ON RAMIFICATION THEORY IN NOETHERIAN RINGS.

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Introduction. The purpose of this paper is to give a fairly general ramification theory for noetherian rings. To illustrate the types of results obtained let us assume that $S$ is a noetherian ring and $R$ is a subring of $S$ such that the kernel $J$ of the mapping $\phi: S \otimes R S \to S$ defined by $\phi(x \otimes y) = xy$ is a finitely generated ideal in $S \otimes R S$. In §2 we show that $S$ is an unramified extension of $R$ (see §1 for definition) if and only if $S$ is a projective $S \otimes R S$-module, or equivalently, if and only if $J$ is a direct summand of $S \otimes R S$. From this it follows that if $\mathfrak{J} = \phi(\mathfrak{H})$, where $\mathfrak{H}$ is the annihilator of $J$ in $S \otimes S$, then a prime ideal $\mathfrak{P}$ in $S$ is ramified if and only if it contains $\mathfrak{J}$. We call $\mathfrak{J}$ the homological different of $S$ over $R$.

The main object of §3 is to show that if $R$ is a noetherian integrally closed domain with field of quotients $K$, $L$ a finite field extension of $K$ and $S$ an integral extension of $R$ in $L$, then $\mathfrak{J}$ is contained in $\mathfrak{D}_{S/R}$, where $\mathfrak{D}_{S/R}$ is the usual different defined using the trace mapping on $L$. There are examples which show that $\mathfrak{J} \neq \mathfrak{D}_{S/R}$ in general. However, in the case that $S$ is $R$-projective, we have that $\mathfrak{J} = \mathfrak{D}_{S/R}$. As an application we prove that if $R$ is a regular local ring of dimension less than or equal to two and $L$ is a separable field extension, then $S$ is unramified over $R$ if and only if each minimal prime ideal is unramified over $R$. This result has been obtained independently by J.-P. Serre (unpublished) and has been used by M. Nagata to prove the theorem in general i.e. for regular local rings of arbitrary dimension [7].

§4 is devoted to showing that under various conditions if $S$ is unramified over $R$, then $S$ is $R$-projective. For instance, this is the case if $R$ is a noetherian, integrally closed domain and $S$ is an integral extension of $R$ in a finite, separable field extension of the field of quotients of $R$.

§5 is devoted to certain homological considerations and is the only place in the paper where any homology theory is used. Perhaps the most striking result obtained here is that if $S$ is unramified over $R$ and $R$-projective and $T$
is a regular ring of finite Krull dimension (i.e. $T$ is a noetherian ring and every local ring of $T$ is a regular local ring) containing $R$, then $S \otimes_R T$ is regular if it is noetherian.

It has recently been brought to our attention that E. Noether had considered the ideals $\mathfrak{J}$, $\mathfrak{M}$ and $\mathfrak{S}_{S/R}$ from a somewhat different point of view in [5].

1. **Notation and terminology.** All rings to be considered in this paper will be assumed commutative with identity element and all modules will be unitary. A ring $S$ together with a ring homomorphism $f: R \to S$ such that $f(1) = 1$ will be called an $R$-algebra. Ideals in $S$ will be denoted by capital German letters and ideals in $R$ by lower case German letters. If $\mathfrak{M}$ is an ideal in $S$, then we call the ideal $f^{-1}(\mathfrak{M})$ in $R$ the contraction of $\mathfrak{M}$ and denote it by $\mathfrak{M} \cap R$. We shall consider $R/\mathfrak{M} \cap R$ to be a subring of $S/\mathfrak{M}$, the identification being given by the monomorphism induced by $f$. If $x \in R$, then we denote $f(x)$ in $S$ by $x$, provided there is no danger of confusion. Thus if $a$ is an ideal in $R$, we shall denote the ideal $S \cdot f(a)$ by $S \cdot a$. It is clear that $(S \cdot a) \cap R$ contains $a$.

A subset $U$ of a ring is called a multiplicative system if a) whenever $u_1$ and $u_2$ are in $U$, the product $u_1u_2$ is in $U$, and b) 0 is not in $U$; 1 is in $U$.

The only multiplicative systems in $R$ that we shall consider are those that do not meet the kernel of $f$. Suppose $U$ is a multiplicative system in $R$ and $U'$ is a multiplicative system in $S$ containing $U$ (i.e. $U'$ contains $f(U)$). Then $f: R \to S$ induces a homomorphism $f': R_U \to S_{U'}$ which makes $S_{U'}$ an $R_U$-algebra. In particular, if $\mathfrak{B}$ is a prime ideal in $S$ and $\mathfrak{p} = \mathfrak{B} \cap R$, then $S_\mathfrak{B}$ is an $R_\mathfrak{p}$-algebra and $R_\mathfrak{p}/\mathfrak{p} R_\mathfrak{p}$ is a subfield of $S_\mathfrak{B}/\mathfrak{B} S_\mathfrak{B}$.

We define the $R$-algebra $S_e$ to be the ring $S \otimes_R S$ with the mapping of $R$ into $S_e$ being given by $r \mapsto r \otimes 1$. $S_e$ is called the enveloping algebra of the $R$-algebra $S$. The map $\phi: S_e \to S$ defined by $\phi(x \otimes y) = xy$ is an $R$-algebra epimorphism. Thus all $S$-modules may be considered as $S_e$-modules. The kernel of $\phi$ will be denoted by $\mathfrak{J}$ which is the ideal generated by $\{1 \otimes x - x \otimes 1\}$, and the annihilator of $\mathfrak{J}$ will be denoted by $\mathfrak{N}$. If $U$ is a multiplicative system, we denote by $U \otimes U$ the multiplicative system in $S_e$ consisting of all elements of the form $u_1 \otimes u_2$ where $u_1$, $u_2$ are in $U$. The map $(S_U)^e \to (S_e)_{U \otimes U}$ defined by $x/u_1 \otimes y/u_2 \to x \otimes y/u_1 \otimes u_2$ is an $R$-algebra isomorphism which we shall use to identify these algebras. If $E$ is an $S$-module, then $E_U = E_{U \otimes U}$. Finally, if $V$ is a multiplicative system in $R$ such that $V$ is contained in $U$, then $S_U \otimes_R S_U = S_U \otimes_R V S_U$.

A prime ideal $\mathfrak{B}$ in the $R$-algebra $S$ is said to be unramified if $\mathfrak{p} = R \cap \mathfrak{B}$ has the following properties:
a) \( pS_B = \mathfrak{B}S_B \);

b) \( S_B/pS_B \) is a separable field extension of \( R_p/pR_p \).\(^2\)

The \( R \)-algebra \( S \) is said to be unramified if

a) every prime ideal in \( S \) is unramified;

b) for each prime ideal \( p \) in \( R \) there are only a finite number of prime ideals \( \mathfrak{B} \) in \( S \) such that \( p = \mathfrak{B} \cap R \).

In other words, \( S \) is unramified if and only if given any prime ideal \( p \) in \( R \) which is the contraction of a prime ideal in \( S \) and \( U = R - p \), then \( S_U/pS_U \) is a separable \( R_U/pR_U \)-algebra.

2. Ramification criteria.

**Lemma 2.1.** If \( S \) is an \( R \)-algebra, the following statements are equivalent:

a) \( S \) is \( S^e \)-projective;

b) the exact sequence \( 0 \rightarrow \mathfrak{g} \rightarrow S^e \xrightarrow{\phi} S \rightarrow 0 \) splits;

c) there is an element \( z \) in \( S^e \) such that \( z(1 \otimes x) = z(x \otimes 1) \) for all \( x \in S \), and \( \phi(z) = 1 \);

d) \( \phi(\mathfrak{N}) = S \).

