A CHARACTERISTIC-FREE EXAMPLE OF A LASCOUX RESOLUTION, AND LETTER-PLACE METHODS FOR INTERTWINING NUMBERS

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Alain Lascoux: un homme de caractère libre dont les travaux se basent en caractéristique zéro.

1. Introduction

Although Alain Lascoux and I never collaborated on a mathematical paper, his ideas have been a constant influence on my students', my collaborators' and my work ever since the appearance of the first draft of his doctoral thesis [12]

His systematic use of characteristic-zero representation theory to find the resolutions of ideals generated by the minors, of any order, of generic matrices, impelled me and my students in that period to develop a characteristic-free theory of Schur and Weyl modules [4]. The attempt by Akin, Weyman and myself to use these characteristic-free methods to reproduce Lascoux' resolutions led to the realization that there were many mysteries hidden in $\mathbb{Z}$-forms of rational representations that had to be uncovered in order to move ahead with this project [3]. Akin and I soon discovered that the study of $\mathbb{Z}$-forms was intimately bound up with the resolutions of Weyl modules [1, 2], the characteristic-zero version of which Lascoux had already presented. The study of such resolutions was helpful in the Roberts-Weyman [13] presentation of the Hashimoto [10] example of the dependence of the Betti numbers of determinantal ideals on characteristic. Further work on these resolutions with Rota [8] led to the use of letter-place methods and place polarizations in a systematic way in this area.

In sections 2 and 3, we will give a few examples of the way in which Lascoux' work has been incoporated into a number of the above-mentioned areas of investigation. Where possible, we will point out the similarities and differences between the classical and 'neoclassical' results. In section 4, we give a very brief indication of how place polarization methods, Capelli identities, and resolutions come into play in the study of intertwining numbers.

2. Examples of $\mathbb{Z}$-forms of rational representations

In [3], we had to prove and use the rather strange fact that the complex

$$(1)\quad 0 \to \Lambda^k \to D_1 \otimes \Lambda^{k-1} \to \cdots \to D_l \otimes \Lambda^{k-l} \to \cdots \to D_k \to 0,$$

in which the maps entail diagonalizing the exterior power and multiplying in the divided power, is exact from Λ^k up to $l = [\frac{k}{2}]$; that is, exact till halfway up. Thus, the kernels (or images) in those dimensions are universal $\mathbb{Z}$-free representations of Gl_n which, when tensored with the rationals, $\mathbb{Q}$, are isomorphic to the Schur modules of the hooks in the corresponding dimensions (since the Schur hooks are the kernels of the corresponding complex where the divided power, D, is replaced by the symmetric power, S). Thus, these integral representations are what Akin, Weyman and the author called $\mathbb{Z}$-forms of the same rational representation of Gl_n. For example, D_k and S_k are non-isomorphic $\mathbb{Z}$-forms of the k^{th} symmetric power.

Another, simpler, way to construct non-isomorphic $\mathbb{Z}$-forms is the following:

Consider the short exact sequence

$$0 \to D_{k+2} \to D_{k+1} \otimes D_1 \to K_{(k+1,1)} \to 0$$

where $K_{(k+1,1)}$ is the Weyl module associated to the hook partition $(k+1, 1)$. If we take an integer, t, and multiply D_{k+2} by t, we get an induced exact sequence and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & D_{k+2} & \to & D_{k+1} \otimes D_1 & \to & K_{(k+1,1)} & \to & 0 \\ & & \downarrow t & & \downarrow & & \downarrow & & \\ 0 & \to & D_{k+2} & \to & E(t; k+1, 1) & \to & K_{(k+1,1)} & \to & 0, \end{array}$$

where $E(t; k+1, 1)$ stands for the cofiber product of D_{k+2} and $D_{k+1} \otimes D_1$. Each of these modules is a $\mathbb{Z}$-form of $D_{k+1} \otimes D_1$, but for t_1 and t_2, two such are isomorphic if and only if $t_1 \equiv t_2 \mod k+2$ (see[2]). In fact, one can easily show that $\mathrm{Ext}^1_A(K_{(k+1,1)}, D_{k+2}) = \mathbb{Z}/(k+2)$, where A stands for the Schur algebra of appropriate degree (namely, $k+2$).

