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A NOTE ON HOMOLOGY IN CATEGORIES

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Introduction

In [1], the existence of the functor Ext in an exact category having sufficiently many projectives and/or injectives was established. Moreover, a fairly general uniqueness theorem on connected sequences of functors was proved in that paper. However, no attempt was made to define Ext without the use of projectives or injectives. A paper by Yoneda [4], however, gives a clear indication of how $\mathrm{Ext}^n(A, C)$ may be defined in an abstract exact category, although this was not the object (nor the result) of that paper.[2] Yoneda showed (using the existence of projectives) that $\mathrm{Ext}^n(A, C)$ is in 1-1 correspondence with the set of "n-fold extensions" of A by C:

$$0 \longrightarrow C \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow A \longrightarrow 0 \quad \text{(exact)}$$

under a suitable equivalence relation.

The need for an abstract homology theory, independent of projectives, was felt when it was realized that the category of sheaves and sheaf homomorphisms over a topological space $\mathfrak{X}$ was an exact category. However, it was felt that sufficiently many projectives did not exist in that category. Grothendieck (unpublished) has established the existence of sufficiently many injective sheaves and has thus been able to define a functor Ext in the category of sheaves.

Recent work by D. Harrison [2] on cohomology based on exact sequences in which kernels are special submodules (e.g., pure submodules) of the domain module, has made it fairly natural to discuss an Ext functor on an exact category which depends upon a specific choice of monomorphisms (e.g., the inclusion map of pure subgroups). Heller [3] has considered such a problem, assuming the existence of projectives. In this paper, we shall define such an Ext without the use of projectives or injectives. We will show that the cohomology groups $H^n(\mathfrak{X}, C)$ of a space $\mathfrak{X}$ with coefficients in a sheaf of abelian groups C over $\mathfrak{X}$ can be interpreted as a special type of Ext.

We felt it best to omit proofs in this note since the methods used in [1],

[1] This work was done with the partial support of NSF grant G-4232.

[2] We have since learned that Yoneda has made this the precise object of another paper.

together with the outline given here, reduce most of the propositions to tedium.

1. Preliminaries

Throughout this paper, $\mathcal{A}$ will be an exact category satisfying the direct sum axiom. The terminology and notation of [1] will be used freely.

Let $\mathcal{I}$ be a subclass of the class of monomorphisms of $\mathcal{A}$. Then associated with $\mathcal{I}$, we have a class $\mathcal{P} = \mathcal{P}(\mathcal{I})$ of epimorphisms consisting of all possible cokernels of the monomorphisms of $\mathcal{I}$. Conversely, given a class $\mathcal{P}$ of epimorphisms of $\mathcal{A}$, we have $\mathcal{I}(\mathcal{P})$ consisting of all possible kernels of maps in $\mathcal{P}$.

DEFINITION. A class of monomorphisms $\mathcal{I}$ will be called an *h. f. class*
(i) if $\mathcal{I} = \mathcal{I}(\mathcal{P}(\mathcal{I}))$
(ii) if $pi = e_A$, then $i \in \mathcal{I}$ (hence also $p \in \mathcal{P}(\mathcal{I})$)
(iii) if α, $\beta \in \mathcal{I}$ and $\alpha\beta$ is defined, then $\alpha\beta \in \mathcal{I}$
(iii′) if γ, $\delta \in \mathcal{P}(\mathcal{I})$ and $\gamma\delta$ is defined, then $\gamma\delta \in \mathcal{P}(\mathcal{I})$
(iv) if α and β are monomorphisms, then $\alpha\beta \in \mathcal{I}$ implies $\beta \in \mathcal{I}$
(iv′) if γ, δ are epimorphisms, and $\gamma\delta \in \mathcal{P}(\mathcal{I})$, then $\gamma \in \mathcal{P}(\mathcal{I})$.