Further, if \( R \) is a field, then \( S \) is \( S^e \)-projective if and only if \( S \) is a separable \( R \)-algebra.

**Proof.** Clearly a) implies b). If b) holds, there is an \( S^e \)-homomorphism \( p: S \rightarrow S^e \) such that \( \phi p \) is the identity. Let \( z = p(1) \). It is easy to see that \( z \) is the desired element to make c) hold, so b) implies c).

Now suppose that c) is true. Then \( z \) is in \( \mathfrak{N} \) so that \( \phi(\mathfrak{N}) \) contains \( \phi(z) = 1 \). Hence \( \phi(\mathfrak{N}) = S \). Thus c) implies d).

If d) holds, there is a \( z \) in \( \mathfrak{N} \) such that \( \phi(z) = 1 \). Define \( p: S \rightarrow S^e \) by \( p(s) = sz \). Since \( z \) is in \( \mathfrak{N} \), \( p \) is an \( S^e \)-homomorphism so that b) holds. Clearly b) implies a) so that d) implies a).

The last statement in the theorem is found in [6, Theorem 1].

**Proposition 2.2.** If the \( R \)-algebra \( S \) is \( S^e \)-projective, then \( S \) is unramified.

\(^2\)If \( R \) is a field, an \( R \)-algebra \( S \) is said to be separable if \( S \) is a finite-dimensional \( R \)-algebra which is a direct sum of separable field extensions of \( R \).
Proof. We must show that if \( p \) is any prime ideal in \( R \) where \( p = \mathfrak{p} \cap R \) for some prime \( \mathfrak{p} \subset S \) and \( U = R - \mathfrak{p} \), then \( S_{U}/pS_{U} \) is a separable \( R_{U}/pR_{U} \)-algebra. Since \( R_{U}/pR_{U} \) is a field, we need only show that \( S_{U}/pS_{U} \) is \( S_{U}/pS_{U} \otimes S_{U}/pS_{U} \)-projective, where the tensor product is taken over \( R_{U}/pR_{U} \).

Since \( S \) is \( S^{e} \)-projective, there is a \( z \) in \( S^{e} \) such that \( \phi(z) = 1 \) and \( z(x \otimes 1) = z(1 \otimes x) \) for all \( x \in S \). We also have the commutative diagram

\[
\begin{array}{cccc}
S \otimes_{R} S & \overset{f}{\longrightarrow} & S_{U} \otimes_{R_{U}} S_{U} & \overset{g}{\longrightarrow} & S_{U}/pS_{U} \otimes_{R_{U}} S_{U}/pS_{U} \\
\phi & \downarrow & \phi' & \downarrow & \phi'' \\
S & \longrightarrow & S_{U} & \longrightarrow & S_{U}/pS_{U}
\end{array}
\]

where the maps are the obvious ones. Let \( z'' = gf(z) \). Then it is clear that \( \phi''(z'') = 1 \) and that \( z'' \) is in the annihilator of the kernel of \( \phi'' \). But since \( S_{U}/pS_{U} \otimes_{R_{U}} S_{U}/pS_{U} = S_{U}/pS_{U} \otimes_{F} S_{U}/pS_{U} \) (where \( F = R_{U}/pR_{U} \)) we have by 2.1 that \( S \) is unramified.

**Proposition 2.3.** Let \( R \) be an integral domain with field of quotients \( K \). If \( S \) is an \( R \)-algebra such that \( S \) is \( S^{e} \)-projective, then \( S \otimes_{R} K \) is a separable \( K \)-algebra. In particular, if \( S \) is an integral domain, then its field of quotients is a separable field extension of \( K \).

**Proof.** It is easy to see that condition c) of 2.1 holds for the \( K \)-algebra \( S \otimes_{R} K \) since it holds for \( S \). If \( S \) is an integral domain, then \( S \otimes_{R} K \) is also and is therefore the field of quotients of \( S \). Hence the second part of the proposition is true.

**Proposition 2.4.** Let \( S \) be a noetherian ring which is an \( R \)-algebra such that every maximal ideal in \( S \) is unramified. Then any \( R \)-derivation \( D \) of \( S \) into a finitely generated \( S \)-module \( E \) is zero.

**Proof.** Let \( \mathfrak{m} \) be a maximal ideal of \( S \) and \( m = \mathfrak{m} \cap R \). Denote by \( E_{\mathfrak{m}} \) the \( S_{\mathfrak{m}} \)-module \( E \otimes_{S} S_{\mathfrak{m}} \) and by \( D_{\mathfrak{m}} : S_{\mathfrak{m}} \rightarrow E_{\mathfrak{m}} \) the derivation induced by \( D : S \rightarrow E \).

Since \( \mathfrak{m} \) is unramified, we have \( mS_{\mathfrak{m}} = \mathfrak{m}S_{\mathfrak{m}} \). Since \( D \) is a derivation over \( R \), \( D_{\mathfrak{m}} \) is a derivation over \( R_{\mathfrak{m}} \) so that \( D_{\mathfrak{m}}(mS_{\mathfrak{m}}) \) is contained in \( mE_{\mathfrak{m}} = \mathfrak{m}E_{\mathfrak{m}} \). Therefore \( D_{\mathfrak{m}} \) induces a derivation

\[
D_{\mathfrak{m}} : S_{\mathfrak{m}}/\mathfrak{m}S_{\mathfrak{m}} \rightarrow E_{\mathfrak{m}}/\mathfrak{m}E_{\mathfrak{m}}
\]

over \( R_{\mathfrak{m}}/mR_{\mathfrak{m}} \). But since \( \mathfrak{m} \) is unramified, \( S_{\mathfrak{m}}/\mathfrak{m}S_{\mathfrak{m}} \) is a separable extension of \( R_{\mathfrak{m}}/mR_{\mathfrak{m}} \) so that \( D_{\mathfrak{m}} = 0 \). Hence \( D_{\mathfrak{m}}(S_{\mathfrak{m}}) \) is contained in \( \mathfrak{m}E_{\mathfrak{m}} \).
Iterating this argument, we have that $D_{\mathfrak{M}}(S_{\mathfrak{M}})$ is contained in $\cap \mathfrak{M}^4 E_{\mathfrak{M}} = 0$ so that $D_{\mathfrak{M}} = 0$.

Since $D_{\mathfrak{M}} = 0$ for every maximal ideal $\mathfrak{M}$ of $S$, it follows easily that $D = 0$.

**Theorem 2.5.** Let $S$ be a noetherian $R$-algebra such that $\mathfrak{J}$ is a finitely generated ideal in $S^e$. Then the following statements are equivalent:

a) $S$ is $S^e$-projective;

b) $S$ is unramified;

c) Every maximal ideal in $S$ is unramified;

d) Every $R$-derivation of $S$ into a finitely generated $S$-module is zero.

**Proof.** We have already shown in 2.2 that a) implies b) and clearly b) implies c). Proposition 2.4 shows that c) implies d). Hence we need only prove that d) implies a).

To show that $S$ is $S^e$-projective, it suffices to show that the exact sequence

$$
0 \to \mathfrak{J} \to S^e \to S \to 0
$$

(1)

splits.

