,j-2}^{(2)} x_{j-2} \cdots x_{i+3} Z_{i+3,i+2}^{(j-i-2)} x_{i+2} Z_{i+2,i+1}^{(j-i-1)} x_{i+1}.$$

Remark. A typical generator of this complex is of the form

$$Z_{i_k+1,i_k}^{(c_k)} x_{i_k} \cdots Z_{i_1+1,i_1}^{(c_1)} x_{i_1} \otimes m, \tag{9}$$

for some sequence $\underline{i}$. The boundary of such a term is

$$\begin{aligned} &(-1)^{k-1} Z_{i_k+1,i_k}^{(c_k)} x_{i_k} \cdots Z_{i_2+1,i_2}^{(c_2)} x_{i_2} \otimes D_{i_1+1,i_1}^{(c_1)} m \\ &\quad + \sum (-1)^{k-\alpha} Z_{i_k+1,i_k}^{(c_k)} x_{i_k} \cdots Z_{i_{\alpha+1}+1,i_{\alpha+1}}^{(c_{\alpha+1})} x_{i_{\alpha+1}} Z_{i_{\alpha+1},i_\alpha}^{(c_\alpha)} Z_{i_\alpha,i_{\alpha-1}}^{(c_{\alpha-1})} x_{i_{\alpha-1}} \cdots Z_{i_1+1,i_1}^{(c_1)} x_{i_1} \otimes m. \end{aligned}$$

The next example suggests how application of the boundary to a generator of form 9 produces another such generator.

Example. In the resolution of $\Lambda^{6,}$ consider the term in dimension 3 of the form

$$Z_{6,5} x_5 Z_{5,4}^{(2)} x_4 Z_{2,1} x_1 \otimes \big(\underline{w}_1 | \mathbf{1}^{(2)}\big) \big(\underline{w}_2 | \mathbf{3}\big) \big(\underline{w}_3 | \mathbf{4}^{(3)}\big) = \mathbf{Y}_{6,3} \mathbf{Y}_{2,0} \otimes \big(\underline{w}_1 | \mathbf{1}^{(2)}\big) \big(\underline{w}_2 | \mathbf{3}\big) \big(\underline{w}_3 | \mathbf{4}^{(3)}\big).$$

Letting $m = (\underline{w}_1 | \mathbf{1}^{(2)})(\underline{w}_2 | \mathbf{3})(\underline{w}_3 | \mathbf{4}^{(3)})$, the boundary of this term is

$$\begin{aligned}
&Z_{6,5}Z_{5,4}^{(2)}x_4Z_{2,1}x_1\otimes m-Z_{6,5}x_5Z_{5,4}^{(2)}Z_{2,1}x_1\otimes m+Z_{6,5}x_5Z_{5,4}^{(2)}x_4Z_{2,1}\otimes m\\
&\quad=Z_{5,4}^{(2)}x_4Z_{2,1}x_1Z_{6,5}\otimes m+Z_{5,4}x_4Z_{2,1}x_1Z_{6,4}\otimes m-Z_{6,5}x_5Z_{2,1}x_1Z_{5,4}^{(2)}\otimes m\\
&\qquad-Z_{6,5}x_5Z_{5,4}^{(2)}x_4Z_{2,1}\otimes m.
\end{aligned}$$

Applying the requisite place polarizations to m, gives

$$0+\mathbf{Y}_{5,3}\mathbf{Y}_{2,0}\otimes\left(\begin{array}{c|r}\underline{w}_1&\mathbf{1}^{(2)}\\\underline{w}_2&\mathbf{3}\\\underline{w}_3&\mathbf{64}^{(2)}\end{array}\right)-\mathbf{Y}_{6,4}\mathbf{Y}_{2,0}\otimes\left(\begin{array}{c|l}\underline{w}_1&\mathbf{1}^{(2)}\\\underline{w}_2&\quad\mathbf{3}\\\underline{w}_3&\mathbf{554}\end{array}\right)+\mathbf{Y}_{6,3}\otimes\left(\begin{array}{c|r}\underline{w}_1&\mathbf{21}\\\underline{w}_2&\mathbf{3}\\\underline{w}_3&\mathbf{4}^{(3)}\end{array}\right).$$

Formalizing the preceding example, we have the following.