To see how such forms are related to resolutions of Weyl modules, consider the partition $(k, 2)$, the associated Weyl module, $K_{(k,2)}$, and its resolution (over the integers):

$$0 \to D_{k+2} \to \{D_{k+2} \oplus D_{k+1} \otimes D_1\} \to D_k \otimes D_2 \to K_{(k,2)} \to 0.$$

Notice that in characteristic zero, the resolution that Lascoux describes would be

$$0 \to D_{k+1} \otimes D_1 \to D_k \otimes D_2 \to K_{(k,2)} \to 0;$$

the Euler-Poincaré characterisitcs of the two are the same, but the first one requires more terms to take care of torsion over the integers.

The map of D_{k+2} into the indicated direct sum is the usual diagonalization into the second summand (or the place polarization ∂_{21}), but is multiplication by 2 into the first. The map from D_{k+2} to $D_k \otimes D_2$ is the second divided power of ∂_{21} or $\partial_{21}^{(2)}$. As a result, we see that the cokernel of this map is what we called above, $E(2; k+1, 1)$, and we would have to rewrite Lascoux' resolution above as

$$0 \to E(2; k+1, 1) \to D_k \otimes D_2 \to K_{(k,2)} \to 0.$$

(This is, of course, the same as his original resolution after tensoring by $\mathbb{Q}$.)

The study of $\mathbb{Z}$-forms, especially as they emerged in our attempts to resolve determinantal ideals beyond the submaximal minors, led to the suspicion that perhaps there is no universal (i.e., integral) resolution for the minors of any order of the generic matrix. Hashimoto [10] finally took what everyone thought must be true: the Betti numbers of these determinantal ideals are independent of characterisitc, and found a counter-example (the case of 2×2 minors of the 5×5 matrix). As things now stand, the resolutions of Lascoux stand as the minimal characterisitc-zero universal ones; recent work of Hashimoto [11] shows that minimal universal resolutions whose terms are tilting modules, do exist.

3. Some examples of three-rowed shapes and skew hooks

One presentation of the Hashimoto counter-example mentioned above, developed by Roberts and Weyman [13], uses the characteristic-free resolution of Weyl modules in certain cases. In this section, we'll look at the resolution of a three-rowed shape, and see how it differs from that constructed by Lascoux in characteristic zero. The resolutions of two-rowed skew shapes were given in [2] and [6], while those for very special three-rowed shapes were given in [7]. The one given here also appeared in [8], but the explicit connection with the Lascoux resolution was not given there.

3.1. **Lascoux and non-Lascoux resolutions for (2,2,2).** The Lascoux resolution of the Weyl module associated to the partition $(2,2,2)$ looks like this:

$$0 \to D_4 \otimes D_2 \otimes D_0 \to \begin{matrix} D_3 \otimes D_3 \otimes D_0 \\ \oplus \\ D_4 \otimes D_1 \otimes D_1 \end{matrix} \to \begin{matrix} D_3 \otimes D_1 \otimes D_2 \\ \oplus \\ D_2 \otimes D_3 \otimes D_1 \end{matrix} \to D_2 \otimes D_2 \otimes D_2 \to 0.$$

Recall that the terms of the Lascoux resolution are read off from the determinantal expansion of the Jacobi-Trudi matrix of the partition, with the position of the terms of the complex determined by the length of the permutations to which they correspond. The correspondence between the terms of the resolution above, and permutations, is as follows:

$$\begin{array}{lcl} D_2 \otimes D_2 \otimes D_2 & \longleftrightarrow & \text{identity} \\ D_3 \otimes D_1 \otimes D_2 & \longleftrightarrow & (12) \\ D_2 \otimes D_3 \otimes D_1 & \longleftrightarrow & (23) \\ D_3 \otimes D_3 \otimes D_0 & \longleftrightarrow & (123) \\ D_4 \otimes D_1 \otimes D_1 & \longleftrightarrow & (213) \\ D_4 \otimes D_2 \otimes D_0 & \longleftrightarrow & (13) \end{array}$$