Observe first that for any $S$-module $E$, $\text{Hom}_S(\mathfrak{J}/\mathfrak{J}^2, E)$ is isomorphic to the group of $R$-derivations of $S$ into $E$. Letting $E = \mathfrak{J}/\mathfrak{J}^2 = S \otimes_{S^e} \mathfrak{J}$, and observing that $\mathfrak{J}/\mathfrak{J}^2$ is a finitely generated $S$-module (since $\mathfrak{J}$ is assumed to be finitely $S^e$-generated), we have $\text{Hom}_S(\mathfrak{J}/\mathfrak{J}^2, \mathfrak{J}/\mathfrak{J}^2) = 0$ and hence $\mathfrak{J}/\mathfrak{J}^2 = 0$, i.e. $\mathfrak{J} = \mathfrak{J}^2$. Since $\mathfrak{J}$ is finitely generated, there is a $y_0$ in $\mathfrak{J}$ such that $x = y_0 x$ for each $x \in \mathfrak{J}$.

Now define a map $p : S^e \to \mathfrak{J}$ by letting $p(1) = y_0$. Then for all $x \in \mathfrak{J}$, we have $p(x) = xp(1) = xy_0 = x$. Thus the sequence (1) above splits and we are done.

**Lemma 2.6.** Let $S$ be a ring, $E$ a finitely generated $S$-module, and $\mathfrak{A}$ the annihilator of $E$. If $U$ is a multiplicative system in $S$, then the annihilator in $S_U$ of $E_U$ is $\mathfrak{A}_U$.

**Proof.** It is obvious that $\mathfrak{A}_U$ is contained in the annihilator of $E_U$. Suppose, then, that $(s/u)E_U = 0$. Let $E$ be generated by $e_1, \ldots, e_n$. Then $E_U$ is also generated by $e_1, \ldots, e_n$ and we have $(s/u)e_i = 0$ for $i = 1, \ldots, n$. But then there is a $u'$ in $U$ such that $u'se_i = 0$ for $i = 1, \ldots, n$ so that $u's$ is in $\mathfrak{A}$. Hence $s$ is in $\mathfrak{A}_U$ and so is $s/u$.

We now define the homological different of the $R$-algebra $S$ to be the ideal $\phi(\mathfrak{N})$ in $S$, and denote it by $\mathcal{S}_{S/R}$. 
Theorem 2.7. Let $S$ be a noetherian $R$-algebra such that $\mathfrak{J}$ in $S^e$ is a finitely generated ideal. A prime ideal $\mathfrak{P}$ in $S$ is unramified if and only if $\mathfrak{P}$ does not contain $\mathfrak{S}_{S/R}$.

Proof. Let $\mathfrak{P}$ be a prime ideal in $S$, $\mathfrak{p} = \mathfrak{P} \cap R$, $U = S - \mathfrak{P}$, $V = R - \mathfrak{p}$. Then $S_U$ is an $R_V$-algebra, $(S^e)_V \otimes_U S_U = S_U \otimes_R S_U = S_U \otimes_{R_V} S_U$, the kernel of map $\tilde{\phi} : S'_V \rightarrow S_U$ is $\mathfrak{J} U \otimes_U$, and $U \otimes_U$ is the annihilator of $\mathfrak{J} U \otimes_U$. Moreover, $S_U$ is a noetherian $R_V$-algebra and $\mathfrak{J} U \otimes_U$ is a finitely generated ideal in $(S_U)^e$.

Suppose $\mathfrak{P}$ does not contain $\mathfrak{S}_{S/R}$. Then $(\mathfrak{S}_{S/R})_U = S_U = \tilde{\phi}(U \otimes_U)$ and so, by 2.1, $S_U$ is a $(S_U)^e$-projective. But then, by 2.5, $S_U$ is an unramified $R'$-algebra. It is thus easy to see that $\mathfrak{P}$ is unramified.

Now suppose that $\mathfrak{P}$ is unramified. Then $\mathfrak{P} S_U$ is unramified over $R_V$ and again by 2.5 (since $\mathfrak{P} S_U$ is the only maximal ideal in $S_U$), $S_U$ is a $(S_U)^e$-projective. Thus by 2.1, $\tilde{\phi}(U \otimes_U) = S_U$. Since $\tilde{\phi}(U \otimes_U) = (\mathfrak{S}_{S/R})_U$, we have $(\mathfrak{S}_{S/R})_U = S_U$ which implies that $\mathfrak{P}$ does not contain $\mathfrak{S}_{S/R}$.

Corollary 2.8. Let $R$ be an integral domain with field of quotients $K$. If $S$ is a noetherian $R$-algebra such that $\mathfrak{J}$ is a finitely generated ideal in $S^e$, and $S \otimes_R K$ is not a separable $K$-algebra, then every prime ideal in $S$ is ramified.

Lemma 2.9. If $S$ is a noetherian ring, and $E$ is a finitely generated $S$-module, then $\bigcap_{\mathfrak{M}} (\bigcap_{\mathfrak{M}} \mathfrak{M}^i E) = 0$, where $\mathfrak{M}$ runs through all maximal ideals of $S$.

Proof. Let $e$ be an element of $\bigcap_{\mathfrak{M}} (\bigcap_{\mathfrak{M}} \mathfrak{M}^i E)$, and let $\mathfrak{A}$ be the annihilator of $e$. If $e \neq 0$, then $\mathfrak{A}$ is a proper ideal of $S$ and so is contained in some maximal ideal $\mathfrak{M}$. Since $e$ is in $\bigcap_{\mathfrak{M}} \mathfrak{M}^i E$, there is an $m$ in $\mathfrak{M}$ such that $(1 - m)e = 0$. However, $(1 - m)$ is not in $\mathfrak{A}$ so that $e$ must be zero.

Proposition 2.10. Let $S$ be a noetherian $R$-algebra, and $\mathfrak{J}$ finitely generated in $S^e$. Then $S$ is unramified if and only if for every maximal ideal $\mathfrak{M}$ of $S$ every $R$-derivation $D : S \rightarrow S/\mathfrak{M}$ is zero.

Proof. Proposition 2.4 shows us that if $S$ is unramified then every $R$-derivation $D : S \rightarrow S/\mathfrak{M}$ is zero. To prove the converse, it is sufficient (by 2.5) to show that every $R$-derivation $D : S \rightarrow E$ is zero, where $E$ is any finitely generated $S$-module.

If $D : S \rightarrow E$ is an $R$-derivation, $E$ finitely generated, and $\mathfrak{M}$ a maximal ideal of $S$, then $E/\mathfrak{M}E$ is a finite-dimensional vector space over $S/\mathfrak{M}$ and so the derivation $\bar{D} : S \rightarrow E/\mathfrak{M}E$ is zero, where $\bar{D}$ is the composition
\[ D \rightarrow E \rightarrow E/ME. \] Therefore \( D(S) \) is contained in \( ME \). Iterating this (i.e. observing that \( ME/ME^{i+1}E \) is a finite-dimensional vector space over \( S/\mathcal{M} \)) we see that \( D(S) \) is contained in \( \bigcap ME^{i}E \). Since this is so for all maximal ideals \( \mathcal{M} \) of \( S \), we have \( D(S) \) is contained in \( \bigcap \mathcal{M}^{i}E \) = 0 (by 2.9). Hence \( D = 0 \).