Proposition 3.1. *The $\mathbb{Z}$-module $\mathbb{X}^n$ is a complex under the action of ∂ and is free with basis given by all elements $b_{\underline{i}}(\underline{w})$ such that each row $w_{i_{l+1}},\dots,w_{i_l}$ (of the bitableau appearing in the expression* (8) *defining $b_{\underline{i}}(\underline{w})$) weakly increases.*

Proof. Assuming that $\mathbb{X}^n$ is closed under ∂, the fact that it is a complex follows from the fact that the square of any skew-derivation is 0.

To prove that $\partial(\mathbb{X}^n)\subseteq\mathbb{X}^n$, we first observe that repeated applications of the identities (5)–(7) gives

$$\partial(\mathbf{Y}_{j,i})=\kappa+\sum_{j>l>i}(-1)^{j-l-1}\mathbf{Y}_{j,l}\mathbf{Y}_{l,i}Z_{l+1,i+1}^{(j-l)},\tag{10}$$

where κ is in the right-sided ideal, K, of B generated by all Z_{m_2,m_1} with $j>m_1>i+1$. This claim follows from the identity

$$\begin{aligned}
&Z_{l,l-1}^{(a+1)}\cdot Z_{l-1,l-2}^{(a+2)}x_{l-2}\cdots x_{i+2}Z_{i+2,i+1}^{(a+l-i-1)}x_{i+1}\\
&\quad=\kappa+Z_{l-1,l-2}^{(1)}x_{l-2}\cdots x_{i+2}Z_{i+2,i+1}^{((l-1)-i-1)}x_{i+1}Z_{l,i+1}^{(a+1)}
\end{aligned}$$

for some $\kappa\in K$. This last is easily proved by induction combined with the fact that for $m_1>l$, we have

$$\begin{aligned}
&Z_{m_2,m_1}^{(b)}\cdot Z_{l,l-1}^{(a+1)}x_{l-1}\cdots x_{i+2}Z_{i+2,i+1}^{(a+l-i-1)}x_{i+1}\\
&\quad=Z_{l,l-1}^{(a+1)}x_{l-1}\cdots x_{i+2}Z_{i+2,i+1}^{(a+l-i-1)}x_{i+1}Z_{m_2,m_1}^{(b)}\in K.
\end{aligned}$$

We conclude by applying two observations. First, the expressions κ and $Z_{l+1,i+1}^{(j-l)}$ in Eq. (10) commute with any $\mathbf{Y}_{h_2,h_1}$ where $i\geqslant h_2$. And second, if j,l,i in Eq. (10) are chosen to be some i_{m+2},i_{m+1},i_m from Eq. (8), then κ annihilates the bitableau appearing in Eq. (8).

Independence of the claimed basis follows from the fact that, treated as reduction rules, Eqs. (5)–(7) (together with the reduction $Z_{j,i}^{(a)} \otimes m \to 1 \otimes (D_{j,i}^{(a)} m)$) are confluent, i.e., they satisfy a "diamond lemma" (see [Be78]). □

Remark. The calculations of the preceding proof show that, in general, if $0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = n$, then $\partial(\mathbf{Y}_{i_k,i_{k-1}} \cdots \mathbf{Y}_{i_1,i_0} \otimes m)$ equals

$$\sum_{v \in \{1,\ldots,n-1\} \setminus \{i_1,\ldots,i_{k-1}\}} (-1)^{r(v)} \mathbf{Y}_{i_k,i_{k-1}} \cdots \mathbf{Y}_{a_v,v} \mathbf{Y}_{v,b_v} \cdots \mathbf{Y}_{i_1,i_0} \otimes D_{v+1,b_v+1}^{(a_v-v)} m, \tag{11}$$

where $b_v = \max\{i \in \{0, i_1, \ldots, i_{k-1}\} \mid i < v\}$, $a_v = \min\{i \in \{i_1, \ldots, i_{k-1}, n\} \mid i > v\}$, and $r(v) = |\{i > v \mid i \notin \{i_1, \ldots, i_k\}\}|$.

The following result, in a slightly different formulation, appears in [B98]. The resolution of the exterior powers was originally described, entirely without letter-place notation, in [AB86].

Proposition 3.2. *The complex $\mathbb{X}^n$ is a resolution of $\Lambda^n(L)$.*

Exactness may be proved via spectral sequences. In the sequel we provide a purely combinatorial definition of $\mathbb{X}^n$ together with a combinatorial proof of Proposition 3.2 relying only on the multilinear algebra presented in Section 2.