PROPOSITION 1.1. *Let $\mathcal{I}$ be an h.f. class. Then*
(a) *if θ is an equivalence, $\theta \in \mathcal{I}$*
(b) *if* $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$ *is exact and splits, then $i \in \mathcal{I}$.*

PROPOSITION 1.2. *If $\alpha_i : A_i \to B_i$, $i = 1,2$ are in $\mathcal{I}$, then so is the induced map $\alpha : A_1 + A_2 \to B_1 + B_2$ (direct sum).*

If $0 \longrightarrow A_1 \xrightarrow{\alpha_1} B_1 \xrightarrow{\beta_1} C \longrightarrow 0$, $0 \longrightarrow A_2 \xrightarrow{\alpha_2} B_2 \xrightarrow{\beta_2} C \longrightarrow 0$ *are exact, we have the exact sequence*

$$0 \longrightarrow R \xrightarrow{\sigma} B_1 + B_2 \xrightarrow{\beta_1 p_1 - \beta_2 p_2} C \longrightarrow 0$$

where p_i are the projections in the direct sum representation of $B_1 + B_2$. We therefore obtain the exact sequence

$$0 \longrightarrow D \xrightarrow{\tau} R \xrightarrow{\beta_1 p_1 \sigma} C \longrightarrow 0 .$$

PROPOSITION 1.3. *If α_1, α_2 above are in $\mathcal{I}$, then so is τ.*

DEFINITIONS. A sequence $\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$ is called *$\mathcal{I}$-exact* if it is an exact sequence and each $d_n = i_n p_n$ where $i_n \in \mathcal{I}$, $p_n \in \mathcal{P}(\mathcal{I})$. An *n-fold extension of A by C* is an $\mathcal{I}$-exact sequence

$$(1) \qquad 0 \longrightarrow C \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow A \longrightarrow 0 .$$

The extension (1) of A by C is *similar* to the extension

$$(2) \qquad 0 \longrightarrow C \longrightarrow E'_n \longrightarrow E'_{n-1} \longrightarrow \cdots \longrightarrow E'_1 \longrightarrow A \longrightarrow 0$$

if there exist maps $E_i \to E'_i$, $i = 1, \cdots, n$, such that

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & C & \longrightarrow & E_n & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & C & \longrightarrow & E'_n & \longrightarrow & \cdots & \longrightarrow & E'_1 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

is commutative. The extension (1) is *equivalent* to the extension (2) if there exists a finite sequence of n-fold extensions $X_0, X_1, \cdots, X_r$ such that X_0 is the extension (1), X_r is the extension (2) and X_i is similar to X_{i+1} or X_{i+1} is similar to X_i for $i = 0, \cdots, r-1$.

Clearly the above defined equivalence is an equivalence relation.

2. The groups $\mathcal{J}$-Ext$^n(A, C)$

In this section, we show that the equivalence classes of the n-fold extensions of A by C form an abelian group, denoted by $\mathcal{J}$-Ext$^n(A, C)$. Moreover, we will see that we obtain in this way an exact connected sequence of functors.

We first define the sum of two n-fold extensions. Let

$$0 \longrightarrow C \xrightarrow{\sigma} E_n \xrightarrow{d_n} E_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} E_1 \xrightarrow{\varepsilon} A \longrightarrow 0\,,$$

$$0 \longrightarrow C \xrightarrow{\sigma'} E'_n \xrightarrow{d'_n} E'_{n-1} \xrightarrow{d'_{n-1}} \cdots \xrightarrow{d'_2} E'_1 \xrightarrow{\varepsilon'} A \longrightarrow 0$$

be n-fold extensions of A by C, and let $X_i = E_i + E'_i$ with direct sum representations

$$E_i \xrightarrow{u_i} X_i \xrightarrow{v_i} E_i$$

$$E'_i \xrightarrow{u'_i} X_i \xrightarrow{v'_i} E'_i\,.$$

Now, we define $F_1 = \operatorname{Ker}(\varepsilon v_1 - \varepsilon' v'_1 : X_1 \to A)$

$F_i = X_i$ for $1 < i < n$

$F_n = \operatorname{Coker}(u_n\sigma - u'_n\sigma' : C \to X_n)$.

