HOMOTOPY EQUIVALENCE OF TWO FAMILIES OF COMPLEXES

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1. Introduction

Many years ago, in [2], a family of complexes was introduced which were to be considered generalizations of the usual Koszul complex, and in [3] they were studied in some detail, especially in relation to the generalized Cohen-Macaulay Theorem and a generalized multiplicity notion (see [8] and [9]). Given a commutative ring R, the idea was to take a map $f : R^m \to R^n$ ($m \geq n$) and, for each integer k, with $1 \leq k \leq n$, to associate a complex related to the map $\Lambda^k f : \Lambda^k R^m \to \Lambda^k R^n$ (we will denote it by $\mathbf{C}(k; f)$ in this article). At about the same time, another—and much more efficient—complex was developed by Eagon and Northcott [6] which was associated to the map $\Lambda^n f$. A number of years later, Eisenbud and one of the authors ([5]) constructed a large family of complexes which were associated to the maps $L_{(k,1^q)}(f) : L_{(k,1^q)}R^m \to L_{(k,1^q)}R^n$ induced on these hooks from the map f. In particular, for $q = 0$, complexes were associated to $\Lambda^k f$ for all $1 \leq k \leq n$ (which we will denote by $\mathbf{T}(k; f)$ in this article), and for $k = n$, the Eagon-Northcott complex mentioned above was reobtained. As was the case of the Eagon-Northcott complex, the complexes in [5] were much slimmer than the corresponding complexes constructed earlier in [2].[1] Although a connection between these complexes was never doubted, none of the authors involved in the construction of these complexes ever established a connection between the more unwieldy—or fatter—ones and the slimmer ones (except for the case $k = n$ which was treated, albeit in a very awkward way, in [4]).

Another gap in the literature has to do with the fact that in [2] it was stated that for every maximal minor μ of f, the complex $\mathbf{C}(n; f)$ carried a homotopy which made multiplication by μ homotopic to zero. In [3] an 'acyclicity-type' proof of the grade-sensitivity of the complexes $\mathbf{C}(k; f)$ was given, which did not involve the use of homotopies, so that the cited homotopy was never written down in either of those papers, and no such homotopy was ever given for these complexes (although explicit use of it was made in [7]).

In this paper, we propose to fill in some of those gaps. In Section 2, we will show that for each maximal minor μ of the map f there is a homotopy on $\mathbf{C}(k; f)$ establishing that multiplication by μ is homotopic to zero (we will occasionally refer to this as the 'homothety homotopy'). In Section 3, we will show the existence of maps $\alpha : \mathbf{T}(k; f) \to \mathbf{C}(k; f)$ and $\theta : \mathbf{C}(k; f) \to \mathbf{T}(k; f)$ such that $\theta\alpha$ is the identity, that is, we will show that the complex $\mathbf{T}(k; f)$ is a summand of $\mathbf{C}(k; f)$. Using this fact, we will show that the homothety homotopy on the latter complex

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[1]This explains the notation $\mathbf{C}(k; f)$ and $\mathbf{T}(k; f)$: the $\mathbf{C}$ stands for 'corpulent', while the $\mathbf{T}$ stands for 'thin'.

can be transported to its summand. Finally, in Section 4 we will show that the composition $\alpha\theta$ is homotopic to the identity. Thus, the fact that the complexes $\mathbf{T}(k;f)$ and $\mathbf{C}(k;f)$ are homotopically equivalent will finally be established.

Throughout the paper, we will use notation that has become more or less standard in this context. We will let $F = R^m$ and $G = R^n$ and note that ΛG^* operates on ΛF (and ΛF on ΛG^*) through f^*. For $\beta \in \Lambda G^*$ and $y \in \Lambda F$, we denote the action of β on y by $\beta(y)$. If we diagonalize an element x, we will generally write its diagonalization as $\sum x_i \otimes x_i^{'}$. If x has degree d, and we want to specify its diagonalization in a fixed bidegree $(l, d-l)$, we will often write this diagonalization as $\sum x_{il} \otimes x_{id-l}^{'}$. [2]

2. The 'Homothety Homotopy'

In this section, we define the homotopy on the complex $\mathbf{C}(k;f)$. This complex is defined as follows:

$$0 \to C^k_{m-n+1} \to \cdots \to C^k_q \to \cdots \quad \to \quad \sum_{s_i \geq 1} \Lambda^{n-k+s_0} G^* \otimes \Lambda^{s_1} G^* \otimes \Lambda^{n+|s|} F \to$$
$$\sum_{s \geq 1} \Lambda^{n-k+s} G^* \otimes \Lambda^{n+s} F \quad \to \quad \Lambda^k F \to \Lambda^k G,$$

where

$$C^k_q = \sum_{s_i \geq 1} \Lambda^{n-k+s_0} G^* \otimes \Lambda^{s_1} G^* \otimes \cdots \otimes \Lambda^{s_{q-2}} G^* \otimes \Lambda^{n+|s|} F, \quad q \geq 2,$$

$|s| = \sum s_i$, and the maps (except for $\Lambda^k f : \Lambda^k F \to \Lambda^k G$) are the bar complex maps associated to the action of the algebra ΛG^* on ΛF.

As the signs of all our maps will be very crucial in all that we are about to do, we will make clear just what we mean by 'boundary map' in this context. Namely, if $a_0 \otimes a_1 \otimes \cdots \otimes a_{q-2} \otimes x \in C^k_q$ for $q \geq 3$, then

$$\begin{aligned} &\partial (a_0 \otimes a_1 \otimes \cdots \otimes a_{q-2} \otimes x) \\ &= a_0 \otimes a_1 \otimes \cdots \otimes a_{q-2}(x) \\ &\quad + \textstyle\sum_{i=0}^{q-3} (-1)^{q-i} a_0 \otimes a_1 \otimes \cdots \otimes a_i \wedge a_{i+1} \otimes \cdots \otimes a_{q-2} \otimes x. \end{aligned} \tag{2.1}$$

We take $\xi \in \Lambda^n G^*$ and $\lambda \in \Lambda^n F$, and we want to show that multiplication in the fat complex $\mathbf{C}(k;f)$ by $\mu = \xi(\lambda)$ is homotopic to zero. We define

$$\sigma_0 : \Lambda^k G \to \Lambda^k F$$

by setting

$$\sigma_0(y) = y(\xi)(\lambda).$$

Then we have

$$\partial \sigma_0(y) = \sum \pm y_i \wedge \xi(y_i^{'} \wedge \lambda) = y \wedge \xi(\lambda) = \mu y. \tag{2.2}$$

Most of the summands disappear in the above sum since the λ is now to be considered as sitting inside $\Lambda^n G$, so that multiplication with $y_i^{'}$ is zero unless degree of $y_i^{'}$

[2] Much of this could be handled more elegantly using letter-place notation. However, since our use of this notation is very limited in this paper, we will avoid introducing the more elaborate letter-place machinery.

is zero. The first equality of 2.2 is a variant of the 'measuring formula' [5], which says that, for $\beta \in \Lambda^i(G^*)$ and $x, y \in \Lambda F$,

$$\beta\,(x \wedge y) = \sum_{l=0}^{i} \pm\beta_l(x) \wedge \beta'_{i-l}(y).$$

The signs $\pm$ in this formula are explicitly given in [5].

We now define

$$\sigma_1(x) = \sum_{l<k}\sum_j x_{j,l}(\xi) \otimes x'_{j,k-l} \wedge \lambda$$

where $x \in \Lambda^k F$. Remember that the l is supposed to indicate the degree of the term, and the j indicates the summation.

In order to see that this works, we prove the following two lemmas:

Lemma 2.1. *For $x \in \Lambda^k F, \xi$ and λ as before, and for each integer l, we have*

$$\sum_j x_{j,l}(\xi)(x'_{j,k-l} \wedge \lambda) = \sum_j \sum_{t=0}^{k-l} (-1)^t \binom{l+t}{l} x'_{j,k-l-t} \wedge x_{j,l+t}(\xi)(\lambda).$$

Proof. By the measuring formula, we see that

$$\begin{aligned}
\sum_j x_{j,l}(\xi)(x'_{j,k-l} \wedge \lambda) &= \sum_j \sum_{i,t} \pm\xi_{i,t}(x'_{j,k-l}) \wedge x_{j,l}(\xi_{i,n-t})(\lambda) \\
&= \sum_j \sum_{i,t} \pm x'_{j,k-l-t} \wedge x''_{j,t}(\xi_{i,t}) x_{j,l}(\xi_{i,n-t})(\lambda) \\
&= \sum_j \sum_t \pm x'_{j,k-l-t} \wedge x_{j,l}(\sum_i x''_{j,t}(\xi_{i,t})\xi_{i,n-t})(\lambda) \\
&= \sum_j \sum_t \pm x'_{j,k-l-t} \wedge x_{j,l}(x''_{j,t}(\xi))(\lambda) \\
&= \sum_j \sum_t \pm \binom{l+t}{l} x'_{j,k-l-t} \wedge x_{j,l+t}(\xi))(\lambda).
\end{aligned}$$

∎

Lemma 2.2. *For $x \in \Lambda^k F$, and ξ and λ as before, we have*

$$\sum_j \sum_{l<k} x_{j,l}(\xi)(x'_{j,k-l} \wedge \lambda) = x \wedge \xi(\lambda) - x(\xi) \wedge \lambda.$$

Proof. As we saw above, for each l we have

$$x_{j,l}(\xi)(x'_{j,k-l} \wedge \lambda) = \sum_{t=0}^{k-l} (-1)^t \binom{l+t}{l} x'_{j,k-l-t} \wedge x_{j,l+t}(\xi)(\lambda).$$

So this says that

$$\sum_{l<k} x_{j,l}(\xi)(x'_{j,k-l} \wedge \lambda) = \sum_\beta (-1)^\beta \sum_l (-1)^l \binom{\beta}{l} x'_{j,k-\beta} \wedge x_{j,\beta}(\xi)(\lambda),$$

and the conclusion follows (because we know what happens to the alternating sum of binomial coefficients). ∎

Now it is easy to see that σ_1 gives us what we need.

The next step is the 'generic' one; that is, once we get this one, the others all are of the same type. We define

$$\sigma_2(\beta \otimes x) = \sum_{l<k-s} \sum_j \beta \otimes x_{j,n+s-k+l}(\xi) \otimes x'_{j,k-l} \wedge \lambda$$

for $\beta \in \Lambda^{n-k+s}G^*, x \in \Lambda^{n+s}F$. There are two 'tricks' to showing that this works: one is to recognize that

$$\beta(x_{j,n+s-k+l})(\xi) = \beta \wedge x_{j,n+s-k+l}(\xi)$$

because ξ is of degree n (we are applying the same formula that we applied in the first equality of 2.2). Then when we compute the boundary of $\sum_{l<k-s} \beta \otimes x_{j,n+s-k+l}(\xi) \otimes x'_{j,k-l} \wedge \lambda$, we can allow $l = k-s$, since the zero degree term in the middle cancels out in the boundary. But then, since the degree of x is greater than n (since $s \geq 1$), the term $x(\xi) = 0$, so when we apply our lemma, we see that this definition of s_2 works.

Now it is easy to see that for $q \geq 2$, we may define

$$\sigma_{q+1}(\beta_0 \otimes \cdots \otimes \beta_{q-1} \otimes x) = \sum_{l<k-|s|} \sum_j \beta_0 \otimes \cdots \otimes \beta_{q-1} \otimes x_{j,n+|s|-k+l}(\xi) \otimes x'_{j,k-l} \wedge \lambda$$

for $\beta_0 \in \Lambda^{n-k+s_0}G^*, \beta_i \in \Lambda^{s_i}G^*$, for $i \geq 1$, and $x \in \Lambda^{n+|s|}F$ ($|s| = \sum_{i\geq 0} s_i$).