By contrast, the terms of the characteristic-free resolution of the same Weyl module, are:

$$
\begin{aligned}
X_0 &= D_2 \otimes D_2 \otimes D_2;\\
X_1 &= Z^{(1)}_{2,1} x D_3 \otimes D_1 \otimes D_2 \oplus Z^{(2)}_{2,1} x D_4 \otimes D_0 \otimes D_2 \oplus\\
&\quad Z^{(1)}_{3,2}\, y D_2 \otimes D_3 \otimes D_1 \oplus Z^{(2)}_{3,2}\, y D_2 \otimes D_4 \otimes D_0;\\
X_2 &= Z^{(1)}_{2,1} x Z^{(1)}_{2,1} x D_4 \otimes D_0 \otimes D_2 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(2)}_{2,1} x D_4 \otimes D_1 \otimes D_1 \oplus Z^{(1)}_{3,2}\, y Z^{(3)}_{2,1} x D_5 \otimes D_0 \otimes D_1 \oplus\\
&\quad Z^{(2)}_{3,2}\, y Z^{(3)}_{2,1} x D_5 \otimes D_1 \otimes D_0 \oplus Z^{(2)}_{3,2}\, y Z^{(4)}_{2,1} x D_6 \otimes D_0 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,2}\, y D_2 \otimes D_4 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z D_3 \otimes D_3 \otimes D_0;\\
X_3 &= Z^{(1)}_{3,2}\, y Z^{(2)}_{2,1} x Z^{(1)}_{2,1} x D_5 \otimes D_0 \otimes D_1 \oplus\\
&\quad Z^{(2)}_{3,2}\, y Z^{(3)}_{2,1} x Z^{(1)}_{2,1} x D_6 \otimes D_0 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,2}\, y Z^{(3)}_{2,1} x D_5 \otimes D_1 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,2}\, y Z^{(4)}_{2,1} x D_6 \otimes D_0 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z Z^{(1)}_{2,1} x D_4 \otimes D_2 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z Z^{(2)}_{2,1} x D_5 \otimes D_1 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z Z^{(3)}_{2,1} x D_6 \otimes D_2 \otimes D_0;\\
X_4 &= Z^{(1)}_{3,2}\, y Z^{(1)}_{3,2}\, y Z^{(3)}_{2,1} x Z^{(1)}_{2,1} x D_6 \otimes D_0 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z Z^{(1)}_{2,1} x Z^{(1)}_{2,1} x D_5 \otimes D_1 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z Z^{(2)}_{2,1} x Z^{(1)}_{2,1} x D_6 \otimes D_0 \otimes D_0 \oplus\\
&\quad Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z Z^{(1)}_{2,1} x Z^{(2)}_{2,1} x D_6 \otimes D_0 \otimes D_0;\\
X_5 &= Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z Z^{(1)}_{2,1} x Z^{(1)}_{2,1} x Z^{(1)}_{2,1} x D_6 \otimes D_0 \otimes D_0.
\end{aligned}
$$

where the subscripts on the X indicate the dimension in which these terms appear. The symbols $Z^{(t)}_{a,b}$ are the formal 'polarization' operators defined in [8], and the letters x, y, z are the separator variables also explained in that paper.

The boundary map for this complex is obtained by polarizing all the separator variables to one. When the separator x disappears between a $Z^{(t)}_{a,b}$ and elements in the tensor product of divided powers, this means $\partial^{(t)}_{a,b}$, or the place polarization operator, applied to that tensor product. The only essentially new terms that we have here are the terms that involve $Z^{(1)}_{3,2}\, y Z^{(1)}_{3,1}\, z$, or more generally terms of the form:

$$Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, Z^{(k_1)}_{2,1}\, x\, \cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, D_{2+1+|k|} \otimes D_{2+1-|k|},$$