In case $n = 1$, we define F_1 to be $\operatorname{Ker}(\varepsilon v_1 - \varepsilon' v'_1)$ modulo $\operatorname{Im}(u_1\sigma - u'_1\sigma')$. It is clear that we obtain maps

$$0 \longrightarrow C \xrightarrow{\kappa} F_n \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_2} F_1 \xrightarrow{\delta} A \longrightarrow 0$$

where $\delta = \varepsilon v_1 \omega = \varepsilon' v' \omega (\omega : F_1 \to X_1$ is the kernel of $\varepsilon v_1 - \varepsilon' v'_1)$,

$$h_i = u_{i-1} d_i v_i + u'_{i-1} d'_i v'_i \qquad \text{for } 1 < i < n$$

$$h_n \tau = u_{n-1} d_n v_n + u'_{n-1} d'_n v'_n$$

(where $\tau : X_n \to F_n$ is the cokernel of $u_n\sigma - u'_n\sigma'$)

$$\kappa = \tau \mathcal{U}_n \sigma = \tau \mathcal{U}'_n \sigma' .$$

The propositions of § 1 guarantee that we end up with an n-fold extension of A by C. Furthermore, it is obvious that this sum is independent of the choice of representatives so that we have addition defined in the set of equivalence classes of n-fold extensions of A by C.

DEFINITION. We denote the set of equivalence classes of n-fold extensions of A by C, together with the addition just defined, by $\mathcal{J}$-$\mathrm{Ext}^n(A, C)$ (or $\mathrm{Ext}^n(A, C)$ when no confusion can arise).

THEOREM 2.1. *$\mathrm{Ext}^n(A, C)$ is an abelian group.*

PROOF. The zero element of $\mathrm{Ext}^n(A, C)$ is represented by $0 \to C \to C + C \to \cdots \to C + C \to C + A \to A \to 0$.

We will postpone the rest of the proof of this theorem until later. First we make some observations about the pairing of $\mathrm{Ext}^n(A, C)$ and $\mathrm{Ext}^m(C, D)$ to $\mathrm{Ext}^{n+m}(A, D)$.

Let $X \in \mathrm{Ext}^n(A, C)$, $Y \in \mathrm{Ext}^m(C, D)$ be represented by

$$0 \longrightarrow C \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow A \longrightarrow 0$$
$$0 \longrightarrow D \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_1 \longrightarrow C \longrightarrow 0$$

respectively. Then denote by $\varphi(X, Y)$ the element of $\mathrm{Ext}^{n+m}(A, D)$ represented by

$$0 \longrightarrow D \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_1 \longrightarrow E_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow A \longrightarrow 0.$$

This is clearly independent of the choice of representatives of X and Y, and we immediately have

PROPOSITION 2.2. *The pairing*

$$\varphi : \mathrm{Ext}^n(A, C) \times \mathrm{Ext}^m(C, D) \longrightarrow \mathrm{Ext}^{n+m}(A, D)$$

is bilinear and associative. Moreover, $\varphi(X, 0) = \varphi(0, Y) = 0$. (The proof is reduced to the case $n = m = 1$.)

Now let f be an element of $H(C, B)$, and let $X \in \mathrm{Ext}^1(A, C)$ be represented by $0 \longrightarrow C \xrightarrow{\alpha} E \xrightarrow{\beta} A \longrightarrow 0$. We define $f_*(X) \in \mathrm{Ext}^1(A, B)$ as follows:

Let $B \xrightarrow{i_1} B + E \xrightarrow{p_1} B$, $E \xrightarrow{i_2} B + E \xrightarrow{p_2} E$ be a direct sum representation, and let $F = \mathrm{Coker}\,(i_1 f - i_2 \alpha : C \to B + E)$. We then have an exact sequence

$$0 \longrightarrow B \xrightarrow{\delta} F \xrightarrow{\gamma} A \longrightarrow 0$$

where $\gamma\tau = \beta p_2$, $\delta = \tau i_1$ and $\tau : B + E \to F$ is the cokernel of $i_1 f - i_2 \alpha$. Since $\gamma\tau = \beta p_2 \in \mathcal{P}(\mathcal{J})$, we have $\gamma \in \mathcal{P}(\mathcal{J})$ so that

$$0 \longrightarrow B \xrightarrow{\delta} F \xrightarrow{\gamma} A \longrightarrow 0$$

is a 1-fold extension of A by B. We define $f_*(X)$ to be the class of this extension in $\mathrm{Ext}^1(A, B)$.

PROPOSITION 2.3. *The map f_* is well defined,* i.e., *is independent of the choice of representative of X. Moreover, $f_*(X+Y) = f_*(X) + f_*(Y)$ and $(f_1 + f_2)_* = f_{1*} + f_{2*}$. In addition, $0_* = 0$.*

We remark here that all these statements can be proved by straightforward application of the methods of [1].

Dually, if $f \in H(B, A)$, one obtains $f_* : \mathrm{Ext}^1(A, C) \to \mathrm{Ext}^1(B, C)$.

We extend the definition of f_* to cover the case $\mathrm{Ext}^n(A, C) \to \mathrm{Ext}^n(A,B)$ as follows: We assume that for $f \in H(C, B)$ and for all D, we have a map $f_* : \mathrm{Ext}^{n-1}(D, C) \to \mathrm{Ext}^{n-1}(D, B)$ which satisfies the conclusions of Proposition 2.3. Let $X \in \mathrm{Ext}^n(A, C)$ and represent X by

$$0 \longrightarrow C \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow A \longrightarrow 0 .$$

Then we have the extensions

$$X_1 : 0 \longrightarrow C \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_2 \longrightarrow D \longrightarrow 0$$
$$Y : 0 \longrightarrow D \longrightarrow E_1 \longrightarrow A \longrightarrow 0 .$$

$X_1 \in \mathrm{Ext}^{n-1}(D, C)$ so that $f_*(X_1) \in \mathrm{Ext}^{n-1}(D, B)$ is defined. Define $f_*(X)$ to be $\varphi(f_*(X_1), Y)$. We therefore obtain

PROPOSITION 2.4. *Let $f \in H(C, B)$. Then for each n and A, we have a homomorphism*

$$f_* : \mathrm{Ext}^n(A, C) \longrightarrow \mathrm{Ext}^n(A, B)$$

such that

(i) $f_*(0) = 0$

(ii) $(f_1 + f_2)_* = f_{1*} + f_{2*}$

(iii) $0_* = 0$

(iv) *if $g \in H(B, D)$, then $(gf)_* = g_* f_*$.*

The dual statement also holds.

The proof of Theorem 2.1 now follows trivially. Addition is clearly commutative and associative. If $X \in \mathrm{Ext}^n(A, C)$, define $-X$ to be $f_*(X)$ where $f \in H(C, C)$, $f = -e_C$. Then clearly $X + -X = 0$ by Proposition 2.4.

3. The exact sequence

Let $0 \longrightarrow A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \longrightarrow 0$ be an $\mathcal{J}$-exact sequence. Then we have a map $\delta_n : \mathrm{Ext}^n(A', C) \to \mathrm{Ext}^{n+1}(A'', C)$ for each $n > 0$ defined as follows:

If X is an element of $\mathrm{Ext}^n(A', C)$, and Y is the class in $\mathrm{Ext}^1(A'', A')$ represented by the extension $0 \longrightarrow A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \longrightarrow 0$ above, then $\delta_n(X) = \varphi(Y, X) \in \mathrm{Ext}^{n+1}(A'', C)$.