3. The Maps α and θ

The maps we want to define are between the complexes $\mathbf{C}(k;f)$ and $\mathbf{T}(k;f)$. The first complex has been described in the section above; we will define the complex $\mathbf{T}(k;f)$ here:

$$\begin{aligned} 0 \to T^k_{m-n+1} \to \cdots \to T^k_q \to \cdots \quad &\to \quad K_{(2,1^{n-k})}G^* \otimes \Lambda^{n+2}F \to \\ \Lambda^{n-k+1}G^* \otimes \Lambda^{n+1}F \quad &\to \quad \Lambda^k F \to \Lambda^k G, \end{aligned}$$

where

$$T^k_q = K_{(q-1,1^{n-k})}G^* \otimes \Lambda^{n+q-1}F \quad q \geq 2,$$

and $K_{(q-1,1^{n-k})}G^*$ denotes the Weyl module associated to the hook partition $(q-1,1^{n-k})$. (Recall that the Weyl—or coSchur—module $K_{(l,1^m)}$ is defined as the image of the map $\Lambda^m \otimes D_l \to \Lambda^{m+1} \otimes D_{l-1}$, where D stands for the divided power. It is an instance of a more general module: cf. e.g. [1].) We should remark that $K_{(1,1^{n-k})}G^* = \Lambda^{n-k+1}G^*$ (and $K_{(0,1^{n-k})}G^* = 0$). The map $\Lambda^{n-k+1}G^* \otimes \Lambda^{n+1}F \to \Lambda^k F$ is the usual action of the ΛG^* on ΛF, and the other maps $K_{(q-1,1^{n-k})}G^* \otimes \Lambda^{n+q-1}F \to K_{(q-2,1^{n-k})}G^* \otimes \Lambda^{n+q-2}F$ are defined as follows: the modules $K_{(q-1,1^{n-k})}G^* \otimes \Lambda^{n+q-1}F$ may be regarded as submodules of $\Lambda^{n-k+1}G^* \otimes D_{q-2}G^* \otimes \Lambda^{n+q-1}F$. By diagonalizing $D_{q-2}G^*$ into $D_{q-3}G^* \otimes D_1 G^*$, and then acting by G^* on ΛF, we see that we have a map

$$\Lambda^{n-k+1}G^* \otimes D_{q-2}G^* \otimes \Lambda^{n+q-1}F \to \Lambda^{n-k+1}G^* \otimes D_{q-3}G^* \otimes \Lambda^{n+q-2}F.$$

It is easy to see that this map, restricted to T^k_q, carries it into T^k_{q-1} (see [5]).

3.1. **The map α.** We may use the observation above to good advantage in defining the map $\alpha : \mathbf{T}(k;f) \to \mathbf{C}(k;f)$. In dimensions 0 and 1, of course, the map is the identity. For ease of notation, we will label these maps α_{-2} and α_{-1} respectively and, in general, we will denote by α_q the map that takes T^k_{q+2} to C^k_{q+2}.

Definition of the map α: For $q = -2$ and -1 we define α_q to be the identity. For $q \geq 0$, we define α_q as the composition

$$\begin{aligned} T^k_{q+2} \quad \hookrightarrow \quad & \Lambda^{n-k+1}G^* \otimes D_qG^* \otimes \Lambda^{n+q+1}F \to \\ & \Lambda^{n-k+1}G^* \otimes \underbrace{D_1G^* \otimes \cdots \otimes D_1G^*}_{q} \otimes \Lambda^{n+q+1}F, \end{aligned}$$

where the right arrow is the q-fold diagonalization of D_qG^*. (We observe that the latter term is a summand of C^k_{q+2}.)

It is a relatively straightforward calculation to see that α, thus defined, is a map of complexes.

3.2. **The map θ.** The map $\theta : \mathbf{C}(k;f) \to \mathbf{T}(k;f)$ is a bit more complicated to define (except in dimensions 0 and 1 where, again, it is defined to be the identity and denoted by θ_{-2} and θ_{-1}). As in the case of the definition of the map α, we will denote by θ_q the map that sends C^k_{q+2} to T^k_{q+2}. We introduce some notation to facilitate its definition.

Notation: Assume that a fixed basis of G^* is given, say $y_1, \ldots, y_n$. An element $y_{j_1} \wedge \cdots \wedge y_{j_l}$ will be written either as $j_1 \wedge \cdots \wedge j_l$ or $j_1 \cdots j_l$ or simply as J. In short, the index on a basis element will be used to denote that element (as is the practice when working with tableaux), and products of elements will be denoted by products of their indices. For $l = 1$, we will usually write j or j_1 instead of J. When working with products in the divided power algebra, we will use tableau notation in order to avoid confusion about whether juxtaposition means the usual product within that algebra, or the divided power when there are repeats. For example, we will write, for $y^{(2)}_{u_1} y_{u_2}$, with $u_1 < u_2$, the tableau $\boxed{u_1 \mid u_1 \mid u_2}$. We will use freely the standard basis, consisting of standard tableaux, for Weyl modules, and a typical basis element of T^k_{q+2} would be denoted by

$$\begin{array}{|c|c|c|c|} \hline j_1 & u_1 & \cdots & u_q \\ \hline j_2 \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+1} \\ \cline{1-1} \end{array} \otimes x$$

where x is an element of $\Lambda^{n+q+1}F$, and $\begin{array}{|c|c|c|c|} \hline j_1 & u_1 & \cdots & u_q \\ \hline j_2 \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+1} \\ \cline{1-1} \end{array}$ is a standard tableau which stands for the image of the element $j_2 \cdots j_{n-k+1} \otimes \boxed{j_1 \mid u_1 \mid \cdots \mid u_q} \in \Lambda^{n-k}G^* \otimes D_{q+1}G^*$ in $\Lambda^{n-k+1}G^* \otimes D_qG^*$ under the map which diagonalizes $D_{q+1}G^* \to G^* \otimes D_qG^*$, and then multiplies $\Lambda^{n-k}G^*$ into $\Lambda^{n-k+1}G^*$ by using the G^* factor.

Recall that 'standard tableau' means that the indices are strictly increasing in the column, and weakly increasing in the row. For reasons that will become apparent later, we will make one more (unusual) convention about our use of tableau notation in the case of rows: *we will assume that the tableau is zero if the top row is not weakly increasing as written.* Thus, in the case of $y_{u_1}^{(2)} y_{u_2}$ with $u_1 > u_2$, we would have to write $\begin{array}{|c|c|c|}\hline u_2 & u_1 & u_1 \\ \hline \end{array}$ to represent it as a tableau.

We are now in a position to define our maps θ_q for all $q \geq -2$.

Definition of the map θ: For $q = -2$ and -1, we have already said that the map is to be the identity. For $q = 0$, and $Y = J \otimes x$, with J a basis element of $\Lambda^{n-k+s} G^*$ and $x \in \Lambda^{n+s} F$, we define

$$\theta_0(Y) = (-1)^{(s-1)(n-k+1)} j_s \cdots j_{n-k+s} \otimes j_1 \cdots j_{s-1}(x).$$

For $q > 0$, and $Y = J \otimes u_1 \otimes \cdots \otimes u_q \otimes x$ with all the u_i basis elements of degree one, and J still of degree $n - k + s$, we define

$$\theta_q(Y) = (-1)^{(s-1)(n-k+1)+(s-1)q} \begin{array}{|c|c|c|c|}\hline j_s & u_q & \cdots & u_1 \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1} \end{array} \otimes j_1 \cdots j_{s-1}(x).$$

It is essential to remember here that the tableau is to be read as equal to zero if the row is not standard. Assume that the map θ_l has been defined on elements $Y' = J \otimes U_1 \otimes \cdots \otimes U_l \otimes x$ for U_i basis elements of arbitrary degree, and $l < q$, and that θ_q has been defined on elements $Z = J \otimes U_1 \otimes \cdots \otimes U_q \otimes x$ with $U_1, \ldots, U_t$ basis elements of arbitrary degree (we make the convention that $U_0 = J$), $U_{t+2}, \ldots, U_q$ basis elements of degree 1, and U_{t+1} basis element of degree $s_{t+1} \leq r$ $(r > 0)$. We now let $Z' = J \otimes U_1 \otimes \cdots \otimes U_{t+1} \otimes \cdots \otimes U_q \otimes x$ with the basis element $U_{t+1} = v \wedge W$, v of degree 1, all of the basis elements $U_{t+2}, \ldots, U_q$ are of degree 1, and degree$(W) = r$. Define

$$\theta_q(Z') = \theta_q \left(B + (-1)^{(q-t-1)r} E \right)$$

where $B = J \otimes U_1 \otimes \cdots \otimes U_t v \otimes W \otimes U_{t+2} \otimes \cdots \otimes U_q \otimes x$, and $E = J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes U_{t+2} \otimes \cdots \otimes U_q \otimes W(x)$.

Notice that in position $t + 1$, the elements B and E are of degree less than or equal to r, while the terms of higher index are not affected in degree. From this we see immediately (using a simple induction proof) that $\theta_q(Z') = 0$ unless $U_{t+2} \geq \cdots \geq U_q$.

While it is trivial to show that the map α is a map of complexes, it is not trivial to prove that the same is true of θ. In order to prove this fact, we must first prove the following two lemmas.

Lemma 3.1. *Let $A = J \otimes U_1 \otimes \cdots \otimes U_q \otimes x$ be a 'basis' element of $\Lambda^{n-k+s_0} G^* \otimes \Lambda^{s_1} G^* \otimes \cdots \otimes \Lambda^{s_q} G^* \otimes \Lambda^{n+|s|} F$, where by 'basis' element we mean that J and all the U_i are basis elements of their corresponding exterior powers (the element x need not be a basis element). Fix an index t with $0 \leq t \leq q-1$, and suppose that $U_{t+1} = v \wedge W$ with v of degree 1 and W of degree r, and that $s_{t+2} = s_{t+3} = \cdots = s_q = 1$. We let $\rho = q - t - 1$, we set $B = J \otimes U_1 \otimes \cdots \otimes U_t v \otimes W \otimes U_{t+2} \otimes \cdots \otimes U_q \otimes x$ and set $E = J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes U_{t+2} \otimes \cdots \otimes U_q \otimes W(x)$. Then*

$$\theta_{q-1} \partial_C A = \theta_{q-1} \partial_C \left\{ B + (-1)^{\rho r} E \right\}.$$

Proof. When $q = 1$, this property is clearly satisfied, so we may assume that $q \geq 2$, and that this all holds for $q-1$. With this inductive assumption, the case $t = q-1$ is easy to prove, so we may assume that $t < q-1$. Since all the work in this lemma involves the application of θ_{q-1}, we will denote it by θ. The proof proceeds by meticulously calculating and keeping track of signs. We will use lower case letters u_i for U_i when $s_i = 1$.

$$\begin{aligned}
\partial_C A \;=\;& J \otimes U_1 \otimes \cdots \otimes U_t \otimes vW \otimes \partial(u_{t+2} \otimes \cdots \otimes u_q \otimes x) \\
&+(-1)^{q-t-1} J \otimes U_1 \otimes \cdots \otimes U_t \otimes vW u_{t+2} \otimes \cdots \otimes u_q \otimes x \\
&+(-1)^{q-t} J \otimes U_1 \otimes \cdots \otimes U_t vW \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes x \\
&+(-1)^{q-t+1} \delta(J \otimes U_1 \otimes \cdots \otimes U_t v) \otimes W \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes x \\
&+(-1)^{q-t+1+\rho r} \delta(J \otimes U_1 \otimes \cdots \otimes U_t) \otimes v \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes W(x).
\end{aligned}$$