**Proposition 2.11.** Let \( S \) and \( T \) be \( R \)-algebras such that \( \phi \) is \( \mathfrak{J} = \ker \)

\[ (S^{e} \rightarrow S) \text{ is finitely generated in } S^{e} \text{ and } S \text{ is noetherian. If } S \text{ is unramified, then } S \otimes_{R} T \text{ is unramified as a } T\text{-algebra. If, further, } T \text{ is unramified, then } S \otimes_{R} T \text{ is unramified as an } R\text{-algebra.} \]

**Proof.** Since \( S \) is unramified, we have by 2.5 that the exact sequence

\[ 0 \rightarrow \mathfrak{J} \rightarrow S^{e} \rightarrow S \rightarrow 0 \]
splits. Therefore the sequence

\[ 0 \rightarrow \mathfrak{J} \otimes_{R} T \rightarrow S^{e} \otimes_{R} T \rightarrow S \otimes_{R} T \rightarrow 0 \]
is exact and splits. But \( S^{e} \otimes_{R} T = (S \otimes_{R} T) \otimes_{T}(S \otimes_{R} T) = (S \otimes_{R} T)^{e} \) as a \( T \)-algebra. Hence \( S \otimes_{R} T \) is \( (S \otimes_{R} T)^{e} \)-projective. Thus, by 2.2, it follows that \( S \otimes_{T} T \) is an unramified \( T \)-algebra.

If, in addition, \( T \) is an unramified \( R \)-algebra, then it easily follows from the fact that \( S \otimes_{R} T \) is an unramified \( T \)-algebra, that \( S \otimes_{R} T \) is an unramified \( R \)-algebra.

**3. The homological different and different.** Throughout this section, \( R \) will be an integral domain with field of quotients \( K \), \( L \) a finite-dimensional \( K \)-algebra, and \( S \) a subring of \( L \) containing \( R \) such that \( S \otimes_{R} K = L \). We shall denote \( \text{Hom}_{R}(S, R) \) by \( S^{*} \) and \( \text{Hom}_{K}(L, K) \) by \( L^{*} \).

We define the map \( \tau: S \otimes_{R} S \rightarrow \text{Hom}_{R}(S^{*}, S) \) to be \( \tau(x \otimes y)(f) = xf(y) \) for \( f \) in \( S^{*} \). By [3; VI, 5.2] if \( S \) is a projective and finitely generated \( R \)-module, then \( \tau \) is an isomorphism. In particular, \( \sigma: L \otimes_{K} L \rightarrow \text{Hom}_{K}(L^{*}, L) \) which is similarly defined, is an isomorphism.

Since \( S \otimes_{R} K = L \), every element of \( S^{*} \) is uniquely extendable to an element of \( L^{*} \). If \( S^{*} \) generates all of \( L^{*} \) over \( K \), then we have a natural map \( \rho: \text{Hom}_{R}(S^{*}, S) \rightarrow \text{Hom}_{K}(L^{*}, L) \). We therefore obtain the following diagram which is easily shown to be commutative:
\[ \text{Hom}_R(S^*, S) \longrightarrow \text{Hom}_R(L^*, L) \]
\[ S \otimes_R S \longrightarrow \text{Hom}_R(S^*, S) \longrightarrow \text{Hom}_R(L^*, L) \overset{\sigma^{-1}}{\longrightarrow} L \otimes_K L \]
\[ \phi \quad \phi' \]
\[ S \longrightarrow L. \]

If \( f \) is in \( \text{Hom}_L(L^*, L) \), it can be seen by standard techniques of linear algebra that \( \phi' \sigma^{-1}(f) = f(\text{Tr}) \), where \( \text{Tr}: L \to K \) is the trace map.

An explicit description of \( \rho \) and of \( \phi' \sigma^{-1} \rho \) can be given as follows: let \( v^1, \cdots, v^n \) be elements of \( S^* \) which form a basis for \( L^* \) over \( K \). Then for \( f \) in \( \text{Hom}_R(S^*, S) \) we have \( \rho(f) \) defined by \( \rho(f)(v^i) = f(v^i) \). If we let \( v_1, \cdots, v_n \) be the basis of \( L \) over \( K \) dual to \( v^1, \cdots, v^n \) and set \( v_i = u_i/r_0 \) with \( u_i \) in \( S \), \( r_0 \) in \( R \) (using the fact that \( S \otimes_R K = L \)), we can easily see that every element \( s \) of \( S \) can be written \( s = \sum r_i v_i \) with \( r_i \) in \( R \) and that \( \text{Tr}(r_0 S) \) is contained in \( R \). Thus if we let \( T': S \to R \) be the restriction of \( r_0 \text{Tr} \) to \( S \), we have for \( f \) in \( \text{Hom}_S(S^*, S) \) that \( \phi' \sigma^{-1} \rho(f) = (1/r_0) f(T') \). If \( \text{Tr}(S) \) were contained in \( R \) (e.g. if \( S \) were integral over \( R \)) then \( \phi' \sigma^{-1} \rho(f) = f(T') \) where \( T' \) is the restriction of \( \text{Tr} \) to \( S \).

We now define the complementary module, \( \mathfrak{C}_{S/R} \), and the different, \( \mathfrak{D}_{S/R} \), as follows:

\[ \mathfrak{C}_{S/R} = \{ x \text{ in } L/\text{Tr}(x S) \text{ is contained in } R \} \]
\[ \mathfrak{D}_{S/R} = \{ x \text{ in } L/x \mathfrak{C}_{S/R} \text{ is contained in } S \}. \]

From the above remarks, we can see that \( \phi' \sigma^{-1} \rho(\text{Hom}_S(S^*, S)) \) is contained in \( \mathfrak{D}_{S/R} \). For suppose \( x \) is in \( \mathfrak{C}_{S/R} \) and \( f \) is in \( \text{Hom}_S(S^*, S) \). Then \( x[(1/r_0)f(T')] = x(\rho(f)(\text{Tr})) = \rho(f)(\text{Tr} \circ x) \), where \( \text{Tr} \circ x: L \to K \) is defined by \( \text{Tr} \circ x(y) = \text{Tr}(xy) \) for \( y \) in \( L \). Since \( \text{Tr} \circ x \) restricted to \( S \) maps \( S \) into \( R \), \( \rho(f)(\text{Tr} \circ x) \) is in \( S \). Therefore \( x \cdot \phi' \sigma^{-1} \rho(f) \) is in \( S \) for all \( x \) in \( \mathfrak{C}_{S/R} \).

Now let us make \( \text{Hom}_R(S^*, S) \) an \( S^e \)-module by defining \( (x \otimes y)f(g) = x \cdot f(g \circ y) \) for \( x, y \) in \( S \), \( f \) in \( \text{Hom}_R(S^*, S) \), \( g \) in \( S^* \). Then \( \tau \) is an \( S^e \)-homomorphism. Furthermore, \( \text{Hom}_S(S^*, S) \) is equal to the set of all \( f \) in \( \text{Hom}_R(S^*, S) \) such that \( \mathfrak{I}f = 0 \). Thus, since \( \mathfrak{N} = \text{annihilator of } \mathfrak{J} \text{ in } S^e \), we have \( \tau(\mathfrak{N}) \) is contained in \( \text{Hom}_S(S^*, S) \).