Moreover, we have a map $\delta_0 : H(A', C) \to \mathrm{Ext}^1(A'', C)$ which is defined by

$$\delta_0(f) = f_*(Y)$$

where $f \in H(A', C)$, Y is as above, and f_* is the map defined in § 2.

Dually, for $\mathcal{J}$-exact sequences

$$0 \longrightarrow C' \xrightarrow{\sigma} C \xrightarrow{\tau} C'' \longrightarrow 0 ,$$

we have maps $\delta'_n : \mathrm{Ext}^n(A, C'') \longrightarrow \mathrm{Ext}^{n+1}(A, C')$ for $n > 0$, and $\delta'_0 : H(A, C'') \longrightarrow \mathrm{Ext}^1(A, C')$.

THEOREM 3.1. *For every pair of $\mathcal{J}$-exact sequences*

$$0 \longrightarrow A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \longrightarrow 0$$

$$0 \longrightarrow C' \xrightarrow{\sigma} C \xrightarrow{\tau} C'' \longrightarrow 0 ,$$

the sequences

$$0 \longrightarrow H(A'', C) \longrightarrow H(A, C) \longrightarrow H(A', C) \longrightarrow \mathrm{Ext}^1(A'', C) \longrightarrow \cdots \longrightarrow$$
$$\mathrm{Ext}^n(A', C) \longrightarrow \mathrm{Ext}^{n+1}(A'', C) \longrightarrow \mathrm{Ext}^{n+1}(A, C) \longrightarrow \mathrm{Ext}^{n+1}(A', C) \longrightarrow \cdots$$

$$0 \longrightarrow H(A, C') \longrightarrow H(A, C) \longrightarrow H(A, C'') \longrightarrow \mathrm{Ext}^1(A, C') \longrightarrow \cdots \longrightarrow$$
$$\mathrm{Ext}^n(A, C'') \longrightarrow \mathrm{Ext}^{n+1}(A, C') \longrightarrow \mathrm{Ext}^{n+1}(A, C) \longrightarrow \mathrm{Ext}^{n+1}(A, C'') \longrightarrow \cdots$$

are exact.

Thus $\mathrm{Ext}^n(A, C)$ is an exact connected sequence of functors on $\mathcal{J}$-exact sequences.

An outline of the proof of this theorem should perhaps be given. In the low dimensions (0 and 1) the proof is straightforward. Assume now that the exactness across the boundary operator has already been demonstrated. Then we show $\mathrm{Ext}^n(A'', C) \to \mathrm{Ext}^n(A, C) \to \mathrm{Ext}^n(A', C)$ is exact as follows.

Let $X \in \mathrm{Ext}^n(A, C)$ be represented by

$$0 \longrightarrow C \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow A \longrightarrow 0 .$$

Then we have

$$0 \longrightarrow C \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_2 \longrightarrow D \longrightarrow 0$$

$$0 \longrightarrow D \longrightarrow E_1 \longrightarrow A \longrightarrow 0$$

exact.

The image X' of X is then obtained by splicing the exact sequences

$$0 \longrightarrow C \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_2 \longrightarrow D \longrightarrow 0$$
$$0 \longrightarrow D \longrightarrow E_1' \longrightarrow A' \longrightarrow 0 .$$

If we assume X' to be zero, we want to show that X is the image of some $X'' \in \mathrm{Ext}^n(A'', C)$. However, we may regard $X' \in \mathrm{Ext}^n(A', C)$ to be the image of $Y \in \mathrm{Ext}^{n-1}(D, C)$ represented by $0 \to C \to E_n \to \cdots \to E_2 \to D \to 0$ under the boundary map with respect to the exact sequence

$$0 \longrightarrow D \longrightarrow E_1' \longrightarrow A' \longrightarrow 0 .$$

Hence, by our assumption of exactness across the boundary map, Y is the image of some $Z \in \mathrm{Ext}^{n-1}(E_1', C)$ represented, say, by

$$0 \longrightarrow C \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow E_1' \longrightarrow 0 .$$