The use of the symbol ∂ has the meaning indicated by Formula 2.1, and involves the action of ΛG^* on ΛF, while the symbol δ denotes the boundary of a bar complex in which the action of ΛG^* on ΛF does not enter. We are also using the convention of leaving out the θ in writing all the above (and following) equalities. For example, the last two lines above are really there by virtue of the definition of this map. Now we continue to compute:

$$\begin{aligned}
\partial_C B \;=\;& J \otimes U_1 \otimes \cdots \otimes U_t v \otimes W \otimes \partial(u_{t+2} \otimes \cdots \otimes u_q \otimes x) \\
&+(-1)^{q-t-1} J \otimes U_1 \otimes \cdots \otimes U_t v \otimes W u_{t+2} \otimes \cdots \otimes u_q \otimes x \\
&+(-1)^{q-t} J \otimes U_1 \otimes \cdots \otimes U_t vW \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes x \\
&+(-1)^{q-t+1} \delta(J \otimes U_1 \otimes \cdots \otimes U_t v) \otimes W \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes x;
\end{aligned}$$

and

$$(-1)^{\rho r} \partial_C E =$$

$$(-1)^{\rho r} \left\{ \begin{array}{c}
J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes \partial(u_{t+2} \otimes \cdots \otimes u_q \otimes W(x)) \\
+(-1)^{q-t-1} J \otimes U_1 \otimes \cdots \otimes U_t \otimes v u_{t+2} \otimes \cdots \otimes u_q \otimes W(x) \\
+(-1)^{q-t} J \otimes U_1 \otimes \cdots \otimes U_t v \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes W(x) \\
+(-1)^{q-t+1} \delta(J \otimes U_1 \otimes \cdots \otimes U_t) \otimes v \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes W(x).
\end{array} \right\}$$

Making obvious cancellations, then applying θ to the term $J \otimes U_1 \otimes \cdots \otimes U_t \otimes vW \otimes \partial(u_{t+2} \otimes \cdots \otimes u_q \otimes x)$ and splitting it up judiciously (as we will be doing to similar terms later), we get further cancellations, and we are left with having to prove the following:

$$\begin{aligned}
& \theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes vW u_{t+2} \otimes \cdots \otimes u_q \otimes x\right) \\
=\;& \theta\left(J \otimes U_1 \otimes \cdots \otimes U_t v \otimes W u_{t+2} \otimes \cdots \otimes u_q \otimes x\right) \\
&+(-1)^{\rho r} \theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v u_{t+2} \otimes \cdots \otimes u_q \otimes W(x)\right) \\
&-(-1)^{\rho r} \theta\left(J \otimes U_1 \otimes \cdots \otimes U_t v \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes W(x)\right).
\end{aligned}$$

(These are simply the terms that are left after the above cancellations.)

Since our definition of θ requires that all our elements (except x) be basis elements, we have to 'rectify' our term vWu_{t+2} in order to see what results when we apply θ. First of all, we notice that if u_{t+2} is equal to v, then it is trivial to prove the desired equality (because several of the terms drop out). If $v < u_{t+2}$, it is clear that

$$\begin{aligned} & \theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes vWu_{t+2} \otimes \cdots \otimes u_q \otimes x\right) \\ = \quad & \theta\left(J \otimes U_1 \otimes \cdots \otimes U_t v \otimes Wu_{t+2} \otimes \cdots \otimes u_q \otimes x\right) \\ & +(-1)^{c(r+1)}\theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes \cdots \otimes u_q \otimes Wu_{t+2}(x)\right). \end{aligned}$$

But we also have

$$\begin{aligned} & (-1)^{\rho r}\theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes vu_{t+2} \otimes \cdots \otimes u_q \otimes W(x)\right) \\ = \quad & (-1)^{\rho r}\theta\left(J \otimes U_1 \otimes \cdots \otimes U_t v \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes W(x)\right) \\ & +(-1)^{\rho r+c}\theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes \cdots \otimes u_q \otimes u_{t+2}W(x)\right), \end{aligned}$$

so we must show that

$$\begin{aligned} & (-1)^{c(r+1)}\theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes \cdots \otimes u_q \otimes Wu_{t+2}(x)\right) \\ = \quad & (-1)^{\rho r+c}\theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes \cdots \otimes u_q \otimes u_{t+2}W(x)\right) \\ = \quad & (-1)^{\rho r+c+r}\theta\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes \cdots \otimes u_q \otimes Wu_{t+2}(x)\right). \end{aligned}$$

Noting that $\rho = c+1$, we see that $\rho r + c + r \equiv c(r+1) \bmod 2$, and our equality is true.

If $v > u_{t+2}$, we observe that $vWu_{t+2} = (-1)^{r+1}u_{t+2}vW$ and use the previous case together with some easy extra work. ∎

Our second lemma that we need for our theorem is relatively straightforward.

Lemma 3.2. *Let $A = J \otimes u_1 \otimes \cdots \otimes u_q \otimes x$ be such that all the u_i are of degree* 1. *Then $\theta_{q-1}\partial_C A = 0$ unless $u_1 \geq u_2 \geq \cdots \geq u_q$.*

Proof. For $q = 1$, there is nothing to prove. For $q = 2$, we assume that $u_1 < u_2$, and we see that

$$\theta_1\partial_C A = \theta_1\left(Ju_1 \otimes u_2 \otimes x - J \otimes u_1u_2 \otimes x + J \otimes u_1 \otimes u_2(x)\right).$$

But

$$\theta_1\left(Ju_1 \otimes u_2 \otimes x + J \otimes u_1 \otimes u_2(x)\right) = \theta_1(J \otimes u_1u_2 \otimes x),$$

and we have it for $q = 2$. To proceed to the general case, recall that if $B = J \otimes u_1 \otimes \cdots \otimes u_{q-1} \otimes x$, then $\theta_{q-1}(B) = 0$ unless $u_1 \geq u_2 \geq \cdots \geq u_{q-1}$.

Assume, then, that we have $u_l \geq u_{l+1} \geq \cdots \geq u_q$, that $u_{l-1} < u_l$, and that $l < q$. Then

$$\begin{aligned} (-1)^q\partial_C A \quad = \quad & \sum_{r=0}^{l-3}(-1)^r J \otimes \cdots \otimes u_r u_{r+1} \otimes \cdots \otimes u_{l-1} \otimes u_l \otimes \ldots \otimes u_q \otimes x \\ & +(-1)^{l-2}J \otimes \cdots \otimes u_{l-2}u_{l-1} \otimes u_l \otimes \cdots \otimes u_q \otimes x \\ & +(-1)^{l-1}J \otimes \cdots \otimes u_{l-2} \otimes u_{l-1}u_l \otimes \cdots \otimes u_q \otimes x \\ & +(-1)^{l+1}J \otimes \cdots \otimes u_{l-1} \otimes u_{l+1}u_l \otimes \cdots \otimes u_q \otimes x \\ & +(-1)^l \sum_{r=1}^{q-l-1}(-1)^{r+1}J \otimes \cdots \otimes u_{l-1} \otimes \cdots \otimes u_{l+r+1}u_{l+r} \otimes \cdots \otimes u_q \otimes x \\ & +(-1)^q J \otimes u_1 \otimes \cdots \otimes u_{q-1} \otimes u_q(x). \end{aligned}$$

Now all of the terms $J \otimes \cdots \otimes u_r u_{r+1} \otimes \cdots \otimes u_{l-1} \otimes u_l \otimes \ldots \otimes u_q \otimes x$ go to zero under θ_{q-1} because our defining identities all take place to the left of u_{l-1} and thus give us linear terms all containing $u_{l-1} \otimes u_l$ which, by the property of θ_{q-1}, all go to zero. The reduction of the element in the third line cancels the term in the second line, and leaves the term

$$\begin{aligned} & (-1)^{l-1+q-l} J \otimes \cdots \otimes u_{l-2} \otimes u_{l-1} \otimes u_{l+1} \cdots \otimes u_q \otimes u_l(x) \\ = \quad & (-1)^{q-1} J \otimes \cdots \otimes u_{l-2} \otimes u_{l-1} \otimes u_{l+1} \cdots \otimes u_q \otimes u_l(x). \end{aligned}$$

The fourth line reduces to

$$\begin{aligned} & (-1)^{l+1} J \otimes \cdots \otimes u_{l-1} u_{l+1} \otimes u_l \otimes \cdots \otimes u_q \otimes x \\ & + (-1)^{l+1+q-l-1} J \otimes \cdots \otimes u_{l-1} \otimes u_{l+1} \otimes \cdots \otimes u_q \otimes u_l(x). \end{aligned}$$

The second term has opposite sign to the identical term above, so they cancel, and we are left with the term in the top line above. But u_{l-1} and u_{l+1} are both less than u_l, and so in the defining identities, the final linearization is going to contain either u_{l-1} or u_{l+1} to the left of u_l and therefore is sent to zero by θ_{q-1}. Thus we are left with the terms

$$\begin{aligned} & (-1)^l \sum_{r=1}^{q-l-1} (-1)^{r+1} J \otimes \cdots \otimes u_{l-1} \otimes u_l \otimes \cdots \otimes u_{l+r+1} u_{l+r} \otimes \cdots \otimes u_q \otimes x \\ & + (-1)^q J \otimes u_1 \otimes \cdots \otimes u_{q-1} \otimes u_q(x). \end{aligned}$$

The terms in the sum, because we assume the indices decrease beyond l, all reduce to linear terms containing $J \otimes \cdots \otimes u_{l-1} \otimes u_l \otimes \cdots$ which are sent by θ_{q-1} to zero. Reducing this term further gives us

$$\begin{aligned} & \pm J \otimes \cdots \otimes u_{l-1} u_{l+r+1} \otimes u_l \otimes u_{l+1} \otimes \cdots \otimes u_{l+r} \otimes \cdots \otimes u_q \otimes x \\ & \pm\ J \otimes \cdots \otimes u_{l-1} \otimes u_{l+r+1} \otimes u_{l+1} \otimes \cdots \otimes u_{l+r} \otimes \cdots \otimes u_q \otimes u_l(x). \end{aligned}$$

Now both of these terms are sent to zero under θ_{q-1}; the first one because further linearizations of the quadratic will reduce to linear terms having either u_{l-1} or u_{l+r+1} to the right of u_l and the second one because, with $r \geq 1$, we have $u_{l+r+1} < u_{l+1}$, and this violation of order gets it sent to zero.

The term $J \otimes u_1 \otimes \cdots \otimes u_{q-1} \otimes u_q(x)$ gets sent to zero, since $l < q$.

We now consider the case $l = q$, which has been left out. In that event, the proof proceeds pretty much as above, with cancellation occurring among the very last terms, and the other terms equalling zero because of the occurrence of $u_{q-1} \otimes u_q$. ∎

With these two lemmas, we are ready to prove the following theorem.

Theorem 3.3. *The map θ defined above is a map of complexes. That is, if we let ∂_C and ∂_T be the boundary maps of $\mathbf{C}(k; f)$ and $\mathbf{T}(k; f)$ respectively, then we have*

$$\partial_T \theta_q = \theta_{q-1} \partial_C$$

for all $q \geq -1$.

Proof. For $q = -1$, this is clear; the first place we must look is at $q = 0$, i.e., we must see that $\partial_T \theta_0 = \partial_C$. But

$$\partial_T \theta_0 (J \otimes x) = (-1)^{(s-1)(n-k+1)} j_s \cdots j_{n-k+s} \left(j_1 \cdots j_{s-1}(x) \right) = J(x),$$

and this last is clearly $\partial_C(J \otimes x)$.

To proceed to the general case, we use the two lemmas above; if we let $Y = J \otimes U_1 \otimes \cdots \otimes U_q \otimes x$, we observe that Lemmas 3.1 and 3.2 permit us to reduce to the case where all the U_i are of degree one (hence we will use lower case u to denote them), and $u_1 \geq \cdots \geq u_q$. In this case, we assume that J is of degree $n-k+s$, and we must prove that

$$(-1)^{(s-1)(n-k+1)+(s-1)q}\partial_T \left(\begin{array}{|c|c|c|c|} \hline j_s & u_q & \cdots & u_1 \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1} \end{array} \otimes j_1 \cdots j_{s-1}(x) \right) = \theta_{q-1}\partial_C(Y).$$

At this point, we must introduce some additional notation. To take account of the fact that not all the u_i need be distinct, we will write

$$Y = J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x$$

with $u_1 > \cdots > u_p$.