Applying Proposition 3.1, we can write down the decomposition of $\mathbb{X}^n$ into a direct sum of tensor products of divided powers as per Proposition 1.1.

Proposition 3.3. *Denoting $X_{\{i_1<\cdots<i_{k-1}\}}$ the $\mathbb{Z}$-linear span of all $b_{\{i_1,\ldots,i_{k-1}\}}(w_1, \ldots, w_n)$ for all sequences $w_1, \ldots, w_n$ of elements in $\mathcal{L}$, gives*

$$\mathbb{X}^n = \bigoplus_{E \subseteq \{1,\ldots,n-1\}} X_E$$

over $U(gl_{\mathcal{L}})$ and $X_{\{i_1<\cdots<i_{k-1}\}} \simeq \mathcal{D}_{i_1-0}(L) \otimes \cdots \otimes \mathcal{D}_{n-i_{k-1}}$ over $U(gl_{\mathcal{L}})$.

4. Bases

The preceding constructions become more combinatorially tractable if we introduce a new basis for the modules X_E. Before defining this basis, we observe that the element $\mathbf{Y}_{n,i_{k-1}} \cdots \mathbf{Y}_{i_1,0}$ preceding the bitableau in (8) can be recovered from the bitableau itself. Thus, for the sequel, we take the definition

$$b_{\{i_1,\ldots,i_{k-1}\}}(w_1, \ldots, w_n) = \left(\begin{array}{r|r} w_{i_1}, \ldots, w_1 & \mathbf{1}^{(i_1)} \\ w_{i_2}, \ldots, w_{i_1+1} & (\boldsymbol{i}_1 + \mathbf{1})^{(i_2-i_1)} \\ \cdots & \cdots \\ w_{i_k}, \ldots, w_{i_{k-1}+1} & (\boldsymbol{i}_{k-1} + \mathbf{1})^{(i_k-i_{k-1})} \end{array} \right). \tag{12}$$

This makes $X_{\{i_1<\cdots<i_{k-1}\}}$ a submodule of $\mathcal{D}([\mathcal{L}|\mathcal{P}])$, specifically the submodule spanned by all $b_{i_1,\ldots,i_{k-1}}(w_1,\ldots,w_n)$ as $\underline{w}$ runs over all sequences in $\mathcal{L}$. The following proposition gives the usual basis for the X_E; its proof is immediate.

Proposition 4.1. *A basis for X_E is given by all $b_E(w_1,\ldots,w_n)$ where $\underline{w}=w_1,\ldots,w_n$ ranges over all sequences such that the strict ascents of $\underline{w}$ lie in E.*

Our new basis will facilitate the definition of ∂ directly on the X_E considered as submodules of $\mathcal{D}([\mathcal{L}|\mathcal{P}])$. This basis is a translation to Weyl modules of a special case of the "compressed" bases defined in [T00]. As before, an element of the new basis is a bitableau. Recall that the bitableau (8) defining the basis element $b_E(\underline{w})$ had $|E|+1$ rows and $\underline{w}$ was restricted to have at most $|E|$ ascents. An element $\bar{b}_E(\underline{w})$ of the new basis will still have at most $|E|$ ascents, but the tableau used to define it will have precisely as many rows as there are ascents in $\underline{w}$.

Definition 4.1. Let $E=\{i_1<\cdots<i_{k-1}\}\subseteq\{1,\ldots,n-1\}$, let $i_k=n$, and let $\underline{w}=w_1,\ldots,w_n$ be a sequence in $\mathcal{L}$. Let $\{i_{a_1}<\cdots<i_{a_l}\}$ denote the set $\{1\leqslant a\leqslant n-1 \mid w_a<w_{a+1}\}\cup\{n\}$. Define the element $\bar{b}_E(w_1,\ldots,w_n)$ to be

$$\left(\begin{array}{r|r} w_{i_{a_1}},\ldots,w_1 & (\boldsymbol{i_{a_1-1}+1})^{(i_{a_1}-i_{a_1-1})}\cdots(\boldsymbol{i_1+1})^{(i_2-i_1)}\mathbf{1}^{(i_1)} \\ w_{i_{a_2}},\ldots,w_{i_{a_1}+1} & (\boldsymbol{i_{a_2-1}+1})^{(i_{a_2}-i_{a_2-1})}\cdots(\boldsymbol{i_{a_1}+1})^{(i_{a_1+1}-i_{a_1})} \\ \cdots & \cdots \\ w_{i_{a_l}},\ldots,w_{i_{a_{l-1}}+1} & (\boldsymbol{i_{a_l-1}+1})^{(\boldsymbol{i_{a_l}}-i_{a_l-1})}\cdots(\boldsymbol{i_{a_{l-1}}+1})^{(i_{a_{l-1}+1}-i_{a_{l-1}})} \end{array}\right). \tag{13}$$