(of which, in this example, there aren't that many). For these we have to use identities of Capelli type, or some easy variants of them ([8]). The boundary map on such a term sends it to

$$\begin{aligned}
&Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, \partial \left\{ Z^{(k_1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, D_{p+1+|k|} \otimes D_{q+1-|k|} \right\} \\
&\pm Z^{(1)}_{3,2}\, y \left\{ Z^{(1)}_{3,1}\, Z^{(k_1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, D_{p+1+|k|} \otimes D_{q+1-|k|} \right\} \\
&\mp Z^{(1)}_{3,2} Z^{(1)}_{3,1}\, z \left\{ Z^{(k_1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, D_{p+1+|k|} \otimes D_{q+1-|k|} \right\},
\end{aligned}$$

and we have to define the terms

$$Z^{(1)}_{3,1}\, Z^{(k_1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, D_{p+1+|k|} \otimes D_{q+1-|k|}$$

and

$$Z^{(1)}_{3,2} Z^{(1)}_{3,1}\, z \left\{ Z^{(k_1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, D_{p+1+|k|} \otimes D_{q+1-|k|} \right\}$$

with $n \geq 2$.

If $n = 2$, $Z^{(1)}_{3,1} v = \partial_{3,1}(v)$, while $Z^{(1)}_{3,2} Z^{(1)}_{3,1}\, z\, v = -Z^{(1)}_{2,1}\, x\, \partial^{(2)}_{3,2}(v) + Z^{(2)}_{3,2}\, y\, \partial_{2,1}(v)$. For $n > 2$, we have

$$\begin{aligned}
Z^{(1)}_{3,1}\, Z^{(k_1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, v \quad = \quad & -Z^{(k_1+1)}_{2,1}\, x\, Z^{(1)}_{3,2}\, Z^{(k_2)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, v + \\
& \binom{k_1 + k_2}{k_2 - 1} Z^{(1)}_{3,2}\, y\, Z^{(k_1+k_2+1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, v,
\end{aligned}$$

and

$$\begin{aligned}
Z^{(1)}_{3,2} Z^{(1)}_{3,1} z\, Z^{(k_1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, v = \quad & -Z^{(1)}_{2,1}\, x\, Z^{(2)}_{3,2} Z^{(k_1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, v + \\
& (k_1 - 1) Z^{(2)}_{3,2}\, y\, Z^{(k_1+1)}_{2,1}\, x \,\cdots\, x\, Z^{(k_{n-2})}_{2,1}\, x\, v.
\end{aligned}$$

3.2. **Reduction of non-Lascoux to Lascoux.** It is of some interest to see how we can discard the 'excess' terms of the characteristic-free resolution to recover that of Lascoux. For example, we want to throw away the terms

$$Z^{(2)}_{2,1} x D_4 \otimes D_0 \otimes D_2 \quad \text{and} \quad Z^{(2)}_{3,2}\, y D_2 \otimes D_4 \otimes D_0.$$

If we look at the image of a term $Z^{(2)}_{2,1} x \otimes v$, where $v \in D_4 \otimes D_0 \otimes D_2$, we see that it is $\partial^{(2)}_{21}(v) = \frac{1}{2}\partial^2_{21}(v)$. So, the image of $Z^{(2)}_{2,1} x \otimes v$ is the same as the image of $\frac{1}{2} Z^{(1)}_{2,1} x \otimes \partial^{(1)}_{21}(v)$. Hence in characteristic zero we can rig up the boundary map taking this into account. Obviously the same kind of thing holds for the term $Z^{(2)}_{3,2}\, y D_2 \otimes D_4 \otimes D_0$ i.e., its image is the same as that of $\frac{1}{2} Z^{(1)}_{3,2} x \otimes \partial^{(1)}_{32}(v)$. Thus, any term that's sent by the 'full' boundary map into one of the 'redundant' terms above, should now be sent into the non-redundant term instead. For example, the following terms should now be sent as follows:

$$Z^{(1)}_{3,2}\, y Z^{(2)}_{2,1} x\, v \to Z^{(1)}_{3,2}\, y\, \partial^{(2)}_{2,1}(v) - Z^{(1)}_{2,1}\, x\, \partial_{3,1}(v) - \frac{1}{2} Z^{(1)}_{2,1}\, x\, \partial_{2,1}\partial_{3,2}(v)$$

and

$$Z^{(1)}_{3,2}\,yZ^{(1)}_{3,1}z\,v \to Z^{(1)}_{3,2}\,y\,\partial_{3,1}(v) - Z^{(1)}_{2,1}\,x\,\partial^{(2)}_{3,2}(v) - \frac{1}{2}Z^{(1)}_{3,2}\,y\,\partial_{3,2}\partial_{2,1}(v).$$