Applying Lemma 3.1 (which we will refer to as the *reduction formulas*), as well as Lemma 3.2, it is straightforward to see that

$$\begin{aligned} \theta_{q-1}\partial_C(Y) \quad &= \quad (-1)^q \sum_{i=1}^{p} \theta_{q-1}\left(Ju_i \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) \\ &\quad + \sum_{i=1}^{p} \theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes u_i(x)\right). \end{aligned}$$

To briefly sketch a proof of this fact, we first observe that

$$\begin{aligned} \partial_C(Y) \quad = \quad & \sum_{i=2}^{p} (-1)^{m_p + \cdots + m_i} J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1} - 1} \otimes u_{i-1}u_i \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x \\ & + J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p - 1} \otimes u_p(x) \\ & + (-1)^q J u_1 \otimes u_1^{\otimes m_1 - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x, \end{aligned}$$

and we want to evaluate θ_{q-1} on each of the terms in the sum for $i = 2, \ldots, p$. Using our reduction formulas, as well as the facts that $u_i < u_{i-1}$ and $u_{i-1}u_i = -u_i u_{i-1}$, we have

$$\begin{aligned} & \theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1} - 1} \otimes u_{i-1}u_i \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) \\ = \quad & -\theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1} - 2} \otimes u_{i-1}u_i \otimes u_{i-1} \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) \\ & + (-1)^{m_p + \cdots + m_i}\theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1} - 1} \otimes u_i^{\otimes m_i} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes u_{i-1}(x)\right). \end{aligned}$$

If we now apply the reduction formulas to the quadratic term

$$-\theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1} - 2} \otimes u_{i-1}u_i \otimes u_{i-1} \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right),$$

we see that we get a new quadratic term plus a linear term that places u_i to the left of u_{i-1}, which θ_{q-1} then carries to zero. Thus, each of these quadratic terms produces only one summand as we continue to apply the reduction formulas, and we may stop 'reducing' when we arrive at the point where u_i multiplies J. Thus we see that

$$\begin{aligned} & \theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1} - 1} \otimes u_{i-1}u_i \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) \\ = \quad & (-1)^{m_1 + \cdots + m_{i-1}}\theta_{q-1}\left(Ju_i \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) \\ & + (-1)^{m_p + \cdots + m_i}\theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1} - 1} \otimes u_i^{\otimes m_i} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes u_{i-1}(x)\right). \end{aligned}$$

Multiplying each such term by $(-1)^{m_p+\cdots+m_i}$, gives us the claimed result.

Since, by our definition (and convention), $\theta_q(Y) = 0$ unless $j_s \le u_p$, we first have to show that $\theta_{q-1}\partial_C(Y) = 0$ if $u_p < j_s$. In that case, it is clear that $\theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes u_i(x)\right) = 0$ for all $i < p$, as is also $\theta_{q-1}\left(Ju_i \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right)$. The latter is true because if $u_p < j_s$, then the $(s+1)th$ factor in Ju_i is either j_s or u_i. In either case (since $u_i > u_p$), the value of θ_{q-1} is zero. Hence we must simply evaluate

$$(-1)^q\theta_{q-1}\left(Ju_p \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p - 1} \otimes x\right) + \theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p - 1} \otimes u_p(x)\right).$$

If $m_p > 1$, then it is again clear that both of the terms above are zero. Hence, we must consider the case that $m_p = 1$. Since $u_p < j_s$, j_s is the $(s+1)th$ factor of Ju_p, so if $u_{p-1} \ge j_s$, we have

$$(-1)^q\theta_{q-1}\left(Ju_p \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{p-1}^{\otimes m_{p-1}} \otimes x\right) = (-1)^{q+s(n-k+1)+s(q-1)+n-k+s} \begin{array}{|c|c|c|c|}\hline j_s & u_{p-1} & \cdots & u_1 \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1}\end{array} \otimes u_p j_1 \cdots j_{s-1}(x)$$

while

$$\theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p - 1} \otimes u_p(x)\right) = (-1)^{(s-1)(n-k+1)+(s-1)(q-1)+s-1} \begin{array}{|c|c|c|c|}\hline j_s & u_{p-1} & \cdots & u_1 \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1}\end{array} \otimes u_p j_1 \cdots j_{s-1}(x).$$

These have opposite sign, so the sum is zero. Of course, if $u_{p-1} < j_s$, each of the terms above is zero, and we have the result in the case that $u_p < j_s$.

Now assume that $u_p \ge j_s$. We will assume the strict inequality; equality requires a separate, but easy, discussion. In this case, we have

$$\begin{aligned}\partial_T\theta_q(Y) &= (-1)^{(s-1)(n-k+1)+(s-1)q}\partial_T\left(\begin{array}{|c|c|c|c|}\hline j_s & u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1}\end{array} \otimes j_1 \cdots j_{s-1}(x)\right) \\ &= (-1)^{(s-1)(n-k+1)+(s-1)q}\sum_i \begin{array}{|c|c|c|c|c|c|}\hline j_s & u_p^{m_p} & \cdots & u_i^{m_i-1} & \cdots & u_1^{m_1} \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1}\end{array} \otimes u_i j_1 \cdots j_{s-1}(x) \\ &\quad + (-1)^{(s-1)(n-k+1)+(s-1)q+s-1} \begin{array}{|c|c|c|c|}\hline u_p & u_p^{m_p-1} & \cdots & u_1^{m_1} \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1}\end{array} \otimes j_1 \cdots j_{s-1} j_s(x).\end{aligned}$$

But we have

$$\theta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes u_i(x)\right) =$$

$$(-1)^{(s-1)(n-k+1)+(s-1)(q-1)+s-1} \begin{array}{|c|c|c|c|c|c|} \hline j_s & u_p^{m_p} & \cdots & u_i^{m_i-1} & \cdots & u_1^{m_1} \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1} \end{array} \otimes u_i j_1 \cdots j_{s-1}(x),$$

which agrees (when we sum) with the first term for $\partial_T \theta_q(Y)$ and, if $u_q \geq j_{s+1}$ (in which case $u_i \geq j_{s+1}$ for all i),

$$(-1)^q \sum_i \theta_{q-1}\left(J u_i \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) =$$

$$(-1)^{q+s(n-k+1)+s(q-1)+n-k-1} \sum_i \begin{array}{|c|c|c|c|c|c|} \hline j_{s+1} & u_p^{m_p} & \cdots & u_i^{m_i-1} & \cdots & u_1^{m_1} \\ \hline u_i \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1} \end{array} \otimes j_1 \cdots j_s(x).$$

However, if $u_p \geq j_{s+1}$, then the term

$$(-1)^{(s-1)(n-k+1)+(s-1)q+s-1} \begin{array}{|c|c|c|c|} \hline u_p & u_p^{m_p-1} & \cdots & u_1^{m_1} \\ \hline j_{s+1} \\ \cline{1-1} \vdots \\ \cline{1-1} j_{n-k+s} \\ \cline{1-1} \end{array} \otimes j_1 \cdots j_{s-1} j_s(x)$$

which arises in the calculation of $\partial_T \theta_q(Y)$, is not a standard tableau: it must be straightened. When straightened, it gives precisely the sum immediately above it. If, though, we assume that $j_s < u_p < j_{s+1}$, then $\theta_{q-1}\left(J u_i \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) = 0$ for all $i \neq p$, and in that case we must simply compute $\theta_{q-1}\left(J u_p \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{p-1}^{\otimes m_{p-1}} \otimes x\right)$. This is easily seen to be equal to the term displayed immediately above, and the proof is complete. ∎

Having obtained the map θ, we are now in a position to transfer the homothety homotopy on the fat complex. That is, we have the homotopy s that takes $\mathbf{C}(k; f)$ to itself with the property that $\partial s + s\partial = \mu$, where μ is a given maximal minor of our original map. Let us denote the complex $\mathbf{C}(k; f)$ by Y, and the slim complex $\mathbf{T}(k; f)$ by X. Let us denote the boundary map on X by ∂_X and that on Y by ∂_Y. Then we have the following lemma.

Lemma 3.4. *Let X and Y be complexes, $\alpha : X \to Y$ and $\theta : Y \to X$ maps of complexes such that $\theta\alpha = id_X$. Let $s : Y \to Y$ be a homotopy on Y which makes the scalar μ homotopic to zero on Y. Then the map $\tau = \theta s \alpha : X \to X$ is a homotopy on X carrying μ to zero.*

Proof. First we have to show that $\partial_X \tau_0 = \mu$. But $\partial_X \tau_0 = \partial_X \theta_1 s_0 \alpha_0 = \theta_0 \partial_Y s_0 \alpha_0 = \theta_0 \mu \alpha_0 = \mu \theta_0 \alpha_0 = \mu$. For $i > 0$, we have to show that

$$\partial_X \tau_i + \tau_{i-1} \partial_X = \mu.$$

But here again we have

$$\begin{aligned}\partial_X \tau_i &= \partial_X \theta_{i+1} s_i \alpha_i = \theta_i \partial_Y s_i \alpha_i = \theta_i \left(\mu - s_{i-1}\partial_Y\right)\alpha_i \\ &= \mu - \theta_i s_{i-1}\partial_Y \alpha_i\end{aligned}$$

while

$$\tau_{i-1}\partial_X = \theta_i s_{i-1}\alpha_{i-1}\partial_X = \theta_i s_{i-1}\partial_Y \alpha_i,$$

and this does it. ∎

As a result of this, we now know that our slim complexes carry the desired homotopy.

4. The Homotopy Equivalence of the Complexes

In this section we prove that the maps α and θ are inverses of each other up to homotopy. That is, we know that $\theta_q \alpha_q = id$. What we want to show, then, is that there is a homotopy, η, on the fat complex, $\mathbf{C}(k; f)$, such that

$$\partial_C \eta + \eta \partial_C = 1 - \alpha\theta.$$

In order to define this homotopy, we have to introduce some auxiliary maps that are closely related to the maps α and θ defined in the previous section.

Definition of the maps $\theta_q^{'}$: For each $q \geq 0$, we define the map

$$\theta_q^{'} : \Lambda^{n-k+s_0} G^* \otimes \Lambda^{s_1} G^* \otimes \cdots \otimes \Lambda^{s_q} G^* \to K_{(q+1,1^{n-k})} G^* \otimes \Lambda^{|s|-q-1} G^*$$

as follows: if all the s_i are equal to 1 for $i > 0$, we adopt the notation of the previous section and set

$$\theta_q^{'}\left(J \otimes u_1 \otimes \cdots \otimes u_q\right) = \pm \begin{array}{|c|c|c|c|c|} \hline j_s & u_q & u_{q-1} & \cdots & u_1 \\ \hline j_{s+1} & \multicolumn{4}{c}{} \\ \cline{1-1} \vdots & \multicolumn{4}{c}{} \\ \cline{1-1} j_{n-k+s} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \otimes j_1 \cdots j_{s-1}.$$

We still adhere to the convention that we adopted in the definition of the maps θ_q : the tableau is to be read as zero if the top row is not weakly increasing. For higher degree terms, we do as we did for the maps θ_q : if $U_{t+1} = v \wedge W$ is a basis element of $\Lambda^{s_{t+1}} G^*$, and $s_{t+2} = \cdots = s_q = 1$, we set

$$\begin{aligned}&\theta_q^{'}\left(J \otimes U_1 \otimes \cdots \otimes U_{t+1} \otimes \cdots \otimes U_q\right) = \\ &\theta_q^{'}\left(J \otimes U_1 \otimes \cdots \otimes U_t v \otimes W \otimes \cdots \otimes U_q\right) \\ &\pm \theta_q^{'}\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes U_{t+2} \otimes \cdots \otimes U_q\right) W.\end{aligned}$$

The notation $\theta_q^{'}\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes U_{t+2} \otimes \cdots \otimes U_q\right) W$ is to be interpreted as follows: the element $\theta_q^{'}\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes U_{t+2} \otimes \cdots \otimes U_q\right)$ is in $K_{(q+1,1^{n-k})} G^* \otimes \Lambda^{|s|-s_{t+1}-q} G^*$ while $W \in \Lambda^{s_{t+1}-1} G^*$. Our notation indicates that we are to multiply the element of $\Lambda^{|s|-s_{t+1}-q} G^*$ by W and we end up in $K_{(q+1,1^{n-k})} G^* \otimes \Lambda^{|s|-q-1} G^*$ as we are supposed to. The signs signified by $\pm$ are equal to those for the corresponding maps θ_q.