Proposition 4.2. *The module X_E has a basis consisting of all $\bar{b}_E(w_1,\ldots,w_n)$ where $w_1,\ldots,w_n$ ranges over all sequences $\underline{w}$ whose ascents lie in E.*

Proof. Since this basis has the same cardinality as that of Proposition 4.1, it suffices to first apply Proposition 2.1, to expand the elements $\bar{b}_E(w_1,\ldots,w_n)$ into the usual basis for X_E, and then apply a rewrite of the same identity to expand the generators of X_E into the basis $\bar{b}_E(w_1,\ldots,w_n)$. □

Notation. As with the bitableaux forming the basis for a Weyl module $K_{\lambda/\mu}$, it will be convenient to denote the basis elements for X_E using (appropriately decorated) Young tableaux. We denote the bitableaux appearing in Eq. (13) by $|[T]|$ where T is defined in two steps as follows. First, we start with T being the Young tableaux of shape λ/μ where $\lambda_{l+1}=\mu_l$, $\lambda_l-\mu_l=i_{a_l}-i_{a_{l-1}}$, and whose row reading-word is $\underline{w}$. We do not draw a box around each letter. Second, we decorate T by inserting a vertical bar between the elements in positions i_j and i_j+1 for all j such that $i_j\notin\{i_{a_1},\ldots,i_{a_l}\}$.

For example, we have

$$\left(\begin{array}{r|r} w_1 & \mathbf{1} \\ w_4, w_3, w_2 & \mathbf{3}^{(2)}, \mathbf{2} \\ w_6, w_5 & \mathbf{6}, \mathbf{5} \\ w_8, w_7 & \mathbf{7}^{(2)} \end{array}\right) = \left\| \begin{bmatrix} & & & w_1 \\ & & w_4w_3|w_2 & \\ & w_6|w_5 & & \\ w_8w_7 & & & \end{bmatrix} \right\|$$

Definition 4.2 (Alternative construction of $\mathbb{X}^n$). Using the definition (12), we could define $\mathbb{X}^n$ to be the linear span of

$$b_{\{i_1,\dots,i_{k-1}\}}(\underline{w}) := \left(\begin{array}{r|r} w_{i_1}, \dots, w_1 & \mathbf{1}^{(i_1)} \\ \cdots & \cdots \\ w_{i_k}, \dots, w_{i_{k-1}+1} & (\boldsymbol{i_{k-1}} + \mathbf{1})^{(i_k - i_{k-1})} \end{array}\right) \tag{14}$$

in $\mathcal{D}([\mathcal{L}|\mathcal{P}])$ for all $w_j \in \mathcal{L}$ and all $0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = n$. We let $U(gl_{\mathcal{L}})$ act on $\mathbb{X}^n$ via its action on $\mathcal{D}([\mathcal{L}|\mathcal{P}])$.

Let $r_{E,v} = |\{j \mid n > j > v \text{ and } j \notin E\}|$. Taking the relation

$$\partial \bar{b}_E(\underline{w}) = \sum_{v \in \{1,\dots,n-1\} \setminus E} (-1)^{r_{E,v}} \bar{b}_{E \cup \{v\}}(\underline{w}),$$

computed by Eq. (11) and expressed in our new basis, as the definition of the boundary, completes the construction of $\mathbb{X}^n$. The fact that ∂ is a map of $U(gl_{\mathcal{L}})$-modules follows from the fact that the polarizations of $\mathcal{L}$ commute with the polarizations of $\mathcal{P}$.