We can go even a little further. Let

\[ \mathfrak{N} = \text{Ker}(S \otimes_R S \to L \otimes_K L = S \otimes_R S \otimes_R K), \]

and let \( \mathfrak{A} \) in \( S^e \) be the annihilator of \( \mathfrak{J}/\mathfrak{N} \) (\( \mathfrak{N} \) is obviously contained in \( \mathfrak{J} \)).
Since $\mathcal{V}$ is the torsion submodule of $S^*$ (as an $R$-module), and since $S$ is torsion-free as an $R$-module, $\tau(w) = 0$ for all $w$ in $\mathcal{V}$. Thus, if $a$ is in $A$, we have $aJ$ is contained in $\mathcal{V}$ and $0 = \tau(aJ) = \tau(a)J$. Thus by the remark above $\tau(a)$ is in $\text{Hom}_S(S^*, S)$ which implies that $\tau(A)$ is contained in $\text{Hom}_S(S^*, S)$.

Combining all the above remarks, and resorting to the commutative diagram above, we have shown

**Proposition 3.1.** Let $R$ be an integral domain with field of quotients $K$, $L$ a finite-dimensional $K$-algebra, and $S$ a subring of $L$ containing $R$ such that $S \otimes_R K = L$ and $S^* \otimes_R K = L^*$. Then if $A$ is the annihilator of $J/\mathcal{V}$, $\phi(A)$ is contained in $D_{S/R}$. In particular, $S_{S/R}$ is contained in $D_{S/R}$.

**Proposition 3.2.** Let $R$, $K$, $L$ and $S$ be as in 3.1 and in addition assume $L$ is a field. Then $L$ is a separable extension of $K$ if and only if $S_{S/R} \neq 0$ (i.e. $\mathcal{N}$ is not contained in $J$).

**Proof.** If $L$ is not separable, the trace map is identically zero, so that for all $f$ in $\text{Hom}_S(S^*, S)$, $p(f)(\text{Tr}) = 0$. Since $S_{S/R} = \phi(\mathcal{N}) = \phi'\sigma^{-1}p(\mathcal{N})$ and $\tau(\mathcal{N})$ is contained in $\text{Hom}_S(S^*, S)$, we have $S_{S/R} = 0$.

If $L$ is separable, the exact sequence

$$0 \to J' \to L \otimes_K L \to L \to 0$$

splits. However, $L = S \otimes_R K$, $L \otimes_K L = S^* \otimes_R K$, and $J' = J \otimes_R K$. Thus the annihilator $\mathcal{N}'$ of $J'$ is $\mathcal{N} \otimes_R K$ (by 2.6), and $\mathcal{N}'$ is not contained in $J'$. Therefore $\mathcal{N}$ is not contained in $J$ and $S_{S/R} \neq 0$.

**Proposition 3.3.** Let $R$ be an integrally closed integral domain with field of quotients $K$, $L$ a separable $K$-algebra, and $S$ a subring of $L$ containing $R$ which is integral over $R$ and such that $S \otimes_R K = L$. Then $\text{Hom}_S(S^*, S)$ is isomorphic to $D_{S/R}$ under the map $f \mapsto f(\text{Tr})$. If $S$ is a projective, finitely generated $R$-module, then $S_{S/R} = D_{S/R}$.

**Proof.** Since $L$ is a separable $K$-algebra, the map $L \to L^*$ given by $x \mapsto \text{Tr} \circ x$ is an isomorphism. Under this isomorphism, $C_{S/R}$ is mapped onto $S^*$. Thus $\text{Hom}_S(S^*, S) \cong \text{Hom}_S(C_{S/R}, S)$ and this latter module is isomorphic to $D_{S/R}$. The composite isomorphism is the one described above, namely $f \mapsto f(\text{Tr})$ ($\text{Tr}$ is here restricted to $S$). Furthermore, by standard arguments, it is easy to see that $S^* \otimes_R K = L^*$ so that all our previous discussion (including commutative diagram) holds. Moreover, if $S$ is finitely generated over $R$ and $R$-projective, then $\tau$ is an isomorphism, $\mathcal{V} = 0$, and $\tau(\mathcal{N}) = \text{Hom}_S(S^*, S)$. Thus in this case $S_{S/R} = D_{S/R}$.
Throughout the rest of this section we shall denote the $R$-module $\text{Hom}_R(E, R)$ by $E^*$, where $E$ is an arbitrary $R$-module.

**Proposition 3.4.** Let $R$ be a noetherian domain such that every proper principal ideal is unmixed (e.g. $R$ is integrally closed). Let $A$ be a finitely generated $R$-module such that $A = A^{**}$, and $B$ a finitely generated torsion-free $R$-module containing $A$ such that $B/A$ is a non-trivial torsion module. Then $\alpha(B/A)$ is unmixed of rank one ($\alpha(B/A)$ is the annihilator of $B/A$ in $R$).

**Proof.** Let $b_1, \ldots, b_t$ be generators of $B$. Then $\alpha(B/A) = \{r \text{ in } R/rb_i \text{ is in } A \text{ for } i = 1, \ldots, t\} = \cap a_i$, where $a_i = \{r \text{ in } R/rb_i \text{ is in } A\}$. Thus if each $a_i$ is unmixed of rank one, then so is $\alpha(B/A)$. We may therefore suppose that $B = A + Rb$, $b$ not in $A$, and $B/A$ is a torsion module.

Observe next that if $h$ is in $A^*$, then $h$ can be extended to a map $\overline{h} : A \otimes_R K \to K$. Moreover, if $x$ in $A \otimes_R K$ is such that $\overline{h}(x) = 0$ for all $h$ in $A^*$, then $x = 0$. As a result, we have that $r$ is in $\alpha(B/A)$ if and only if $r\overline{h}(b) = \overline{h}(rb)$ is in $R$ for all $h$ in $A^*$. For if $r$ is in $\alpha(B/A)$, then $rb$ is in $A$ so that $\overline{h}(rb) = \overline{h}(rb)$ is in $R$. Conversely, if $\overline{h}(rb)$ is in $R$ for all $h$ in $A^*$, then the map $A^* \to R$ given by $h \to \overline{h}(rb)$ is an element of $A^{**} = A$ so that $\overline{h}(rb) = h(a_0)$ for some $a_0$ in $A$ and all $h$ in $A^*$. Thus $\overline{h}(rb - a_0) = 0$ for all $h$ in $A^*$ and by the above remarks, $rb - a_0$ i.e. $r$ is in $\alpha(B/A)$.

Since $A^*$ is finitely generated, say by $h_{1}, \ldots, h_n$, we see that $\alpha(B/A) = \{r \text{ in } R/\overline{h_i}(rb) \text{ is in } R \text{ for } i = 1, \ldots, n\}$. Let $\overline{h_i}(b) = u_i/v$. Then $r$ is in $\alpha(B/A)$ if and only if $r$ is in $\cap (v) : u_i$ i.e. $\alpha(B/A) = \cap (v) : u_i$. Now by assumption on $R$, $(v)$ is an unmixed ideal of rank one so that $(v) : u_i$ is also. Thus $\alpha(B/A)$ is unmixed of rank one.

**Corollary 3.5.** Let $R$ be an integrally closed noetherian integral domain with field of quotients $K$, $L$ a separable field extension of $K$, and $S$ the integral closure of $R$ in $L$. Then if $S/R \neq S$, $S/R$ must be of rank one.

This follows from 3.4 by letting $R = A = S$ and $B = C_{S/R}$ and observing that $\mathfrak{D}_{S/R} = \alpha(C_{S/R}/S)$.