Definition of the maps $\alpha_q^{'}$: For each $q \geq 0$ we define the map

$$\alpha_q^{'} : K_{(q+1,1^{n-k})}G^* \to \Lambda^{n-k+1}G^* \otimes \Lambda^1 G^* \otimes \cdots \otimes \Lambda^1 G^*$$

as the composition

$$K_{(q+1,1^{n-k})}G^* \hookrightarrow \Lambda^{n-k+1}G^* \otimes D_q G^* \to \Lambda^{n-k+1}G^* \otimes \underbrace{D_1 G^* \otimes \cdots \otimes D_1 G^*}_{q}.$$

As the reader can see, this is just the map α_q with the exterior power of F stripped away.

In addition to these maps, we introduce one more piece of notation. If $u_1, u_2, \ldots, u_q$ is a sequence of indices, then $\boxed{u_1 \mid u_2 \mid \cdots \mid u_q}$ denotes an element of $D_q G^*$. We denote by $\Delta\left(\boxed{u_1 \mid u_2 \mid \cdots \mid u_q}\right)$ the total diagonalization of this element in $\underbrace{D_1 G^* \otimes \cdots \otimes D_1 G^*}_{q}$. Keep in mind that, because of our earlier convention on reading tableaux, this is zero unless we have $u_1 \leq u_2 \leq \cdots \leq u_q$.

Now we are ready to define the homotopy $\eta : \mathbf{C}(k; f) \to \mathbf{C}(k; f)$.

Definition of the homotopy: For the sake of notational convenience, we start the homotopy with $\eta_{-2} : \Lambda^k G \to \Lambda^k F$, and set it equal to zero. Similarly, we set $\eta_{-1} : \Lambda^k F \to \sum_{s\geq 1} \Lambda^{n-k+s}G^* \otimes \Lambda^{n+s}F$ to be zero. We define η_0 : $\sum_{s\geq 1} \Lambda^{n-k+s}G^* \otimes \Lambda^{n+s}F \to \sum_{s_i \geq 1} \Lambda^{n-k+s_0}G^* \otimes \Lambda^{s_1}G^* \otimes \Lambda^{n+|s|}F$ as

$$\eta_0(J \otimes x) = \theta_0^{'}(J) \otimes x.$$

For an element $A = J \otimes u_1 \otimes \cdots \otimes u_q \otimes x$, with all the u_i of degree one, and J of degree $n-k+s$, we define

$$\eta_q(A) = \sum_{i=0}^{q} (-1)^i \left(\alpha_i^{'} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{q-i+2}\right)\left(\theta_i^{'} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{q-i+1}\right) \left(J \otimes \Delta\left(\boxed{u_q \mid u_{q-1} \mid \cdots \mid u_1}\right) \otimes x\right).$$

Assume that η_q has been defined on elements $Y = J \otimes U_1 \otimes \cdots \otimes U_q \otimes x$ with $U_1, \ldots, U_t$ of arbitrary degree, U_{t+1} of degree $\leq r$, and U_i of degree one for $i > t+1$. Let A be an element: $A = J \otimes U_1 \otimes \cdots \otimes U_t \otimes U_{t+1} \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes x$ with $U_1, \ldots, U_t$ basis elements of arbitrary degree, $U_{t+1} = vW$ with degree$(v) = 1$, degree$(W) = r$, and the degree of u_i equal to one for $i > t+1$. We define

$$\eta_q(A) =$$

$$\begin{cases} \eta_q(\Gamma) + (-1)^{q-t}\Big\{\sum_{i=0}^{q-t-1}(-1)^{i(r-1)} \sum_\lambda J \otimes U_1 \otimes \cdots \otimes U_t \otimes \Delta\left(\boxed{u_{\lambda_1} \mid \cdots \mid u_{\lambda_i} \mid v}\right) \\ \qquad \otimes W \otimes \Delta\left(\boxed{u_q \mid \cdots \mid \widehat{u}_{\lambda_1} \mid \cdots \mid \widehat{u}_{\lambda_i} \mid \cdots \mid u_{t+2}}\right) \otimes x\Big\} \\ \text{if } u_{t+2} \geq u_{t+3} \geq \cdots \geq u_q; \\ \\ 0 \quad \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \Gamma &= J \otimes U_1 \otimes \cdots \otimes U_t v \otimes W \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes x + \\ &\quad (-1)^{r(q-t-1)} J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes u_{t+2} \otimes \cdots \otimes u_q \otimes W(x), \end{aligned}$$

and $\lambda = (\lambda_1, \ldots, \lambda_i)$ is a strictly descending sequence of indices between $t+2$ and q.

As in the definition of the maps θ_q, we resort to this inductive definition, making heavy use of the fact that there is no upper bound on the degree of J at any point.

The fact that this does provide the desired homotopy is not immediately obvious. In fact, the rest of the section is devoted to the proof of this fact. First we will prove two lemmas which are special cases of the main theorem.

Lemma 4.1. *Let* $Y = J \otimes U_1 \otimes \cdots \otimes U_t \otimes u_{t+1} \otimes \cdots \otimes u_q \otimes x$, *with* u_i *of degree one for* $i > t$. *Then* $\eta_{q-1}\partial_C(Y) = Y$ *unless* $u_q \leq u_{q-1} \leq \cdots \leq u_{t+1}$.

Proof. The first place that we have to prove anything is for $q = 2$. In that case we have, if $Y = J \otimes u_1 \otimes u_2 \otimes x$,

$$\begin{aligned} \eta_1\partial_C(Y) &= \eta_1(Ju_1 \otimes u_2 \otimes x) \\ &\quad -\eta_1(J \otimes u_1u_2 \otimes x) + \eta_1(J \otimes u_1 \otimes u_2(x)). \end{aligned}$$

But if we assume that $u_2 > u_1$,

$$\begin{aligned} &\eta_1(J \otimes u_1u_2 \otimes x) = \eta_1(Ju_1 \otimes u_2 \otimes x) + \\ &\eta_1(J \otimes u_1 \otimes u_2(x)) - J \otimes u_1 \otimes u_2 \otimes x, \end{aligned}$$

and so for $q = 2$, we are done. Now we proceed to prove this result by induction on q.

Suppose our first 'wrong' inequality occurs in the last place, that is, suppose that $u_{q-1} < u_q$. We see immediately that we have the result, for

$$\eta_{q-1}\partial_C(Y) =$$

$$\begin{aligned} &\sum_{i=1}^{q-2}(-1)^{q-i+1}\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_{i-1}U_i \otimes \cdots \otimes u_{q-1} \otimes u_q \otimes x\right) \\ &+\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_{q-2}u_{q-1} \otimes u_q \otimes x\right) \\ &-\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_{q-2} \otimes u_{q-1}u_q \otimes x\right) \\ &+\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes u_{q-1} \otimes u_q(x)\right). \end{aligned}$$

By our definition of η_{q-1}, we see that the terms

$$\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_{i-1}U_i \otimes \cdots \otimes u_{q-1} \otimes u_q \otimes x\right)$$

are zero. On the other hand, we know that

$$\begin{aligned} &-\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_{q-2} \otimes u_{q-1}u_q \otimes x\right) \\ = \; &-\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_{q-2}u_{q-1} \otimes u_q \otimes x\right) \\ &-\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_{q-2} \otimes u_{q-1} \otimes u_q(x)\right) \\ &+J \otimes U_1 \otimes \cdots \otimes u_{q-1} \otimes u_q \otimes x. \end{aligned}$$

Thus the surviving terms above are simply Y, and we are done.

Now suppose that for some l with $t+1 < l < q$, we have $u_{l-1} < u_l$, but $u_q \leq u_{q-1} \leq \cdots \leq u_l$. As in the previous section, we will rewrite Y as follows:

$$Y = J \otimes U_1 \otimes \cdots \otimes U_{l-2} \otimes u \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x$$

with $m_1 + \cdots + m_p = q - l + 1, u < u_1$, and $u_p < \cdots < u_1$. Set $\mu_i = m_p + \cdots + m_i$; we have

$$\eta_{q-1}\partial_C\left(Y\right) =$$

$$\sum_{i=1}^{l-2}(-1)^{q-i+1}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{i-1}U_i\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)$$
$$+(-1)^{q-l+2}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)$$
$$+(-1)^{q-l+1}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}\otimes uu_1\otimes u_1^{\otimes m_1-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)$$
$$+\sum_{i=2}^{p}(-1)^{\mu_i}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_{i-1}u_i\otimes u_i^{\otimes m_i-1}\\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x\end{array}\right)$$
$$+\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{p-1}^{\otimes m_{p-1}}\otimes u_p^{\otimes m_p-1}\otimes u_p(x)\right).$$

By the same argument as above, we see that the summands from 1 to $l-2$ are all zero. Also, the last term is zero, since we are assuming now that $l<q$ (i.e., that $u_1>u_p$, or, if $p=1$, then $m_p>1$). So this leaves us with the problem of evaluating

$$(-1)^{q-l+2}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)$$
$$+(-1)^{q-l+1}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}\otimes uu_1\otimes u_1^{\otimes m_1-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)$$
$$+\sum_{i=2}^{p}(-1)^{\mu_i}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_{i-1}u_i\otimes u_i^{\otimes m_i-1}\\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x\end{array}\right).$$

Now

$$\begin{aligned} & (-1)^{q-l+1}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}\otimes uu_1\otimes u_1^{\otimes m_1-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)\\ = & (-1)^{q-l+1}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}u\otimes u_1\otimes u_1^{\otimes m_1-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)\\ & +(-1)^{q-l+1+q-l}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}\otimes u\otimes u_1^{\otimes m_1-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes u_1(x)\right)\\ & +\sum_{\rho\geq 0}\sum_{(n_p,\ldots,n_2)} J\otimes U_1\otimes\cdots\otimes U_{l-2}\otimes\Delta\left(\boxed{\begin{array}{c|c|c|c} u_p^{n_p} & \cdots & u_2^{n_2} & u\end{array}}\right)\otimes u_1\otimes\\ & \qquad \Delta\left(\boxed{\begin{array}{c|c|c|c} u_p^{m_p-n_p} & \cdots & u_2^{m_2-n_2} & u_1^{m_1-1}\end{array}}\right)\otimes x,\end{aligned}$$

where $\rho=n_2+\cdots+n_p$ and $(n_p,\ldots,n_2)$ runs over all sequences of non-negative integers such that $n_i\leq m_i$ for $i=2,\ldots,p$. [3]Consequently, we see that

$$\eta_{q-1}\partial_C(Y)=$$

$$\sum_{i=2}^{p}(-1)^{\mu_i}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_{i-1}u_i\otimes u_i^{\otimes m_i-1}\\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x\end{array}\right)$$
$$-\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_{l-2}\otimes u\otimes u_1^{\otimes m_1-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes u_1(x)\right)$$
$$+\sum_{\rho\geq 0}\sum_{(n_p,\ldots,n_2)} J\otimes U_1\otimes\cdots\otimes U_{l-2}\otimes\Delta\left(\boxed{\begin{array}{c|c|c|c} u_p^{n_p} & \cdots & u_2^{n_2} & u\end{array}}\right)\otimes u_1\otimes$$
$$\Delta\left(\boxed{\begin{array}{c|c|c|c} u_p^{m_p-n_p} & \cdots & u_2^{m_2-n_2} & u_1^{m_1-1}\end{array}}\right)\otimes x.$$

---

[3]We should explain that we use the notation $\boxed{\begin{array}{c|c|c|c} u_p^{n_p} & \cdots & u_2^{n_2} & u\end{array}}$ to indicate the element of the divided power algebra represented by this tableau. Thus the exponents mean that the element is repeated in the tableau. We could also write $u_p^{(n_p)}\cdots u_2^{(n_2)}u$ but, while this is equal to $uu_p^{(n_p)}\cdots u_2^{(n_2)}$ in the divided power algebra, the tableau $\boxed{\begin{array}{c|c|c|c} u & u_p^{n_p} & \cdots & u_2^{n_2}\end{array}}$ represents the element 0.