It is easily seen that $0 \to E_1' \to E_1 \to A'' \to 0$ is $\mathcal{J}$-exact and that X is the image of the element in $\mathrm{Ext}^n(A'', C)$ represented by

$$0 \longrightarrow C \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow E_1 \longrightarrow A'' \longrightarrow 0 .$$

Now the proof of exactness across the boundary usually involves two steps. However, one step is dual to the other ; so we will merely outline the proof of exactness of

$$\mathrm{Ext}^n(A, C) \longrightarrow \mathrm{Ext}^n(A', C) \longrightarrow \mathrm{Ext}^{n+1}(A'', C) .$$

Let $X \in \mathrm{Ext}^n(A', C)$ be represented by

$$0 \longrightarrow C \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow A' \longrightarrow 0 .$$

Then the image X'' of X is represented by

$$0 \longrightarrow C \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow A \longrightarrow A'' \longrightarrow 0 .$$

The trick here is to use induction on the number of links required to get from the the trivial zero element of $\mathrm{Ext}^{n+1}(A'', C)$ to X''. The proof then is similar to the preceding one.

REMARK. A simple example of the above theory is the following :

Let $\mathcal{A}$ be the category of R-modules, R a commutative ring. A submodule A of an R-module B will be pure if $rb \in A$ implies $rb = ra$ for some $a \in A (r \in R,\ b \in B)$. Now let $\mathcal{J}$ be the h. f. class generated by all inclusions $A \subset B$ where A is a pure submodule of B (it is clear that one can talk about the h. f. class generated by a class of monomorphisms).

Then $\mathcal{I}$ consists of all monomorphisms whose images are pure submodules of the range.

Another example that may be of some interest is the following :

Let $\mathfrak{X}$ be a topological space, A a sheaf of rings over $\mathfrak{X}$, and A' a subsheaf of rings. If $\mathcal{A}$ is the category of all A-sheaves, we can define $\mathcal{I}$ to be the class of all monomorphisms which yield A'-splitting exact sequences. If $\mathfrak{X}$ were a differentiable manifold, A and A' could be the sheaves of germs of continuous and differentiable functions, respectively.

4. Cohomology with respect to a functor

Let $T(A, C)$ be a covariant additive functor, and let $\mathcal{Q}$ be the class of all objects Q such that $T(Q, C)$ is an exact functor of C. We can now define a family $\mathcal{I}_T$ by saying that a monomorphism α belongs to $\mathcal{I}_T$ if for every $Q \in \mathcal{Q}$, the exact sequence $0 \longrightarrow A' \xrightarrow{\alpha} A \longrightarrow A'' \longrightarrow 0$ yields an exact sequence $0 \to H(A'', Q) \to H(A, Q) \to H(A', Q) \to 0$. $\mathcal{I}_T$ is easily seen to be an h. f. class, and we therefore obtain our functors $\mathcal{I}_T\text{-Ext}^n(A, C)$. If, moreover, every object A can be mapped by an element of $\mathcal{I}_T$ into an object of $\mathcal{Q}$, then $\mathcal{I}_T\text{-Ext}^n(A, C)$ is obviously isomorphic to the groups obtained by taking $\mathcal{Q}$-resolutions of C (i. e., $0 \to C \to Q_0 \to Q_1 \to \cdots$, $Q_i \in \mathcal{Q}$, is a $\mathcal{Q}$-resolution of C, and $H(A, Q_0) \to H(A, Q_1) \to \cdots$ is a complex whose homology is $\mathcal{I}_T\text{-Ext}^n(A, C)$).

The above discussion can be generalized to include functors of any number of variables, and of any variance. Moreover, we could define another class $\mathcal{I}'_T$ dually to the way $\mathcal{I}_T$ was defined above.