**Proposition 3.6.** With $R$, $K$, $L$ and $S$ as above, we have $\mathfrak{D}_{S/R} = S$ if and only if every minimal prime ideal of $S$ is unramified.

**Proof.** By 2.7, every minimal prime ideal of $S$ is unramified if and only if rank $S_{S/R}$ is greater than one. Since $\mathfrak{D}_{S/R}$ contains $S_{S/R}$, we have that if every minimal prime is unramified, then $\mathfrak{D}_{S/R}$ has rank greater than one. Thus, by 3.5, $\mathfrak{D}_{S/R} = S$.

---

*We would like to thank O. Goldman for suggesting this proposition to us.*
Conversely, let $\mathcal{D}_{S/R} = S$, $\mathfrak{P}$ be a minimal prime of $S$, and $\mathfrak{p} = \mathfrak{P} \cap R$. Then $\mathfrak{p}$ is a minimal prime of $R$ and $R_\mathfrak{p}$ is a regular local ring of dimension one. Since $S_\mathfrak{p}$ is a finitely generated, torsion-free $R_\mathfrak{p}$-module, $S_\mathfrak{p}$ is $R_\mathfrak{p}$-free. We have by 3.3 that $\mathcal{E}_{S_\mathfrak{p}/R_\mathfrak{p}} = \mathcal{D}_{S_\mathfrak{p}/R_\mathfrak{p}}$. But it is easily seen that $\mathcal{E}_{S_\mathfrak{p}/R_\mathfrak{p}} = \mathcal{E}_{S/R} \otimes_S S_\mathfrak{p}$ and $\mathcal{D}_{S_\mathfrak{p}/R_\mathfrak{p}} = \mathcal{D}_{S/R} \otimes_S S_\mathfrak{p}$. Therefore $\mathcal{E}_{S_\mathfrak{p}/R_\mathfrak{p}} = S_\mathfrak{p}$ and thus $\mathfrak{P}$ is unramified.

**Corollary 3.7.** Let $R$, $K$, $S$ and $L$ be as in 3.5 and assume further that $S$ is $R$-projective. Then $S$ is unramified if and only if every minimal prime ideal of $S$ is unramified.

**Proof.** Since $S$ is $R$-projective and finitely generated, we have $\mathcal{E}_{S/R} = \mathcal{D}_{S/R}$. Furthermore, $S$ is unramified if and only if $\mathcal{E}_{S/R} = S$ (by 2.5). Thus 3.6 implies 3.7.

The following theorem has also been obtained independently by Serre.

**Theorem 3.8.** Let $R$ be a regular local ring of dimension less than or equal to two, and let $K$, $S$, and $L$ be as above. Then $S$ is unramified if and only if every minimal prime ideal of $S$ is unramified.

**Proof.** We need only show that $S$ is $R$-projective, for then we may apply 3.7. However, by [2, 2.10] it is sufficient to show that $S$ is a Macaulay ring ($S$ is semi-local). Since $\dim S \leq 2$, and since $S$ is integrally closed, we have that every principal ideal of $S$ is unmixed and that every ideal of rank two that is generated by two elements is unmixed. Hence $S$ is Macaulay and therefore $R$-projective.

4. **On being free.** Throughout this section, $R$ will be an integrally closed local domain with maximal ideal $m$ and field of quotients $K$. The residue class field $R/m$ will be denoted by $F$.

**Proposition 4.1.** Let $L$ be a separable $K$-algebra and $S$ an integral extension of $R$ in $L$ such that $S \otimes_R K = L$ and is unramified. Then $\mathcal{D}_{S/R} = S$.

**Proof.** Since $S$ is unramified, $\mathcal{D}_{S/R} = S$. However, since $\mathcal{D}_{S/R}$ is contained in $\mathcal{D}_{S/R}$, we have $\mathcal{D}_{S/R} = S$.

**Lemma 4.2.** Let $S$ be an $R$-algebra containing $R$ which is torsion-free over $R$ and such that

a) $S$ is finitely generated over $R$,

b) there is an element $t$ in $S/mS$ such that $S/mS = F[t]$.

If $\theta$ in $S$ is such that $\theta \mapsto t$ under the natural map $S \to S/mS$, then
\( S = R[\theta] \) and \( \{1, \theta, \cdots, \theta^{n-1}\} \) is a free basis for \( S \) over \( R \) (where \( n = [S/mS : F] \)).

**Proof.** Since \( S/mS = F[t] \), we have that \( \{1, t, \cdots, t^{n-1}\} \) is a basis for \( S/mS \) over \( F \). Since \( R \) is a local ring, this implies that \( \{1, \theta, \cdots, \theta^{n-1}\} \) generates \( S \) over \( R \) and is a minimal generating set for \( S \) over \( R \). Also, since \( R \) is integrally closed, we know that the minimal polynomial \( f \) in \( K[x] \) for \( \theta \) has its coefficients in \( R \). We will show that degree \( f = n \), hence that \( \{1, \theta, \cdots, \theta^{n-1}\} \) is a basis for \( K[\theta] \) over \( K \). This will imply that \( \{1, \theta, \cdots, \theta^{n-1}\} \) is a free basis for \( S \) over \( R \).

Let \( \bar{f} \) in \( F[X] \) be the corresponding polynomial of \( f \). Then \( \bar{f}(t) = 0 \) so that \( \deg \bar{f} \equiv n \). On the other hand, since \( \theta^n \) is in \( S \), we have \( \theta^n = \sum_{i=0}^{n-1} r_i \theta^i \), \( r_i \) in \( R \). Therefore \( \deg f \equiv n \) and we are done.

**Proposition 4.3.** Let \( S \) be a torsion-free \( R \)-algebra containing \( R \) which is unramified and finitely generated over \( R \). Then \( S \) is a free \( R \)-module on \( n \) generators (where \( n = [S/mS : F] \)). Moreover, if \( L \) is the full ring of quotients of \( S \), \( [L : K] = n \) and \( S \) is integrally closed in \( L \).

**Proof.** Let us assume first that \( F \) is an infinite field. Then it is well known [4] that \( S/mS = F[t] \). Thus, by 4.2, we have that \( S \) is \( R \)-free with basis \( \{1, \theta, \cdots, \theta^{n-1}\} \).

Now suppose \( F \) is finite. Let \( X \) be an indeterminate, and consider the local domain \( R[X]_{m^*} = R' \), where \( m^* \) is the extension of \( m \) to \( R[X] \). The maximal ideal \( m' \) of \( R' \) is \( R' \cdot m^* \) and \( R'/m' = F' = F(X) \). \( R' \) is integrally closed since \( R[X] \) is and rings of quotients of integrally closed rings are integrally closed.

We now have \( R' \) contained in \( S' \), where \( S' = S[X]_{m^*} \). \( S' \) a finitely generated torsion-free \( R' \)-module, and \( [S'/m'S' : F'] = n \). If we show that \( S' \) is unramified over \( R' \), and use the fact that \( F' \) is infinite, we will have that \( S' \) is a free \( R' \)-module on \( n \) generators. This will imply that \( S \) is \( R \)-free on \( n \) generators for if \( s_1, \cdots, s_m \) is a minimal generating set for \( S \) over \( R \), it is also one for \( S' \) over \( R' \), hence a free basis for \( S' \) over \( R' \) and therefore a free basis for \( S \) over \( R \), with \( m = n \).