(Notice that since all of this action is taking place above t-level, we are never encountering anything but linear terms as we proceed. Notice too that in our last double sum, we have not put in any sign that depends upon ρ; that is because we are assuming our formula for our reductions has a sign $\rho(r-1)$ in it, and so these signs disappear when $r = 1$.)

Clearly what remains to be done is to calculate

$$\sum_{i=2}^{p}(-1)^{\mu_i}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_{i-1}u_i\otimes u_i^{\otimes m_i-1} \\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x \end{array}\right).$$

As in the proof of Theorem 3.3, we are going to use 'reduction formulas' to move our element u_i to the left until it hits up against u_1. As we do this, certain terms are going to be sent to zero under η_{q-1}. As opposed to the situation in Theorem 3.3, however, we will pick up terms involving total diagonalizations; these are the 'correction terms' that figure into the definition of our homotopy. To illustrate, we see that

$$\begin{array}{rl} & \eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_{i-1}u_i\otimes u_i^{\otimes m_i-1} \\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x \end{array}\right) \\ = & -\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_iu_{i-1}\otimes u_i^{\otimes m_i-1} \\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x \end{array}\right) \\ = & -\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-2}\otimes u_{i-1}u_i\otimes u_{i-1}\otimes u_i^{\otimes m_i-1} \\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x \end{array}\right) \\ & +(-1)^{\mu_i}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_i^{\otimes m_i} \\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes u_{i-1}(x) \end{array}\right) \\ & -(-1)^{\mu_i}\sum_{\rho_i\geq 0}\sum_{(n_p,\ldots,n_i)} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes \\ & \qquad \Delta\left(\boxed{u_p^{n_p}}\boxed{\cdots}\boxed{u_i^{n_i}}\boxed{u_i}\right)\otimes u_{i-1}\otimes\Delta\left(\boxed{u_p^{m_p-n_p}}\boxed{\cdots}\boxed{u_i^{m_i-n_i-1}}\right)\otimes x. \end{array}$$

We immediately see that the terms

$$(-1)^{\mu_i}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_i^{\otimes m_i} \\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes u_{i-1}(x) \end{array}\right)$$

are zero for $i > 2$, and that for $i = 2$, the term is zero unless $m_1 = 1$. Since this term is to be multiplied by $(-1)^{\mu_i}$, it occurs with a positive sign, and cancels the corresponding term in the original calculation of $\eta_{q-1}\partial_C(Y)$ above.

We now want to continue to eliminate the quadratic term (i.e., the one involving $\otimes u_{i-1}u_i\otimes u_{i-1}\otimes$); the other remaining term, though by no means elegant, is here to stay. But as in Theorem 3.3, when we apply our reduction process, we will get another quadratic term, then a term which is carried to zero by η_{q-1} (because we will have u_i to the left of u_{i-1}), and another correction term (which cannot be discarded). Continuing with this type of reduction, and letting Z denote $J\otimes U_1\otimes$

$\cdots \otimes U_{l-2}$, we arrive finally to the conclusion that

$$
\begin{aligned}
&(-1)^{\mu_i}\eta_{q-1}\left(\begin{array}{c} Z\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}-1}\otimes u_{i-1}u_i\otimes u_i^{\otimes m_i-1}\\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x\end{array}\right)\\
=\;&(-1)^{q-l+1}\eta_{q-1}\left(\begin{array}{c} Z\otimes uu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}}\otimes u_i^{\otimes m_i-1}\\ \otimes\ldots\otimes u_p^{\otimes m_p}\otimes x\end{array}\right)\\
&-\sum_{\lambda=1}^{i-1}\sum_{\alpha_\lambda=1}^{m_\lambda} Z\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{\lambda-1}^{\otimes m_{\lambda-1}}\otimes u_\lambda^{\otimes m_\lambda-\alpha_\lambda}\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p}&\cdots&u_i^{n_i}&u_i\\\hline\end{array}\right)\otimes\\
&\qquad u_\lambda\otimes\Delta\left(\begin{array}{|c|c|c|c|c|c|}\hline u_p^{m_p-n_p}&\cdots&u_i^{m_i-n_i-1}&u_{i-1}^{m_{i-1}}&\cdots&u_\lambda^{\alpha_\lambda-1}\\\hline\end{array}\right)\otimes x,
\end{aligned}
$$

so that we see that (making a few changes in our indices of summation in the "non-eta" terms)

$$\eta_{q-1}\partial(Y)=$$

$$
\begin{aligned}
&\sum Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p}&\cdots&u_2^{n_2}&u\\\hline\end{array}\right)\otimes u_1\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{m_p-n_p}&\cdots&u_2^{m_2-n_2}&u_1^{m_1-1}\\\hline\end{array}\right)\otimes x\\
&-\sum_{\lambda=i+1}^{p}\sum_{i=1}^{p-1}\sum_{\alpha_i=0}^{m_i-1} Z\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}}\otimes u_i^{\otimes\alpha_i}\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{\beta_p}&\cdots&u_\lambda^{\beta_\lambda+1}\\\hline\end{array}\right)\otimes\\
&\qquad u_i\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-\beta_p}&\cdots&u_\lambda^{m_\lambda-\beta_\lambda-1}&\cdots&u_i^{m_i-\alpha_i-1}\\\hline\end{array}\right)\otimes x\\
&+(-1)^{q-l+1}\sum_{i=2}^{p}\eta_{q-1}\left(Z\otimes uu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}}\otimes u_i^{\otimes m_i-1}\otimes\ldots\otimes u_p^{\otimes m_p}\otimes x\right),
\end{aligned}
$$

where $\beta_\gamma\geq 0$. Thus, what we have left is to evaluate

$$(-1)^{q-l+1}\sum_{i=2}^{p}\eta_{q-1}\left(Z\otimes uu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}}\otimes u_i^{\otimes m_i-1}\otimes\ldots\otimes u_p^{\otimes m_p}\otimes x\right),$$

and this clearly depends on the relative sizes of u and u_i for $i=2,\ldots,p$.

First, let us assume that $u<u_p<\cdots<u_1$. In that case, we see that

$$
\begin{aligned}
&(-1)^{q-l+1}\sum_{i=2}^{p}\eta_{q-1}\left(Z\otimes uu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}}\otimes u_i^{\otimes m_i-1}\otimes\ldots\otimes u_p^{\otimes m_p}\otimes x\right)\\
=\;&\sum_{i=2}^{p} Z\otimes u\otimes u_i\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p}&\cdots&u_i^{m_i-1}&\cdots&u_1^{m_1}\\\hline\end{array}\right)\otimes x.
\end{aligned}
$$

(The terms involving η disappear because either u or u_i would appear before the u_1, and that would make η vanish.) This then yields

$$
\begin{aligned}
&\eta_{q-1}\partial(Y)=Z\otimes u\otimes\sum_{i=1}^{p}u_i\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p}&\cdots&u_i^{m_i-1}&\cdots&u_1^{m_1}\\\hline\end{array}\right)\otimes x\\
&-\sum_{\lambda=i+1}^{p}\sum_{i=1}^{p-1}\sum_{\alpha_i=0}^{m_i-1} Z\otimes u\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_{i-1}^{\otimes m_{i-1}}\otimes u_i^{\otimes\alpha_i}\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{\beta_p}&\cdots&u_\lambda^{\beta_\lambda+1}\\\hline\end{array}\right)\otimes\\
&\qquad u_i\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-\beta_p}&\cdots&u_\lambda^{m_\lambda-\beta_\lambda-1}&\cdots&u_i^{m_i-\alpha_i-1}\\\hline\end{array}\right)\otimes x.
\end{aligned}
$$

A simple argument shows us that the term that we are subtracting is equal to all of the positive term except for the element Y. For we have

$$\sum_{i=1}^{p} u_i \otimes \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p} & \cdots & u_i^{m_i-1} & \cdots & u_1^{m_1}\\ \hline\end{array}\right) = \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p} & \cdots & u_i^{m_i} & \cdots & u_1^{m_1}\\ \hline\end{array}\right),$$

and this latter term runs through all $(q-l)$-fold tensor products of content $(m_1, ..., m_p)$. The terms that we are subtracting can clearly be seen to run through all such tensor products of the same content, except for the term $u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p}$.Thus, the difference is just Y. Let us record this fact as a separate identity:

$$\begin{aligned} Y = Z \otimes u \otimes \textstyle\sum_{i=1}^{p} u_i \otimes \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p} & \cdots & u_i^{m_i-1} & \cdots & u_1^{m_1}\\ \hline\end{array}\right) \otimes x \\ - \textstyle\sum_{\lambda=i+1}^{p} \sum_{i=1}^{p-1} \sum_{\alpha_i=0}^{m_i-1} Z \otimes u \otimes u_1^{\otimes m_1} \otimes \cdots \otimes \\ u_{i-1}^{\otimes m_{i-1}} \otimes u_i^{\otimes \alpha_i} \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{\beta_p} & \cdots & u_\lambda^{\beta_\lambda+1}\\ \hline\end{array}\right) \otimes \\ u_i \otimes \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-\beta_p} & \cdots & u_\lambda^{m_\lambda-\beta_\lambda-1} & \cdots & u_i^{m_i-\alpha_i-1}\\ \hline\end{array}\right) \otimes x. \end{aligned} \tag{4.1}$$

We will now give a full argument to handle the case that $u_p < u < u_{p-1} < \cdots < u_1$. Once we do this, the general argument will be clear. The thing we must do in this situation is to see what gets added and what subtracted from the terms in (4.1) above. A term that gets added to the positive terms, due to the fact that we may now consider the tableau $\begin{array}{|c|c|}\hline u_p^{\beta_p+1} & u\\ \hline\end{array}$, is $Z \otimes \Delta\left(\begin{array}{|c|c|}\hline u_p^{\beta_p+1} & u\\ \hline\end{array}\right) \otimes u_1 \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-\beta_p-1} & \cdots & u_1^{m_1}\\ \hline\end{array}\right) \otimes x$. Since we no longer have $Z \otimes u \otimes u_p \otimes \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-1} & \cdots & u_i^{m_i} & \cdots & u_1^{m_1}\\ \hline\end{array}\right) \otimes x$, we must subtract it from our positive terms. At the same time, we must consider the contribution from

$$(-1)^{q-l+1}\eta_{q-1}\left(Z \otimes uu_p \otimes u_1 \otimes \cdots \otimes u_{p-1} \otimes x\right),$$

which is $-Z \otimes u_p^{\otimes \beta_p+1} \otimes u \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-\beta_p-1} & \cdots & u_1^{m_1}\\ \hline\end{array}\right) \otimes x$. However, each of the terms

$$(-1)^{q-l+1}\eta_{q-1}\left(Z \otimes uu_i \otimes u_1 \otimes \cdots \otimes u_{i-1} \otimes u_{i+1} \otimes \ldots \otimes u_p \otimes x\right),$$

with $1 < i < p$, contributes a positive term that it did not contribute in the preceding case, namely

$$Z \otimes \Delta\left(\begin{array}{|c|c|}\hline u_p^{\beta_p+1} & u\\ \hline\end{array}\right) \otimes u_i \otimes \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-\beta_p-1} & \cdots & u_i^{m_i} & \cdots & u_1^{m_1}\\ \hline\end{array}\right) \otimes x.$$

But now it is easy to see that the terms added cancel out the terms subtracted, and this completes the proof in this case. Clearly, to attack the general case, we assume that $u_p < \cdots < u_{i+1} \leq u < u_i < \cdots < u_1$ and notice that in each step the terms added cancel those subtracted, as in the case just treated. This completes the proof of our lemma. ∎

This next lemma tells us that our explicit definition of η for terms all of whose elements are of degree one satisfies the homotopy identity.