With this modification, the terms of the type

$$Z^{(1)}_{2,1}xZ^{(1)}_{2,1}xD_4 \otimes D_0 \otimes D_2$$

and

$$Z^{(1)}_{3,2}\,yZ^{(1)}_{3,2}\,yD_2 \otimes D_4 \otimes D_0$$

are automatically sent to zero. As a result, when we go to the last term that counts in this complex, we see that under the boundary map in the big complex we get

$$Z^{(1)}_{3,2}\,yZ^{(1)}_{3,1}z\,Z^{(1)}_{2,1}\,x\,v \to Z^{(1)}_{3,2}\,yZ^{(1)}_{3,1}z\,\partial_{2,1}(v) + Z^{(1)}_{3,2}\,y\,Z^{(2)}_{2,1}x\,\partial_{3,2}(v) - Z^{(1)}_{3,2}\,yZ^{(1)}_{3,2}\,y\,\partial^{(2)}_{2,1}(v) - Z^{(1)}_{2,1}\,xZ^{(1)}_{2,1}\,x\,\partial^{(2)}_{3,2}(v),$$

so that under the modified boundary, we can simply define the boundary map on this term to be

$$Z^{(1)}_{3,2}\,yZ^{(1)}_{3,1}z\,Z^{(1)}_{2,1}\,x\,v \to Z^{(1)}_{3,2}\,yZ^{(1)}_{3,1}z\,\partial_{2,1}(v) + Z^{(1)}_{3,2}\,y\,Z^{(2)}_{2,1}x\,\partial_{3,2}(v).$$

Of course it remains to prove that in characteristic zero this is exact; we indicate how to do this in the next subsection.

3.3. **Question of exactness.** Here, we indicate more schematically how we modify the maps to take advantage of divisibility in $\mathbb{Q}$. Divide the terms of our big complex into the sum of those of the Lascoux complex, and the others. That is, we look at the terms of the complex as:

$$\begin{aligned} X_0 &= A_0 \\ X_1 &= A_1 \oplus B_1 \\ X_2 &= A_2 \oplus B_2 \\ X_3 &= A_3 \oplus B_3 \\ X_j &= B_j \quad \text{for j} = 4, 5, \end{aligned}$$

where the As are the sums of the 'Lascoux' terms, and the Bs are the sums of the others.

Let σ_1 be the map

$$B_1 \to A_1$$

defined by

$$Z^{(2)}_{2,1}xv \longmapsto \frac{1}{2}Z^{(1)}_{2,1}x\partial_{2,1}(v) \quad \text{and} \quad Z^{(2)}_{3,2}\,y\,w \longmapsto \frac{1}{2}Z^{(1)}_{3,2}\,y\partial_{3,2}(w)$$

where $v \in D_4 \otimes D_0 \otimes D_2$ and $w \in D_2 \otimes D_4 \otimes D_0$. This situation we've already discussed, but we should point out that the map σ_1 satisfies the identity:

$$\delta_{A_1A_0}\sigma_1 = \delta_{B_1B_0},$$

where by $\delta_{A_1A_0}$ we mean the component of the boundary of the fat complex that carries A_1 to A_0. We'll use notation $\delta_{A_{i+1}A_i}$, $\delta_{A_{i+1}B_i}$ etc. in the same way. We define

$$\partial_1 : A_1 \to A_0$$

as

$$\partial_1 = \delta_{A_1A_0}.$$

But now we're in position to define

$$\partial_2 : A_2 \to A_1$$

by

$$\partial_2 = \delta_{A_2A_1} + \sigma_1\delta_{A_2B_1}.$$

We get immediately

Fact 3.1. *The composition* $\partial_1\partial_2 = 0$.