Remark. Employing the above definitions, the resolution $\mathbb{X}(E_0, E_1)$, defined in Section 1 as a subquotient complex of $\mathbb{X}^n$, is naturally isomorphic to the span, inside $\mathcal{D}([\mathcal{L}|\mathcal{P}])$, of all X_E for $E_0 \subseteq E \subseteq E_1$ with boundary given by

$$\partial \bar{b}_E(\underline{w}) = \sum_{v \in E_1 \setminus E} (-1)^{r_{E,v}} \bar{b}_{E \cup \{v\}}(\underline{w}). \tag{15}$$

Returning to the example following Definition 3.2, we have

$$\partial \left\| \begin{bmatrix} & bb \\ abc|c & \end{bmatrix} \right\| = \left\| \begin{bmatrix} & bb \\ a|bc|c & \end{bmatrix} \right\| - \left\| \begin{bmatrix} & bb \\ ab|c|c & \end{bmatrix} \right\| + \left\| \begin{bmatrix} & b|b \\ abc|c & \end{bmatrix} \right\|.$$

Proof that $\mathbb{X}(E_0, E_1)$ is a complex. It suffices to check that $\mathbb{X}^n$ is a complex. We check this directly by showing that for any $E \subseteq \{1, \dots, n-1\}$, the image under ∂^2 of any basis element $\bar{b}_{E'}(\underline{w})$ is 0 in X_E. We may assume $E = E' \uplus \{v_1\} \uplus \{v_2\}$, for some $v_1, v_2 \in \{1, \dots, n-1\}$, since otherwise the component of $\partial^2(\bar{b}_{E'}(\underline{w}))$ in X_E is automatically 0. Now we compute that the component of $\partial^2(\bar{b}_{E'}(\underline{w}))$ in X_E is the same as the component in X_E of $\partial((-1)^{r_{E',v_1}} \bar{b}_{E' \uplus \{v_1\}}(\underline{w}) + (-1)^{r_{E',v_2}} \bar{b}_{E' \uplus \{v_2\}}(\underline{w}))$, and this component is the same as the component in E of

$$\left((-1)^{r_{E' \uplus \{v_1\}, v_2}} (-1)^{r_{E',v_1}} + (-1)^{r_{E' \uplus \{v_2\}, v_1}} (-1)^{r_{E',v_2}}\right) \bar{b}_{E' \uplus \{v_1, v_2\}}(\underline{w}).$$

Assuming in the preceding notation that $v_1 \geqslant v_2$, we have $r_{E' \uplus \{v_1\}, v_2} = r_{E', v_2} - 1$ and $r_{E' \uplus \{v_2\}, v_1} = r_{E', v_1}$. So since $r_{E' \uplus \{v_1\}, v_2} + r_{E', v_1} = r_{E', v_2} - 1 + r_{E' \uplus \{v_2\}, v_1}$, the coefficient of $\bar{b}_{E' \uplus \{v_1, v_2\}}(\underline{w})$ above vanishes and the component of $\partial^2(\bar{b}_{E'}(\underline{w}))$ in X_E is 0. □

We can now establish the structure of the resolutions $\mathbb{X}(E_0, E_1)$ as mapping cones.

Proof of Theorem 1.3. For each $E_0 \subseteq E \subseteq E_1 \backslash \{v\}$, consider $\varphi : \mathbb{X}(E_0, E_1 \backslash \{v\}) \to \mathbb{X}(E_0 \cup \{v\}, E_1)$ given by

$$\varphi_E : X_E \to X_{E \cup \{v\}} \quad \text{such that} \quad \varphi_E\bigl(\bar{b}_E(\underline{w})\bigr) = (-1)^{r_{E,v}} \bar{b}_{E \cup \{v\}}(\underline{w})$$

for each basis element $\bar{b}_E(\underline{w})$ of X_E. Since φ_E may be defined by the action of the divided power of a polarization of the places $\mathcal{P}$, the maps φ_E are $U(gl_{\mathcal{L}})$-equivariant. We claim that the map $\tilde{\varphi}$ given by $\tilde{\varphi}_E = (-1)^{|(E_1 \backslash \{v\}) \backslash E|} \varphi_E$ is a map of complexes. Observing that the sign appearing in the definition of φ_E above simply records the dimension of the module X_E in the resolution $\mathbb{X}(E_0, E_1 \backslash \{v\})$, the claim immediately follows from the calculation on page 38 that $\mathbb{X}^n$ and hence $\mathbb{X}(E_0, E_1)$ is a complex.