5. Comparison with cohomology theories

Let $\mathfrak{X}$ be a topological space, and let $\mathcal{A}$ be the category of sheaves of abelian groups over $\mathfrak{X}$ (or of R-modules where R is a principal ideal domain). Let $\mathcal{Q} = \{F \otimes S_0\}$ where F runs through $\mathcal{A}$ and S_0 is the sheaf of zero-dimensional Alexander-Spanier cochains. Define $\mathcal{I}$ to be the set of all monomorphisms $\alpha : A' \to A$ such that $\text{Hom}(A, Q) \to \text{Hom}(A', Q) \to 0$ is exact for all $Q \in \mathcal{Q}$. Denote by $\bar{H}^n(\mathfrak{X}, C)$ the group $\mathcal{I}\text{-Ext}^n(R, C)$ where R is the simple sheaf with stalk R and C is a sheaf of R-modules.

THEOREM 5.1. *If $\mathfrak{X}$ is a paracompact space, then* $H^n(\mathfrak{X}, C) \approx \bar{H}^n(\mathfrak{X}, C)$.

The proof is a simple application of the uniqueness theorem in [1] to the observation that the monomorphism $F \to F \otimes S_0$ is contained in $\mathcal{I}$ for every sheaf F. Thus H and $\bar{H}$ are both exact connected sequences of functors vanishing on $\mathcal{Q}$, and every element of $\mathcal{A}$ can be $\mathcal{I}$-imbedded in an element of $\mathcal{Q}$.

An open question is the following : Let V be an algebraic variety (in the senes of Serre), and let $\mathcal{A}$ be the category of coherent algebraic sheaves over V. Grothendieck has shown that $H^n(V, C)$ is isomorphic to $\overline{\mathrm{Ext}^n}(\mathcal{O}, C)$ where $\overline{\mathrm{Ext}^n}$ is obtained by going to the category of algebraic sheaves (sheaves of $\mathcal{O}$-modules) and taking injective resolutions. How does $\mathrm{Ext}^n(\mathcal{O}, C)$, in the category $\mathcal{A}$, compare with $\overline{\mathrm{Ext}^n}(\mathcal{O}, C)$? For dimensions 0 and 1 they are the same.

6. Appendix

In [1] it was stated that the notions of infinite direct sums and products were not easily expressible in exact categories. We should like to give here categorical definitions of these terms as well as of direct and inverse limits.

Let

(D) $$A_\alpha \xrightarrow{i_\alpha} A \xrightarrow{p_\alpha} A_\alpha$$

be a direct family (i. e., $p_\alpha i_\alpha = \delta_\alpha^\beta$).

DEFINITIONS. (D) is a *direct sum representation* of A if for every family of maps $\{f_\alpha : A_\alpha \to C\}$, there exists a unique map $f : A \to C$ such that $f i_\alpha = f_\alpha$.

(D) is a *direct product representation* of A if for every family of maps $\{f_\alpha : C \to A_\alpha\}$, there is a unique map $f : C \to A$ such that $p_\alpha f = f_\alpha$.

Now let Λ be a directed set, $\{A_\lambda ; \pi_\lambda^\mu\}$ a directed system of objects, and $\{B_\lambda ; \rho_\mu^\lambda\}$ an inverse system of objects.

DEFINITIONS. The direct limit of $\{A_\lambda ; \pi_\lambda^\mu\}$ is a pair $(A, \{h_\lambda\})$ where A is an object of $\mathcal{A}$, $h_\lambda : A_\lambda \to A$, and satisfying the condition that if $\{\alpha_\lambda : A_\lambda \to C\}$ is a family of maps such that $\alpha_\mu \pi_\lambda^\mu = \alpha_\lambda$ for all $\lambda < \mu$, then there exists a unique map $\alpha : A \to C$ such that $\alpha h_\lambda = \alpha_\lambda$ for all λ.

The inverse limit of $\{B_\lambda ; \rho_\mu^\lambda\}$ is a pair $(B, \{k_\lambda\})$, $k_\lambda : B \to B_\lambda$, satisfying dual hypotheses to those of direct limits.

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