Since \( S \) is unramified, \( S \) is \( S^e \)-projective so that the exact sequence

\[ 0 \rightarrow \mathcal{J} \rightarrow S^e \rightarrow S \rightarrow 0 \]

splits. Therefore, the exact sequence

\[ 0 \rightarrow \mathcal{J} \otimes_R R' \rightarrow S^e \otimes_R R' \rightarrow S \otimes_R R' \rightarrow 0 \]
splits. Since \( S' = S[X] \otimes_{R[X]} R' = S \otimes_R (R[X] \otimes_{R[X]} R') = S \otimes_R R' \) and \((S')^e = S' \otimes_R S' = S^e \otimes_R R'\), we see that \( S' \) is \((S')^e\)-projective and therefore \( S' \) is unramified over \( R' \). This then shows that \( S \) is \( R \)-free on \( n \) generators. The rest of the proposition follows from standard arguments [4].

**Theorem 4.4.** Let \( R \) be a noetherian integral domain (not necessarily local) with field of quotients \( K \), and \( L \) a separable \( K \)-algebra. If \( S \) is an unramified integral extension of \( R \) in \( L \) such that \( S \otimes_R K = L \), then \( S \) is \( R \)-projective.

**Proof.** By standard localization arguments, this result follows from 4.3.

**Proposition 4.5.** Let \( S \) be a local ring containing \( R \) which is unramified and finitely generated over \( R \). Then \( S \) is \( R \)-free.

**Proof.** Since \( S \) is unramified, \( S \otimes_R K \) is a separable \( K \)-algebra. Let \( S' = \text{Im}(S \to S \otimes_R K) \). Then \( S' \) is torsion-free and finitely generated over \( R \), and we have the exact sequence

\[
0 \to t(S) \to S \to S' \to 0,
\]

where \( t(S) \) is the \( R \)-torsion submodule of \( S \), and is finitely generated over \( R \). If we can show that \( t(S)/mt(S) = 0 \), we will have \( t(S) = 0 \). Therefore \( S \cong S' \) and so \( S \) will be torsion-free, hence free (by 4.3). Since \( S \) is unramified, \( S/mS \) is a field and therefore the map \( S/mS \to S'/mS' \), being an epimorphism, must be an isomorphism. Moreover, \( S/mS \) is a separable extension of \( F \) so that \( S'/mS' \) is also. Hence \( S' \) is unramified (by 2.5 it is sufficient to test ramification of \( S' \) by its unique maximal ideal) and by 4.3 is free over \( R \). Therefore the sequence (E) splits over \( R \) so that the sequence

\[
0 \to t(S)/mt(S) \to S/mS \to S'/mS' \to 0
\]

is exact. Since \( S/mS \cong S'/mS' \), we have \( t(S)/mt(S) = 0 \), hence \( t(S) = 0 \) and \( S \cong S' \).

**Proposition 4.6.** Let \( R \) be analytically normal (i.e. \( \hat{R} \), the completion of \( R \), is also an integrally closed local domain) and let \( S \) be a ring containing \( R \) which is unramified and finitely generated over \( R \). Then \( S \) is \( R \)-free.

**Proof.** \( Sm \) is the radical of \( S \), so that \( S \) contains \( \hat{R} \) and \( S \) is a finitely generated \( \hat{R} \)-module. Now \( S = S_1 + \cdots + S_n \) (direct sum), where each \( S_i \) is a local ring which is an \( \hat{R} \)-algebra. In fact, each \( S_i \) contains a copy of \( \hat{R} \). It can also be easily seen that each \( S_i \) is an unramified \( \hat{R} \)-algebra (since \( S \) is unramified over \( R \)). Therefore, by 4.5, each \( S_i \) is free over \( \hat{R} \), which implies
that $S$ is $R$-free. Since $S$ is finitely generated over $R$, $S$ being $R$-free implies that $S$ is $R$-free \cite[Theorem 3.2]{1}.

Since an integrally closed geometric local ring is analytically normal, we see that an unramified integral, finitely generated extension ring $S$ of an integrally closed geometric local ring $R$ is $R$-free. Hence if $R$ is a normal affine ring (not necessarily local), $S$ is $R$-projective.

**Lemma 4.7.** Let $R$ be a noetherian ring (not necessarily an integrally closed local domain) and let $S$ be a ring containing $R$ which is finitely generated as an $R$-module. If $S$ is $R$-projective, then $R$ is a direct summand of $S$ as an $R$-module.

**Proof.** From the exact sequence

$$0 \to R \to S \to S/R \to 0$$

it is clearly sufficient to prove that $S/R$ is $R$-projective. Let $m$ be a maximal ideal of $R$. Then the exact sequence

$$0 \to R_m \to S_m \to (S/R)_m \to 0$$

splits since $S_m$ is a projective (hence free) $R_m$-module, and 1 is part of a free basis for $S_m$ over $R_m$. Therefore $(S/R)_m$ is free for every maximal ideal $m$ and so by \cite[VII, Exercise 11]{3} $S/R$ is $R$-projective.

**Proposition 4.8.** Let $R \subset S \subset T$ be noetherian rings with $T$ a finitely generated projective unramified $R$-algebra. Then $S$ is unramified over $R$ if and only if $T$ is $S$-projective.

**Proof.** Suppose $T$ is $S$-projective. Then we have the commutative diagram

$$
\begin{array}{ccc}
S \otimes_R S & \longrightarrow & T \otimes_R T \\
\downarrow & & \downarrow \\
S & \longrightarrow & T.
\end{array}
$$

Since $T$ is $S$-projective, $T \otimes_T T$ is $S \otimes_T S$-projective, and $S$ is a direct summand of $T$ as an $S$- hence also as an $S \otimes_R S$-module. But $T$ is $T \otimes_T T$-projective since $T$ is an unramified $R$-algebra. Hence $S$ is $S \otimes_R S$-projective.

By \cite[IX, Proposition 2.3]{3} (letting $\Lambda = \Gamma = S$, $A = S$, $\Sigma = R$, $B = T$) we have that since $S$ is $S \otimes_R S$-projective (being unramified) and $T$ is $R$-projective, then $T$ is $S$-projective.
5. Ramification and homology.

Proposition 5.1. Let $S$ and $T$ be $R$-algebras such that $S$ is $R$-projective and $S$ is $S^e$-projective. If $E$ is an $S \otimes_R T$-module, then $\text{hd}_S \otimes_R T E = \text{hd}_T E$ and thus $\text{gl. dim } S \otimes_R T \leq \text{gl. dim } T$.

Further, if $S$ is $R$-free, then $\text{gl. dim } S \otimes_R T = \text{gl. dim } T$.

Proof. By [3; XVI, sec. 4] we have the spectral sequence

$$H^p(S, \text{Ext}^q_T(E, C)) \Rightarrow \text{Ext}^n_{S \otimes_R T}(E, C),$$

where $C$ is an $S \otimes_R T$-module. Since $S$ is $S^e$-projective, this spectral sequence collapses to $H^0(S, \text{Ext}^n_T(E, C)) \simeq \text{Ext}^n_{S \otimes_R T}(E, C)$. From the fact that $C$ is an arbitrary $S \otimes_R T$-module, it follows that $\text{hd}_S \otimes_R T E \leq \text{hd}_T E$.