To show that $\mathbb{X}(E_0, E_1)$ is the mapping cone of $\tilde{\varphi}_E$, it suffices to check the identities

$$\begin{gathered}
\mathbb{X}(E_0, E_1)_i = \mathbb{X}(E_0 \cup \{v\}, E_1)_i \oplus \mathbb{X}(E_1, E_1 \backslash \{v\})_{i-1}, \\
\partial_{\mathbb{X}(E_0, E_1)}|_{\mathbb{X}(E_0 \cup \{v\}, E_1)} = \partial_{\mathbb{X}(E_0 \cup \{v\}, E_1)}, \\
\partial_{\mathbb{X}(E_0, E_1)}|_{\mathbb{X}(E_0, E_1 \backslash \{v\})} = \partial_{\mathbb{X}(E_0, E_1 \backslash \{v\})} + \varphi.
\end{gathered}$$

The first identity follows from the fact that X_E has dimension $|E_1 \backslash E|$ in $\mathbb{X}(E_0, E_1)$ and $\mathbb{X}(E_0 \cup \{v\}, E_1)$, whereas the dimension of X_E in $\mathbb{X}(E_0 \cup \{v\}, E_1)$ is $|(E_1 \backslash \{v\}) \backslash E|$. The second two identities are immediate consequences of formula (15) for ∂. □

We remark that this expression of $\mathbb{X}(E_0, E_1)$ as a mapping cone provides the basis of an alternate, inductive proof for the exactness of $\mathbb{X}(E_0, E_1)$. One assumes exactness of the smaller complexes $\mathbb{X}(E_0 \cup \{v\}, E_1)$ and $\mathbb{X}(E_1, E_1 \backslash \{v\})$ except at the 0th dimension term. Then one uses the known fact that $\tilde{\varphi}_{E_1 \backslash \{v\}}$ is the lift of a map from the 0th homology of $\mathbb{X}(E_1, E_1 \backslash \{v\})$ to the 0th homology of $\mathbb{X}(E_0 \cup \{v\}, E_1)$ whose cokernel is precisely the Weyl module one desires to have as the homology of $\mathbb{X}(E_0, E_1)$.

5. The homotopy

We produce a splitting homotopy S for the resolutions $\mathbb{X}(E_0, E_1)$ of the skew hooks. This proves the exactness of these resolutions.

Definition 5.1. Given $E_0 \subseteq E_1 \subseteq \{1, \ldots, n-1\}$, we will define a $\mathbb{Z}$-linear map S: $\mathbb{X}(E_0, E_1) \to \mathbb{X}(E_0, E_1)$ by giving its action on the basis elements $\bar{b}_E(\underline{w})$ for all $E_0 \subseteq E \subseteq E_1$.

Given a basis element $\bar{b}_E(\underline{w})$, let D be the set of weak descents in $\underline{w}$, specifically $D = \{1 \leqslant i \leqslant n-1 \mid w_i \geqslant w_{i+1}\}$. If $(D \cap E_1)\backslash E_0 \neq \emptyset$, then let v be the minimal element of $(D \cap E_1)\backslash E_0$. We define S by

$$S\bigl(\bar{b}_E(\underline{w})\bigr) = \begin{cases} (-1)^{r_{E\backslash\{v\},v}} b_{E\backslash\{v\}}(\underline{w}) & \text{if } v \text{ exists and } v \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Example. We display the action of the homotopy (on basis elements) for the resolution $\mathbb{X}(\emptyset, \{1,3\})$ of the Weyl module $K_{(3,3,2)/(2,1,0)}$:

$$\bigl|[w_5w_4|w_3w_2|w_1]\bigr| \mapsto \bigl|[w_5w_4|w_3w_2w_1]\bigr| \mapsto 0,$$

$$\bigl|[w_5w_4w_3w_2|w_1]\bigr| \mapsto -\bigl|[w_5w_4w_3w_2w_1]\bigr| \mapsto 0,$$

$$\left|\left[\begin{array}{ll} & w_3w_2|w_1 \\ w_5w_4 & \end{array}\right]\right| \mapsto \left|\left[\begin{array}{ll} & w_3w_2w_1 \\ w_5w_4 & \end{array}\right]\right| \mapsto 0,$$

$$\left|\left[\begin{array}{ll} & w_1 \\ w_5w_4|w_3w_2 & \end{array}\right]\right| \mapsto -\left|\left[\begin{array}{ll} & w_1 \\ w_5w_4w_3w_2 & \end{array}\right]\right| \mapsto 0,$$

$$\left|\left[\begin{array}{lll} & & w_1 \\ & w_3w_2 & \\ w_5w_4 & & \end{array}\right]\right| \mapsto 0.$$

Necessarily, we assume that in each decorated tableau above, there is a weak descent $w_i \geqslant w_{i+1}$ iff we have written w_i and w_{i+1} in the same row.