Proof. We have

$$\begin{aligned}\partial_1\partial_2(a) = \delta_{A_1A_0}\{\delta_{A_2A_1}(a) + \sigma_1\delta_{A_2B_1}(a)\} = \\ \delta_{A_1A_0}\delta_{A_2A_1}(a) + \delta_{A_1A_0}\sigma_1\delta_{A_2B_1}(a).\end{aligned}$$

But since $\delta_{A_1A_0}\sigma_1 = \delta_{B_1B_0}$, we see that

$$\delta_{A_1A_0}\sigma_1\delta_{A_2B_1}(a) = \delta_{B_1B_0}\delta_{A_2B_1}(a),$$

so

$$\delta_{A_1A_0}\delta_{A_2A_1}(a) + \delta_{A_1A_0}\sigma_1\delta_{A_2B_1}(a) = \delta_{A_1A_0}\delta_{A_2A_1}(a) + \delta_{B_1B_0}\delta_{A_2B_1}(a).$$

But this is zero since it is the boundary of the fat complex applied to a. $\square$

Before we can prove exactness at A_1, we have to define a map

$$\sigma_2 : B_2 \to A_2$$

such that

$$(\mathrm{C}_2) \qquad \delta_{\mathrm{B_2A_1}} + \sigma_1\delta_{\mathrm{B_2B_1}} = \{\delta_{\mathrm{A_2A_1}} + \sigma_1\delta_{\mathrm{A_2B_1}}\}\sigma_2.$$

We define this map as follows:

$$\begin{array}{lcl}
Z^{(1)}_{2,1}\, x\, Z^{(1)}_{2,1}\, x\, v & \longmapsto & 0 \\
Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,2}\, y\, v & \longmapsto & 0 \\
Z^{(1)}_{3,2}\, y\, Z^{(3)}_{2,1}\, x\, v & \longmapsto & \frac{1}{3}\{Z^{(1)}_{3,2}\, y\, Z^{(2)}_{2,1}\, x\, \partial_{2,1}(v)\} \\
Z^{(2)}_{3,2}\, y\, Z^{(3)}_{2,1}\, x\, v & \longmapsto & \frac{1}{3}\{Z^{(1)}_{3,2}\, y\, Z^{(2)}_{2,1}\, x\, \partial_{3,1}(v) - Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, \partial^{(2)}_{2,1}(v)\} \\
Z^{(2)}_{3,2}\, y\, Z^{(4)}_{2,1}\, x\, v & \longmapsto & -\frac{1}{2}\{Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, \partial^{(3)}_{2,1}(v)\}
\end{array}$$

where the element v is in the appropriate tensor product of divided powers as given in the description of the fat complex.

Fact 3.2. *The map* σ_2 *defined above satisfies the condition* (C_2).

Proof. Trivial. □

Fact 3.3. *We have exactness at* A_1*.*

Proof. This is just a diagram chase. □

Using σ_2 we can now also define

$$\partial_3 : A_3 \to A_2$$

by

$$\partial_3 = \delta_{A_3A_2} + \sigma_2\delta_{A_3B_2}.$$

Not too surprisingly, we have

Fact 3.4. $\partial_2\partial_3 = 0$.

Of course, what we need now is a map $\sigma_3 : B_3 \to A_3$ similar to the σ's above, i.e., satisfying

$$(\mathrm{C}_3) \qquad \delta_{\mathrm{B_3A_2}} + \sigma_2\delta_{\mathrm{B_3B_2}} = \{\delta_{\mathrm{A_3A_2}} + \sigma_2\delta_{\mathrm{A_3B_2}}\}\sigma_3.$$

We define such a σ_3 as follows:

$$\begin{array}{lcl}
Z^{(1)}_{3,2}\, y\, Z^{(2)}_{2,1}\, x\, Z^{(1)}_{2,1}\, x\, v & \longmapsto & 0 \\
Z^{(2)}_{3,2}\, y\, Z^{(3)}_{2,1}\, x\, Z^{(1)}_{2,1}\, x\, v & \longmapsto & \frac{1}{3}\{Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, Z^{(1)}_{2,1}\, x\, \partial^{(2)}_{2,1}(v)\} \\
Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,2}\, y\, Z^{(3)}_{2,1}\, x\, v & \longmapsto & -\frac{1}{3}\{Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, Z^{(1)}_{2,1}\, x\, \partial^{(1)}_{2,1}(v)\} \\
Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,2}\, y\, Z^{(4)}_{2,1}\, x\, v & \longmapsto & -\frac{1}{3}\{Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, Z^{(1)}_{2,1}\, x\, \partial^{(2)}_{2,1}(v)\} \\
Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, Z^{(1)}_{2,1}\, x\, v & \longmapsto & \frac{1}{3}\{Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, Z^{(1)}_{2,1}\, x\, \partial^{(1)}_{2,1}(v)\} \\
Z^{(1)}_{3,2}\, y\, Z^{(1)}_{3,1}\, z\, Z^{(3)}_{2,1}\, x\, v & \longmapsto & 0
\end{array}$$

where the element v again is taken in the appropriate tensor product of divided powers as described earlier.

Fact 3.5. *The map* σ_3 *defined above satisfies the condition* (C_3)*.*

Proof. Trivial. □

We have yet to prove that we have exactness at A_2 and that ∂_3 is a monomorphism. But first we explicitly define the boundary maps in the complex

$$0 \to A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

The map ∂_1 is clear: it's just the operation of the indicated polarization operators on the argument. The map ∂_2 is defined as:

$$\partial_2\left(Z^{(1)}_{3,2}\, yZ^{(2)}_{2,1}xv\right) = Z^{(1)}_{3,2}\, y\, \partial^{(2)}_{2,1}(v) - Z^{(1)}_{2,1}\, x\, \partial^{(1)}_{3,1}(v) - \frac{1}{2}Z^{(1)}_{2,1}\, x\, \partial^{(1)}_{2,1}\partial^{(1)}_{3,2}(v);$$

$$\partial_2\left(Z^{(1)}_{3,2}\, yZ^{(1)}_{3,1}\, z\, v\right) = Z^{(1)}_{3,2}\, y\, \partial^{(1)}_{3,1}(v) - \frac{1}{2}Z^{(1)}_{3,2}\, y\, \partial^{(1)}_{3,2}\partial^{(1)}_{2,1}(v) + Z^{(1)}_{2,1}\, x\, \partial^{(2)}_{3,2}(v).$$

And finally, the map ∂_3 is defined as:

$$\partial_3\left(Z_{3,2}^{(1)}\,yZ_{3,1}^{(1)}\,z\,Z_{2,1}^{(1)}\,x\,v\right)=Z_{3,2}^{(1)}\,yZ_{3,1}^{(1)}\,z\,\partial_{2,1}^{(1)}(v)+Z_{3,2}^{(1)}\,yZ_{2,1}^{(2)}\,x\,\partial_{3,2}^{(1)}(v).$$

Since one component of the map ∂_3 is a diagonalization of D_2 into $D_1 \otimes D_1$, it is clear that ∂_3 is injective. Hence, the only remaining exactness to prove is at A_2. To handle this, we suppose that we have $V \in A_1$ with $\partial_2(V) = 0$. We want to produce an element $W \in A_3$ such that $\partial_3(W) = V$.

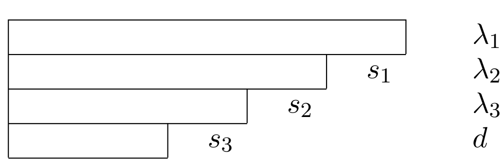
It is easy to see that we can add to V an element $W \in B_2$, such that under the boundary of the characteristic-free complex, $V + W$ goes to zero, while $\sigma_2(W) = 0$. Using the fact that this 'fat' complex is acyclic, we know that there are elements $\alpha \in A_3, \beta \in B_3$ such that $\alpha+\beta$ goes to $V+W$ under the 'fat' boundary map. It is then easy to show that $\alpha + \sigma_3(\beta)$ goes to V, and our Lascoux 'reduction' is exact.