But considering $S \otimes_R T$ as a $T$-algebra, we have by [3; XVI, Exercise 5] that

$$\text{hd}_T E \leq \text{hd}_T S \otimes_R T + \text{hd}_S \otimes_R T E.$$ 

Since $S$ is $R$-projective, it follows that $S \otimes_R T$ is $T$-projective. Therefore $\text{hd}_T S \otimes_R T = 0$ and thus $\text{hd}_T E \leq \text{hd}_S \otimes_R T E$, which gives the desired equality.

From the fact that $\text{hd}_S \otimes_R T E = \text{hd}_T E$ for arbitrary $S \otimes_R T$-modules $E$ it follows that $\text{gl. dim } S \otimes_R T \leq \text{gl. dim } T$. Further, if $S$ is $R$-free and $A$ is a $T$-module, then $\text{hd}_T A = \text{hd}_T S \otimes_R A$ since $S \otimes_R E$ is a direct sum of copies of $A$. But $\text{hd}_T S \otimes_R A = \text{hd}_S \otimes_R T S \otimes_R A$ by the previous arguments. Thus $\text{hd}_T A = \text{hd}_T S \otimes_R A$, which means that $\text{gl. dim } T \leq \text{gl. dim } S \otimes_R T$.

Corollary 5.2. Let $S$ and $T$ be noetherian $R$-algebras such that $S$ is unramified and $R$-projective and $\mathfrak{g}$ is a finitely generated ideal in $S^e$. If $T$ is a regular ring of finite (Krull) dimension and $S \otimes_R T$ is noetherian, then $S \otimes_R T$ is a regular ring of (Krull) dimension less than or equal to that of $T$.

Further, if $S$ is $R$-free (e.g. $R$ a local ring) then the dimensions of $S \otimes_R T$ and $T$ are equal.

Proof. By 2.5 we have that $S$ is $S^e$-projective. Since $T$ is a regular ring of finite dimension, we have by [1, Corollary 4.8] that $\text{gl. dim } T < \infty$. Therefore it follows from 5.1 that $\text{gl. dim } S \otimes_R T \leq \text{gl. dim } T$, which means that $S \otimes_R T$ is a regular ring of dimension less than or equal to that of $T$. The rest of the corollary follows from the fact that if $\text{gl. dim } S \otimes_R T = \text{gl. dim } T$, then the dimensions of $S \otimes_R T$ and $T$ are equal.

Proposition 5.3. Let $S$ be an $R$-algebra, where $R$ is an integral domain
with field of quotients $K$ such that $S^e$ is noetherian and $0 < [S \otimes_R K : K] < \infty$. Then $\text{hd}_{S^e} S = 0$ or $\infty$.

Proof. First we observe that $\text{hd}_{S^e} S \geq \text{hd}_{L^e} L$ where $L = S \otimes K$ and $L^e = (S \otimes_R K) \otimes_K (S \otimes_R K)$. Since $[L : K] < \infty$, it is well known that if $L$ is not a separable $K$-algebra, then $\text{hd}_{L^e} L = \infty$. Thus we may assume that $L$ is a separable $K$-algebra. Further, let us assume that $S$ is not $S^e$-projective. Therefore we have that $\mathcal{J} \neq (0)$ and the ideal generated by $\mathcal{J}$ and $\mathcal{N}$ is not $S^e$. By 2.6 we know that $\mathcal{N} \otimes_R K$ is the annihilator of the kernel of $L^e \to L$. From the fact that $L$ is a separable $K$-algebra we know that $\text{hd}_{L^e} L = 0$ and hence by 2.1 $\mathcal{N} \otimes_R K \neq (0)$. Therefore $\mathcal{N} \neq (0)$ and thus $\mathcal{J}$ consists entirely of zero divisors in $S^e$.

Let $\mathcal{M}$ be a maximal ideal in $S^e$ containing $\mathcal{J}$ and $\mathcal{N}$. Then the ideal $\mathcal{J}m$ in the local ring $S^e_m$ is not zero and consists entirely of zero-divisors. Therefore we conclude from the exact sequence

$$0 \to \mathcal{J}m \to (S^e)m \to S_m \to 0$$

that the annihilator of $S_m$ as an $(S^e)_m$-module is not zero and consists entirely of zero-divisors. Hence by [2, 6.2] we have that $\text{hd}_{(S^e)_m} S_m = \infty$. Since $\text{hd}_{S^e} S \geq \text{hd}_{(S^e)_m} S_m$, we have that $\text{hd}_{S^e} S = \infty$.

Appendix.

PROPOSITION A.1. Let $R$ be a noetherian ring and $T$ an $R$-algebra which is a finitely generated module. If $S_1$ and $S_2$ are unramified subalgebras of $T$, then the subalgebra generated by $S_1$ and $S_2$ is an unramified $R$-algebra. Thus $T$ contains an unramified $R$-algebra, which contains all the unramified $R$-subalgebras of $T$.

Proof. By 2.11 we know that $S_1 \otimes_R S_2$ is an unramified $R$-algebra. Therefore $\text{Im}(S_1 \otimes S_2 \to T)$ is an unramified $R$-subalgebra of $T$, which establishes the first part of the proposition. The second part follows from the chain conditions in $T$.

We next observe that Proposition 4.5 is true without assuming that $S$ is a local ring. As in the proof of Proposition 4.5, it suffices to show that $t(S)$, the torsion submodule of $S$, is zero.

Since $S$ is unramified, $S \otimes_R K$ is a separable $K$-algebra. Let $S' = \text{Im}(S \to S \otimes K)$. Then $S'$ is a torsion-free $R$-algebra which is unramified since it is the image of an unramified $R$-algebra. Since $t(S)$ is a finitely
generated \( R \)-torsion module, there is a non-zero \( x \) in \( R \) such that \( xt(S) = 0 \). Now let \( \mathfrak{M} \) be a maximal ideal in \( S \). Then \( t(S) \subset \mathfrak{M} \) for if not, we have that \( S_{\mathfrak{M}} \) and \( t(S)_{\mathfrak{M}} = x(S_{\mathfrak{M}}) = 0 \) which is impossible.

Since \( S' \) is \( R \)-projective, we have that
\[
0 \to t(S)/mt(S) \to S/mS \to S'/mS' \to 0
\]
is exact. Thus
\[
0 \to (t(S)/mt(S))_{\mathfrak{M}} \to (S'/mS)_{\mathfrak{M}} \to (S'/mS')_{\mathfrak{M}} \to 0
\]
is exact. Since both \( S \) and \( S' \) are unramified, the map \((S/mS)_{\mathfrak{M}} \to (S'/mS')_{\mathfrak{M}} \to 0\) is a field epimorphism, hence an isomorphism. Thus \((t(S)/mt(S))_{\mathfrak{M}} = 0\) for all maximal ideals \( \mathfrak{M} \) of \( S \) which implies that \( t(S)/mt(S) = 0 \). Hence \( t(S) = 0 \), which means that \( S \cong S' \) and thus we are done.

**Proposition A.2.** Let \( R \subset S \subset T \) be noetherian, integrally closed domains such that \( T \) is a finitely generated \( R \)-module. Then \( T \) is unramified over \( R \) if and only if \( T \) is unramified over \( S \) and \( S \) is unramified over \( R \).

**Proof.** Suppose \( T \) is unramified over \( R \). Then \( T \) is unramified over \( S \). Since \( S \) is integrally closed we know by Proposition 4.6, that \( T \) is \( S \)-projective. Thus the result follows from Proposition 4.8.

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**REFERENCES.**