Theorem 5.1. *Given $E_0 \subseteq E \subset E_1 \subseteq \{1,\ldots,n-1\}$ and $x \in X_E$, we have $S\partial(x) + \partial S(x) = x$.*

Proof. It suffices to prove the proposition when $x = \bar{b}_E(\underline{w})$ for any $\underline{w}$ with strict ascent set $\overline{D} \subseteq E$.

Case 1. Suppose v exists and $v \notin E$, so $S(\bar{b}_E(\underline{w})) = 0$. We compute

$$\partial\bigl(\bar{b}_E(\underline{w})\bigr) = \sum_{u \in E_1 \backslash E} (-1)^{r_{E,u}} \bar{b}_{E\cup\{u\}}(\underline{w}) \tag{16}$$

and observe that since the v appearing in the definition of $S(\bar{b}_F(\underline{w}))$ depends only on $\underline{w}$ and not on F, S must vanish on every term of the right-hand side except $(-1)^{r_{E,v}} \bar{b}_{E\cup\{v\}}(\underline{w})$.

Case 2. Suppose v exists and $v \in E$. Then $S(\bar{b}_E(\underline{w})) = (-1)^{r_{E\backslash\{v\},v}} \bar{b}_{E\backslash\{v\}}(\underline{w})$. In this case, the homotopy is nonzero on all terms on the right-hand side of Eq. (16), and we find

$$S\partial\bigl(\bar{b}_E(\underline{w})\bigr) = \sum_{u \in E_1 \backslash E} (-1)^{r_{E,u}} (-1)^{r_{(E\cup\{u\})\backslash\{v\},v}} \bar{b}_{(E\cup\{u\})\backslash\{v\}}(\underline{w}). \tag{17}$$

On the other hand, we calculate $S(\bar{b}_E(\underline{w})) = (-1)^{r_{E\backslash\{v\},v}} b_{E\backslash\{v\}}(\underline{w})$, so

$$\partial S\big(\bar{b}_E(\underline{w})\big) = \sum_{u\in(E_1\backslash E)\cup\{v\}} (-1)^{r_{E\backslash\{v\},v}}(-1)^{r_{E\backslash\{v\},u}} b_{(E\backslash\{v\})\cup\{u\}}(\underline{w}). \tag{18}$$

The basis element $b_E(\underline{w})$ appears in Eq. (18) with coefficient 1 when $u = v$ and is absent from Eq. (17). The coefficient of any other basis element $b_{(E\backslash\{v\})\cup\{u\}}$ that could appear in $S\partial(b_E(\underline{w})) + \partial S(b_E(\underline{w}))$ vanishes for $u \neq v$. Letting $F = E\backslash\{v\}$, this amounts to applying the definition of $r_{F,u}$ to check that

$$(-1)^{r_{F,v}}(-1)^{r_{F,u}} + (-1)^{r_{F\cup\{v\},u}}(-1)^{r_{F\cup\{u\},v}} = 0.$$

Case 3. Finally, suppose $D \cap E_1 \subseteq E_0$ or, equivalently, $E_1 \subseteq E_0 \cup \overline{D}$. Then we have $S(x) = 0$. But $E_1 \supseteq E \supseteq E_0$ and $E \supseteq \overline{D}$, so $E = E_1$, where we are permitted homology. □

We note that Definition 5.1 still leads to a homotopy if instead of choosing v to be *minimal* in $(D\cap E_1)\backslash E_0$, we choose it to be the earliest element of $(D\cap E_1)\backslash E_0$ appearing in some pre-determined permutation, σ, of $\{1,\ldots,n-1\}$. It is an interesting exercise to show directly that the result of averaging (over the symmetric group on $\{1,\ldots,n-1\}$) all such homotopies is itself a homotopy, albeit no longer a characteristic-free one.

Proof of Theorem 1.2. It suffices to prove that homology of $\mathbb{X}(E_0, E_1)$ is as claimed. We do this by augmenting the boundary with a map $\varepsilon : X_{E_1} \to K_{\lambda/\mu}$, defining S on $K_{\lambda/\mu}$, and proving that $S\varepsilon(x) + \partial S(x) = x$ for all $x \in X_{E_1}$.