4. Intertwining numbers

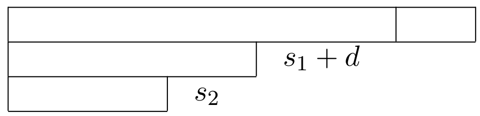
In this last section, we give a brief indication of the beginnings of a place-polarization approach to calculating intertwining numbers. A complete description of the zero-dimensional intertwining number in the three-rowed case is given in [5]. This was based on an approach already suggested in [2], but in trying to proceed to partitions having four or more rows, the calculations became very unwieldy. These calculations involve elaborate straightenings of non-standard tableaux, so it seemed worthwhile to seek another method that would cut down on the straightenings that must be performed.

In [8], we had made extensive use of place polarizations, and had made an explicit correspondence between the algebra of such polarizations and the Heisenberg-Weyl algebra. The new idea in attacking the intertwining numbers is to express some of the basic targets of our investigation in terms of place polarizations, and to then simplify our problem by proving some identities in the Heisenberg-Weyl algebra. This investigation is just now under way, so we will simply give a little example of the sort of thing we're talking about.

Let us start with a four-rowed tableau λ of the form



and manufacture from it the three-rowed tableau μ:



$$\mu_1 = \lambda_1 + d$$
$$\mu_2 = \lambda_2$$
$$\mu_3 = \lambda_3$$

Our aim is to compute the abelian group $\mathrm{Ext}^1_A(K_\lambda, K_\mu)$, where A is the Schur algebra of appropriate degree. If we let $\square$ denote the presentation map of the partition λ, it is made up of place polarizations $\partial_{21}^{(l)}, \partial_{32}^{(l)}$ or $\partial_{43}^{(l)}$ for a range of positive integers l. Since the group we want to find is determined by the invariant factors of the matrix associated to the map $\mathrm{Hom}_A(\square, K_\mu)$ (see [5]), and since the domain and range of $\square$ consists of sums of tensor products of divided powers, this amounts to calculating a basis for certain weight submodules of K_μ, and then making explicit the entries of the matrix of the map above with respect to these bases.

The weight submodule of K_μ corresponding to weight λ is parametrized by three parameters $3_1, 4_1, 4_2$, where n_i indicates the number of n's in row i. The homomorphism corresponding to these parameters, $\alpha^{3_1,4_1,4_2}$, may be described as the composition of place polarizations:

$$\alpha^{3_1,4_1,4_2} = \partial_{34}^{(d-4_1-4_2)}\partial_{24}^{(4_2)}\partial_{14}^{(4_1)}\partial_{23}^{(d-3_1-4_1-4_2)}\partial_{13}^{(3_1)}\partial_{12}^{(d-3_1-4_1)}.$$

The only restrictions on the parameters are that

$$3_1 + 4_1 + 4_2 \leq d$$

and

$$4_2 \leq s_2.$$

We see immediately that:

$$\begin{array}{ll} 2_1 = d - 3_1 - 4_1 & 2_2 = s_3 + s_2 + 3_1 + 4_1 \\ 3_2 = d - 3_1 - 4_1 - 4_2 & 3_3 = s_3 + 4_1 + 4_2 \\ 4_3 = d - 4_1 - 4_2. & \end{array}$$

The idea of using the place polarizations is that composing the alphas with $\square$ just amounts to composing the given composite of place polarizations with either $\partial_{21}^{(l)}, \partial_{32}^{(l)}$ or $\partial_{43}^{(l)}$ for some positive integer l. Since all of these polarizations are taking place at the level of the generators of the Weyl modules involved, that is, in the tensor product of appropriate divided powers, we want to throw away those terms that are obviously in the kernel of the Weyl map defining K_μ. But those are any terms which are composites of place polarizations that have either a $\partial_{21}^{(l)}$ or $\partial_{32}^{(l)}$ at the extreme left end of the composites. Thus when we take an alpha and compose it with one of the $\partial_{21}^{(l)}, \partial_{32}^{(l)}$ or $\partial_{43}^{(l)}$, we want to use the Heisenberg-Weyl identities to bring these all the way to the left, if we can.

That is the general strategy behind the simplification, and there is some evidence that this can be carried out. At least, this is a program on which a group of us are currently engaged.[1] As a more detailed description of the fragments we have so far obtained would take more space than would be justified by the results, we will forgo giving one here.

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