Lemma 4.2. *Let $Y = J \otimes u_1 \otimes \cdots \otimes u_q \otimes x$, with all the u_i of degree 1. Then*

$$\partial_C \eta_q(Y) + \eta_{q-1}\partial_C(Y) = Y - \alpha_q \theta_q(Y).$$

Proof. We will again write $Y = J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x$ with $u_1 > \cdots > u_p$ and $\sum m_i = q$. By a process identical to the one used to prove the lemma above, and using the identity (4.1) above, it can be easily shown that

$$\eta_{q-1}\partial_C(Y) =$$

$$(-1)^q \sum_{i=1}^{p} \eta_{q-1}\left(Ju_i \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1}} \otimes u_i^{\otimes m_i - 1} \otimes \ldots \otimes u_p^{\otimes m_p} \otimes x\right) +$$
$$\sum_{i=1}^{p} \eta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \ldots \otimes u_p^{\otimes m_p} \otimes u_i(x)\right) +$$
$$Y - J \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes x.$$

We will assume first that $u_p > j_{s+1}$.

What we have to compute now is $\partial_C \eta_q(Y) + \eta_{q-1}\partial_C(Y)$, and show that it equals $Y - \alpha_q\theta_q(Y)$. Let us look first at

$$(-1)^{q+1}\partial_C \alpha_0^{'}\theta_0^{'}\left(J \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes x\right).$$

Since $\alpha_0^{'}$ is the identity, and $\theta_0^{'}(J) = (-1)^{(s-1)(n-k+1)} j_s \cdots j_{n-k+s} \otimes j_1 \cdots j_{s-1}$, we see that

$$(-1)^{q+1}\partial_C \alpha_0^{'}\theta_0^{'}\left(J \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes x\right) =$$

$$J \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes x - (-1)^{\sigma} j_s \cdots j_{n-k+s} \otimes j_1 \cdots j_{s-1}\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes x$$
$$+(-1)^{q+1}\sum_{i=1}^{p}(-1)^{\sigma} j_s \cdots j_{n-k+s} \otimes j_1 \cdots j_{s-1} \otimes \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p} & \cdots & u_i^{m_i-1} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes u_i(x),$$

(where we have set $\sigma = (s-1)(n-k+1)$). Many terms in the boundary disappear due to the fact that $D_2 \to D_1 \otimes D_1 \to \Lambda^2$ is zero. In any event, we see that adding $(-1)^{q+1}\partial_C \alpha_0^{'}\theta_0^{'}\left(J \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes x\right)$ to our expression for $\eta_{q-1}\partial_C(Y)$ gives us

$$Y - (-1)^{\sigma} j_s \cdots j_{n-k+s} \otimes j_1 \cdots j_{s-1}\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes x +$$
$$(-1)^{q+1}\sum_{i=1}^{p}(-1)^{\sigma} j_s \cdots j_{n-k+s} \otimes j_1 \cdots j_{s-1} \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots u_i^{m_i-1} \cdots & u_1^{m_1} \\ \hline\end{array}\right) u_i(x) +$$
$$(-1)^q \sum_{i=1}^{p} \eta_{q-1}\left(Ju_i \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_{i-1}^{\otimes m_{i-1}} \otimes u_i^{\otimes m_i - 1} \otimes \ldots \otimes u_p^{\otimes m_p} \otimes x\right) +$$
$$\sum_{i=1}^{p} \eta_{q-1}\left(J \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \ldots \otimes u_p^{\otimes m_p} \otimes u_i(x)\right).$$

For maximum coverage, we will consider the case that $j_s \le u_p$ (for otherwise, $\theta_q(Y) = 0$). After handling this case, we will indicate what must be added and subtracted in the event that $u_p < \cdots < u_i < j_s \le u_{i-1} < \cdots < u_1$.

For $l > 0$, we have

$$(-1)^{q-l+1}\partial_C \alpha_l^{'}\theta_l^{'}\left(J \otimes \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array}\right) \otimes x\right) =$$

$$(-1)^{q-l+1}\sum_{\mathbf{n}}\partial_C\alpha_i^{'}\theta_i^{'}\left(J\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x\right)$$

$$= \quad (-1)^{q-l+1}\sum_{\mathbf{n}}\partial_C\left(\begin{array}{c}(-1)^{\sigma+(s-1)l}\alpha_i^{'}\left(\begin{array}{|c|c|c|c|}\hline j_s & u_p^{n_p} & \cdots & u_1^{n_1}\\\hline \vdots & & & \\ \cline{1-1} j_{n-k+s} & & & \\ \cline{1-1}\end{array}\right)\otimes j_1\cdots j_{s-1}\otimes \\ \Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x\end{array}\right)$$

where $\mathbf{n}=(n_1,\ldots,n_p)$ runs through all p-tuples of non-negative integers with $n_1+\cdots+n_p=l$. We see that

$$(-1)^{q-l+1}\sum_{\mathbf{n}}\partial_C\left(\begin{array}{c}(-1)^{\sigma+(s-1)l}\alpha_i^{'}\left(\begin{array}{|c|c|c|c|}\hline j_s & u_p^{n_p} & \cdots & u_1^{n_1}\\\hline \vdots & & & \\ \cline{1-1} j_{n-k+s} & & & \\ \cline{1-1}\end{array}\right)\otimes j_1\cdots j_{s-1} \\ \otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x\end{array}\right) = \text{(up to sign)}$$

$$(a)_l: \quad (-1)^{q-l+1}\sum_{\mathbf{n}}\alpha_i^{'}\theta_l^{'}\left(J\otimes u_1^{\otimes n_1}\otimes\cdots\otimes u_p^{\otimes n_p}\right)\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)(x)\pm$$

$$(b)_l \quad : \quad \sum_{\mathbf{n}} j_s\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes$$
$$j_1\cdots j_{s-1}\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x\pm$$

$$(c)_l \quad : \quad \sum_{\mathbf{n}}\sum_{\lambda} u_\lambda j_{s+1}\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|c|c|c|}\hline j_s & u_p^{n_p} & \cdots & u_\lambda^{n_\lambda-1} & \cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes$$
$$j_1\cdots j_{s-1}\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x\pm$$

$$(d)_l \quad : \quad \sum_{\mathbf{n}} j_s\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1}\\\hline\end{array}\right) j_1\cdots j_{s-1}\otimes$$
$$\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x\pm$$

$$(e)_l \quad : \quad \sum_{\mathbf{n}}\sum_{\lambda} u_\lambda j_{s+1}\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|c|c|c|}\hline j_s & u_p^{n_p} & \cdots & u_\lambda^{n_\lambda-1} & \cdots & u_1^{n_1}\\\hline\end{array}\right) j_1\cdots j_{s-1}\otimes$$
$$\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x$$

A simple calculation shows that

$$\alpha_l^{'}\theta_l^{'}\left(J\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p} & \cdots & u_i^{m_i-1} & \cdots & u_1^{m_1}\\\hline\end{array}\right)\otimes u_i(x)\right)$$

cancels $(a)_l$, and that $(d)_{l+1}$ cancels $(b)_l$. To handle the next cancellations, let us rewrite the term $(e)_l$ as:

$$
\begin{aligned}
(e_1)_l \qquad & \sum_{\mathbf{n}}\sum_{\lambda} u_\lambda j_{s+1}\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_\lambda^{n_\lambda-1} & \cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes\\
& \qquad j_1\cdots j_{s-1}j_s\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x+\\
(e_2)_l \qquad & \sum_{\mathbf{n}}\sum_{\lambda} u_\lambda j_{s+1}\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline j_s & u_p^{n_p}\cdots & u_\lambda^{n_\lambda-1} & u_\mu^{n_\mu-1}\cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes\\
& \qquad u_\mu j_1\cdots j_{s-1}j_s\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x.
\end{aligned}
$$

Then we see that $(e_2)_{l+1}$ cancels $(c)_l$. Finally, let us evaluate the terms

$$(-1)^q\sum_{i=1}^{p}\eta_{q-1}\left(Ju_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)$$

that remain from $\eta_{q-1}\partial_C(Y)$. Again we apply $\alpha_l'\theta_l'$ to each of the terms

$$Ju_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x$$

and add them to the terms that we have left. At this point that is simply $(e_1)_l$. We get

$$
\begin{aligned}
&\alpha_i'\theta_i'\left(Ju_i\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p} & \cdots & u_i^{m_i-1} & \cdots & u_1^{m_1}\\\hline\end{array}\right)\otimes x\right)=\\
&\sum_{\mathbf{n}}\alpha_i'\theta_i'\left(\begin{array}{c} Ju_i\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes\\ \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x\end{array}\right)=\\
&\sum_{\mathbf{n}} u_i j_{s+1}\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes j_1\cdots j_s\otimes\\
&\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x\\
&+\sum_{\mathbf{n}} u_\lambda u_i j_{s+2}\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_\lambda^{n_\lambda-1} & \cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes j_1\cdots j_s\otimes\\
&\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x.
\end{aligned}
$$

For the usual reasons, the terms

$$
\begin{aligned}
&\sum_{\mathbf{n}} u_\lambda u_i j_{s+2}\cdots j_{n-k+s}\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_\lambda^{n_\lambda-1} & \cdots & u_1^{n_1}\\\hline\end{array}\right)\otimes j_1\cdots j_s\otimes\\
&\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x
\end{aligned}
$$

add up to zero, and the term above that clearly cancels the term $(e_1)_{l+1}$, and it is easy to complete the steps to show that we have the desired result. In particular, we see that the term $(a)_q$ does not get cancelled, and that is why the sum we are left with is $Y-\alpha_q\theta_q(Y)$.

All this was done under the hypothesis that $j_s\le u_p$. However, just as we did in the previous lemma, we simply have to keep track of what we add and subtract from the terms above under the assumption that $u_p<\cdots<u_i<j_s\le u_{i-1}<\cdots<u_1$, to get the result in general.For example, if we had $i=p$, we would have to study the terms above under the restriction that in our sequences $\mathbf{n}$, the entry $n_p=0$. But this presents no new difficulties. ∎

Now we can state and prove the main theorem.

Theorem 4.3. *The map η defined above is a homotopy of the identity map on $\mathbf{C}(k; f)$ with the map $\alpha\theta$. That is, for all $q \geq -1$, we have*

$$\partial_C \eta_q + \eta_{q-1}\partial_C = 1 - \alpha_q \theta_q.$$

Proof. It is easy to verify this in the lower dimensions: for $q = -1$, we get zero on both sides. For $q = 0$ we take $J \otimes x \in \Lambda^{n-k+s}G^* \otimes \Lambda^{n+s}F$ and check that

$$\partial_C \left(-(-1)^{(s-1)(n-k+1)} j_s \cdots j_{n-k+s} \otimes j_1 \cdots j_{s-1} \otimes x \right) =$$

$$-(-1)^{(s-1)(n-k+1)} j_s \cdots j_{n-k+s} \otimes j_1 \cdots j_{s-1}(x) + J \otimes x.$$

In fact, the signs have been rigged so that this does check.