We define ε by $\varepsilon(b_{E_1}(\underline{w})) = [T]$ where T is the tableau of shape λ/μ with row-reading word $\underline{w}$. This map is well-known to be a homomorphism of $U(gl_{\mathcal{L}})$-modules. One can verify this using superalgebra techniques by constructing ε as a product of negatively signed place polarizations. Informally, we can describe (up to sign) the action of ε as replacing each subsequence $(\boldsymbol{i_j}+\mathbf{1})^{(i_{j+1}-i_j)}$ on the right-hand side of the bitableaux in either expression (13) or (14) with the sequence $\lambda_j^-,\ldots,(\mu_j+2)^-,(\mu_j+1)^-$.

Let $\bar{b}_{E_1}(\underline{w})$ be a basis element of X_{E_1}, let $\overline{D} \subseteq E_1$ be the set of strict ascents of $\underline{w}$, and D its set of weak descents. Following Case (2) of the preceding proof, if there exists a minimal element v of $(D \cap E_1)\backslash E_0$ and if $\varepsilon(\bar{b}_{E_1}(\underline{w})) = 0$, then $S\varepsilon(\bar{b}_{E_1}(\underline{w})) + \partial S(\bar{b}_{E_1}(\underline{w})) = \bar{b}_{E_1}(\underline{w})$. But the existence of v guarantees that we can write $\bar{b}_{E_1}(\underline{w})$ in the following form:

$$\bar{b}_{E_1}(\underline{w}) = \left(\begin{array}{c|c} \cdots & \cdots \\ \cdots & \cdots(\boldsymbol{v}+\mathbf{1})^{(b)}(\boldsymbol{v}-\boldsymbol{a}+\mathbf{1})^{(a)}\cdots \\ \cdots & \cdots \end{array}\right)$$

for some $a, b \geqslant 1$. Thus, choosing m such that $v = i_m$, $\varepsilon\bar{b}_{E_1}(\underline{w})$ has two copies of λ_{m+1}^- in the displayed row. Since $\lambda_{m+1}^-\lambda_{m+1}^- = 0$, we indeed have $\varepsilon(\bar{b}_{E_1}(\underline{w})) = 0$.

Suppose however that $(D \cap E_1)\backslash E_0$ is empty. Then we are in Case 3 of the preceding proof and so $E_1\backslash E_0 \subseteq \overline{D} \subseteq E_1$. But this is precisely the condition required to make $\varepsilon(\bar{b}_{E_1}(\underline{w}))$ the basis element $[T]$ corresponding to the semi-standard Young tableau T

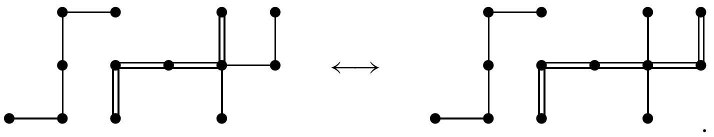
of shape λ/μ with row-reading word $\underline{w}$. We complete the proof by defining $S([T]) = \bar{b}_{E_1}(\underline{w})$. $\square$

Remark. The above homotopy establishes a complete (directed) matching among the basis elements in $\mathbb{X}(E_0, E_1)$ that do not appear in $S\varepsilon(E_1)$. Since matched elements appear in adjacent stages of the resolution, their weights cancel when we compute the formal character of the homology at X_{E_1} as an alternating sum of the formal characters of the stages of the resolution. In fact, on the level of formal characters, our homotopy is the Gessel–Viennot sign reversing involution [GV85]. To see this, consider the following bijection between basis elements and collections of lattice paths. Define the height of the sth horizontal step (reading from the right) of the jth lattice path from the right to be $w_{i_{j-1}+s}$.

Example. In our previous example, the homotopy acted (in part) by

$$\left\|\begin{bmatrix} & \mathbf{22|2} \\ \mathbf{13} & \end{bmatrix}\right\| \mapsto \left\|\begin{bmatrix} & \mathbf{222} \\ \mathbf{13} & \end{bmatrix}\right\|$$

which corresponds to the sign-reversing involution



6. Extensions

The results of this paper hold if $\mathbb{Z}$ is replaced by any commutative ring $\mathfrak{r}$ with unity via minor modifications of the above constructions and proofs. Additionally, the resolution and homotopy presented above extend to the supersymmetric generalization of Schur and Weyl modules.

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