In the general case, we consider the element $Y = J \otimes U_1 \otimes \cdots \otimes U_t \otimes U_{t+1} \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x$ with $U_{t+1} = v \wedge W$, and $v, u_1, \ldots, u_p$ all of degree one, as usual, with $u_1 > \cdots > u_p$. We assume that t is some integer lying between 0 and $q-1$, and we set

$$\begin{aligned}
B(Y) &= J \otimes U_1 \otimes \cdots \otimes U_t v \otimes W \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x \\
E(Y) &= J \otimes U_1 \otimes \cdots \otimes U_t \otimes v \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes W(x) \\
\Gamma(Y) &= B(Y) + (-1)^{\sigma} E(Y)
\end{aligned}$$

with $\sigma = r(q-t-1)$, where degree $(w) = r$. Furthermore, we assume that we have established the fact that η_i is a homotopy of desired type for $i < q$, and that η_q is also such on elements of type Y in which the degree of U_{t+1} is less than or equal to r. We want to establish the homotopy identity for the element Y whose $(t+1)th$ term is of degree $r+1$. We see immediately, since $\theta_q(Y) = \theta_q(\Gamma(Y))$, that we have to establish

$$\partial_C \eta_q \left(Y - \Gamma(Y)\right) + \eta_{q-1}\partial_C \left(Y - \Gamma(Y)\right) = Y - \Gamma(Y).$$

Using a by now familiar type of argument, we see that

$$\eta_{q-1}\partial_C\left(Y\right) =$$

$$\begin{aligned}
&(-1)^{q-t-1}\sum_{i=1}^{p} \eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes vWu_i \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) \\
&+\sum_{i=1}^{p} \eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_t \otimes vW \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_i^{\otimes m_i - 1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes u_i(x)\right) \\
&+(-1)^{q-t}\eta_{q-1}\left(J \otimes U_1 \otimes \cdots \otimes U_t vW \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x\right) \\
&+(-1)^{q-t}\sum_{i=1}^{t} \eta_{q-1}\left(L_i\right) + Y - J \otimes U_1 \otimes \cdots \otimes U_t \otimes vW \otimes \Delta(\boxed{u_p^{m_p} \mid \cdots \mid u_1^{m_1}}) \otimes x,
\end{aligned}$$

where we set

$$L_i = J \otimes U_1 \otimes \cdots \otimes U_{i-1}U_i \otimes \cdots \otimes U_t \otimes vW \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_p^{\otimes m_p} \otimes x;$$

$$\eta_{q-1}\partial_C\left(B(Y)\right) =$$

$$
\begin{aligned}
&(-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t v\otimes W u_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)\\
&+\sum_{i=1}^{p}\eta_{q-1}\left(B\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes u_i(x)\right)\right)\\
&+(-1)^{q-t}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t vW\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)\\
&+(-1)^{q-t}\sum_{i=1}^{t}\eta_{q-1}\left(B(L_i)\right)+B(Y)-B\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes\Delta(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1}\\\hline\end{array})\otimes x\right);
\end{aligned}
$$

where by $B(X)$ we mean the obvious, and

$$\eta_{q-1}\partial_C\left(E(Y)\right)=$$

$$
\begin{aligned}
&(-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes v u_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)\\
&(-1)^{r}\sum_{i=1}^{p}\eta_{q-1}\left(E\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes u_i(x)\right)\right)\\
&+(-1)^{q-t}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t v\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)\\
&+(-1)^{q-t}\sum_{i=1}^{t}\eta_{q-1}\left(E(L_i)\right)+E(Y)-E\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes\Delta(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1}\\\hline\end{array})\otimes x\right),
\end{aligned}
$$

where by $E(X)$ we also mean the obvious. Then

$$\eta_{q-1}\partial_C\left(Y-\Gamma(Y)\right)=$$

$$
\begin{aligned}
&\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes u_i(x)\right)-\\
&\sum_{i=1}^{p}\eta_{q-1}\left(\Gamma\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes u_i(x)\right)\right)+\\
&(-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vWu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)-\\
&(-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t v\otimes Wu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)-\\
&(-1)^{\sigma+q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)-\\
&(-1)^{\sigma+q-t}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t v\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)+\\
&(-1)^{q-t}\sum_{i=1}^{t}\eta_{q-1}\left(L_i-\Gamma(L_i)\right)+\left(Y-\Gamma(Y)\right)-\\
&\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes\Delta(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1}\\\hline\end{array})\otimes x-\\ \Gamma\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes\Delta(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1}\\\hline\end{array})\otimes x\right)\end{array}\right)
\end{aligned}
$$

The first two rows are, by the definition of our homotopy, equal to

$$\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{(r-1)\rho}Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & v\\\hline\end{array}\right)\otimes W\otimes$$
$$\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes u_i(x),$$

where we have set

$$Z=J\otimes U_1\otimes\cdots\otimes U_t$$

and $|\mathbf{n}|=n_1+...+n_p$.

The terms $(-1)^{q-t}\sum_{i=1}^{t}\eta_{q-1}\left(L_i-\Gamma(L_i)\right)$ are also easy to write down; they are simply

$$(-1)^{q-t}\sum_{i=1}^{t}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{(r-1)\rho}J\otimes U_1\otimes\cdots\otimes U_{i-1}U_i\otimes\cdots\otimes U_t\otimes$$
$$\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & v\\\hline\end{array}\right)\otimes W\otimes\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\otimes x\right).$$

Thus all of our terms but the following have been converted to "non-η" terms, and we have to examine their sum:

$$(-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vWu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)-$$
$$(-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_tv\otimes Wu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)-$$
$$(-1)^{\sigma+q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)-$$
$$(-1)^{\sigma+q-t}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_tv\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right).$$

There are two possibilities we have to consider: $u_i<v$ and $u_i>v$. (The case $u_i=v$ is easy to dispose of: two of the four summands above are zero, and the other two are very easy to handle.) As usual, we will make the assumption that $v>u_i$ for all i, and remark what happens to various terms that we have to add and/or subtract in the other cases.

Since vW is a basis element, v is less than all the factors of W, so u_ivW is a basis element and we have

$$(-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vWu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)$$
$$=\quad(-1)^{r+q-t}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_tu_i\otimes vW\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)+$$
$$(-1)^{r+q-t+(r+1)(q-t-2)}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes vW(x)\right)+$$
$$(-1)^{r-1}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{r\rho}Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i\\\hline\end{array}\right)\otimes vW\otimes$$
$$\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x.$$

(The penultimate term has no summation in it because the terms of the same sort coming from $i > 1$ all vanish.)

Next we see that

$$
\begin{aligned}
& (-1)^{\sigma+q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)\\
= \quad & (-1)^{\sigma+q-t}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_tu_i\otimes v\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)+\\
& (-1)^{\sigma-2}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes vW(x)\right)+\\
& (-1)^{\sigma+q-t}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho} Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i\\\hline\end{array}\right)\otimes v\otimes\\
& \qquad\qquad \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes W(x).
\end{aligned}
$$

When we collect our terms we have

$$
\begin{aligned}
& (-1)^{r+q-t}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_tu_i\otimes vW\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)+\\
& (-1)^{\sigma+q-t}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_tu_i\otimes v\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)-\\
& (-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_tv\otimes Wu_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)-\\
& (-1)^{\sigma+q-t}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_tv\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)+\\
& (-1)^{r-1}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{r\rho}Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i\\\hline\end{array}\right)\otimes vW\otimes\\
& \qquad\qquad \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x+\\
& (-1)^{\sigma+q-t}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho} Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i\\\hline\end{array}\right)\otimes v\otimes\\
& \qquad\qquad \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes W(x).
\end{aligned}
$$

Next, we see that

$$\begin{aligned}
& (-1)^{q-t-1}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t v\otimes W u_i\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)\\
= \; & (-1)^{q-t-1+r}\sum_{i=1}^{p}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t v u_i\otimes W\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\right)+\\
& (-1)^{q-t-1+r+r(q-t-2)}\eta_{q-1}\left(J\otimes U_1\otimes\cdots\otimes U_t v\otimes\otimes u_1^{\otimes m_1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\right)+\\
& (-1)^{r}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{(r-1)\rho}J\otimes U_1\otimes\cdots\otimes U_t v\otimes\\
& \Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i\\\hline\end{array}\right)\otimes W\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x
\end{aligned}$$

so that our collected terms now are

$$\begin{aligned}
& (-1)^{r+q-t}\sum_{i=1}^{p}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes U_t u_i\otimes vW\otimes u_1^{\otimes m_1}\otimes\cdots\otimes\\ u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\end{array}\right)+\\
& (-1)^{\sigma+q-t}\sum_{i=1}^{p}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes U_t u_i\otimes v\otimes u_1^{\otimes m_1}\otimes\cdots\otimes\\ u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes W(x)\end{array}\right)-\\
& (-1)^{q-t+r}\sum_{i=1}^{p}\eta_{q-1}\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes U_t u_i v\otimes W\otimes u_1^{\otimes m_1}\otimes\cdots\otimes\\ u_i^{\otimes m_i-1}\otimes\cdots\otimes u_p^{\otimes m_p}\otimes x\end{array}\right)+\\
& (-1)^{r-1}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{r\rho}Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i\\\hline\end{array}\right)\otimes vW\otimes\\
& \qquad\qquad \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x+\\
& (-1)^{\sigma+q-t}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i\\\hline\end{array}\right)\otimes v\otimes\\
& \qquad\qquad \Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes W(x))+\\
& (-1)^{r}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{(r-1)\rho}J\otimes U_1\otimes\cdots\otimes U_t v\otimes\\
& \Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i\\\hline\end{array}\right)\otimes W\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x.
\end{aligned}$$

Finally, we have that the first three rows add up to

$$\begin{aligned}
& (-1)^{r-1}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{(r-1)\rho}J\otimes U_1\otimes\cdots\otimes U_t u_i\otimes\\
& \Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & v\\\hline\end{array}\right)\otimes W\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1}\\\hline\end{array}\right)\otimes x,
\end{aligned}$$

so that we end up with our collected terms summing to

$$(-1)^{r-1}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{r\rho}Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i \\ \hline\end{array}\right)\otimes vW\otimes$$

$$\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1} \\ \hline\end{array}\right)\otimes x+$$

$$(-1)^{\sigma+q-t}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i \\ \hline\end{array}\right)\otimes v\otimes$$

$$\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1} \\ \hline\end{array}\right)\otimes W(x)+$$

$$(-1)^{r}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{(r-1)\rho}J\otimes U_1\otimes\cdots\otimes U_t v\otimes$$

$$\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & u_i \\ \hline\end{array}\right)\otimes W\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1} \\ \hline\end{array}\right)\otimes x+$$

$$(-1)^{r-1}\sum_{i=1}^{p}\sum_{\rho\geq 0}\sum_{|\mathbf{n}|=\rho}(-1)^{(r-1)\rho}J\otimes U_1\otimes\cdots\otimes U_t u_i\otimes$$

$$\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & v \\ \hline\end{array}\right)\otimes W\otimes\Delta\left(\begin{array}{|c|c|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_i^{m_i-n_i-1} & \cdots & u_1^{m_1-n_1} \\ \hline\end{array}\right)\otimes x.$$

All of the above was by way of calculating $\eta_{q-1}\partial_C\left(Y-\Gamma(Y)\right)$. We now have to add to what we have obtained, the terms of $\partial_C\eta_q\left(Y-\Gamma(Y)\right)$. But we know that

$$\eta_q\left(Y-\Gamma(Y)\right)=$$

$$\sum_{\rho\geq 0}\sum_{\mathbf{n}}(-1)^{(r-1)\rho}Z\otimes\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & v \\ \hline\end{array}\right)\otimes W\otimes$$

$$\Delta\left(\begin{array}{|c|c|c|}\hline u_p^{m_p-n_p} & \cdots & u_1^{m_1-n_1} \\ \hline\end{array}\right)\otimes x.$$

If we take $\partial_C\eta_q\left(Y-\Gamma(Y)\right)$, we see that the boundary applied to the Z part of these terms just gives us our terms related to our L_i terms above. The boundary term corresponding to the action on x is covered by terms calculated at the outset.The terms

$$\left(\begin{array}{c} J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes\Delta(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array})\otimes x- \\ \Gamma\left(J\otimes U_1\otimes\cdots\otimes U_t\otimes vW\otimes\Delta(\begin{array}{|c|c|c|}\hline u_p^{m_p} & \cdots & u_1^{m_1} \\ \hline\end{array})\otimes x\right)\end{array}\right)$$

that we found earlier are taken care of in part by the $W(x)$ terms we have found just above. The rest of the terms that we have laboriously calculated deal with the interactions of the v with W, and with the multiplication of these with their immediate neighbors that occur when we apply the boundary. In short, we can check off, term by term, the results of applying the boundary and our collected terms, taking into account a certain amount of internal cancellation (such as the $W(x)$ terms just mentioned).

All of the above was predicated on the assumption that $v>u_1$. If we had $u_1>v>u_2$, say, then the terms involving $\Delta\left(\begin{array}{|c|c|c|c|}\hline u_p^{n_p} & \cdots & u_1^{n_1} & v \\ \hline\end{array}\right)$ would have the requirement that $n_1=0$. On the other hand, when we calculated the "η" terms of our $\eta_{q-1}\partial_C\left(Y-\Gamma(Y)\right)$, we would have had to keep the v in front (other signs involved in 'rectifying' the product Wu_1 would have cancelled each other out), and we would have had a corresponding gain or loss of our resulting "non-η'' terms. Thus we have our desired result. ∎

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