Satellites and Universal Functors

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SATelliteS AND Universal FunCTORS

By David A. Buchsbaum

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Introduction

In [3], Cartan and Eilenberg showed that if $\Lambda$ is a ring (with identity element) and $A$ and $C$ are two (left) $\Lambda$-modules, then one can define for each integer $n \geq 0$, an abelian group $\text{Ext}_{\Lambda}^n(A, C)$, with $\text{Ext}_{\Lambda}^n(A, C) = \text{Hom}_{\Lambda}(A, C)$ the group of $\Lambda$-homomorphisms of $A$ into $C$. Moreover, if $\alpha: A \to A'$, $\gamma: C' \to C$ are $\Lambda$-homomorphisms, then one has induced maps $\gamma_*: \text{Ext}^n(A, C') \to \text{Ext}^n(A, C)$ and $\alpha^*: \text{Ext}^n(A', C) \to \text{Ext}^n(A, C)$. The definition of the functors $\text{Ext}^n(A, C)$ relied heavily on the existence of sufficiently many projective modules in the category of all $\Lambda$-modules.

In a recent note [1], we showed that it is possible to define the functors $\text{Ext}^n(A, C)$ in an arbitrary exact category $\mathcal{A}$ without the use of projective or injective objects. In fact, we defined functors $\mathcal{I}$-$\text{Ext}^n(A, C)$ where $\mathcal{I}$ is a family of monomorphisms of $\mathcal{A}$ satisfying certain conditions (an h.f. class [1]). It was therefore natural to suppose that if $T$ is an additive functor from the category $\mathcal{A}$ to the category $\mathcal{B}$, then one should be able to define the derived (or satellite) functors of $T$, [3], without recourse to injectives or projectives. We show in this note that this can be done, provided we assume that direct (or inverse) limits exist in the category $\mathcal{B}$. Derived functors $\mathcal{I}$-$S^nT$, relative to an h. f. class of monomorphisms $\mathcal{I}$ can also be obtained.

By considering universal functors, we can show that if $\mathcal{X}$ is a topological space, and $\Phi$ is a family of paracompact supports in the sense of Cartan [2], then $H^q(\mathcal{X}, C)$ can be expressed as a satellite of $\Gamma_\Phi$, where $H^q(\mathcal{X}, C)$ denotes the $\Phi$-cohomology of $\mathcal{X}$ with coefficients in the sheaf $C$, and $\Gamma_\Phi(C)$ is the module of sections of $C$ with supports in $\Phi$. We also show that if $V$ is a projective variety, then $H^q(V, C) \approx \text{Ext}^q(\mathcal{O}, C) \approx S^q\Gamma(C)$ where $C$ is a coherent algebraic sheaf over $V$, $\mathcal{O}$ is the sheaf of local rings of $V$, and $\text{Ext}^q(\mathcal{O}, C)$ and $S^q\Gamma(C)$ are defined strictly in the category of coherent algebraic sheaves.

1. Definition of $S^nT$

Let $T$ be a contravariant additive functor from the exact category $\mathcal{A}$ to the exact category $\mathcal{B}$. Denote by $P$ the exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0.$$
Now let \( \mathcal{D}_A = \{P\} \) be the totality of all such exact sequences in \( A \). We introduce a partial ordering in \( \mathcal{D}_A \) as follows: \( P' < P \) if there exists a map from \( P \) to \( P' \) over \( A \), i.e., if there is a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & M \\
g & \downarrow & f \\
0 & \rightarrow & M' \rightarrow P' \rightarrow A \rightarrow 0
\end{array}
\]

This is equivalent to saying that \( P' < P \) if there is a map \( g: M \rightarrow M' \) such that under the map \( g_x: \text{Ext}^i(A, M) \rightarrow \text{Ext}^i(A, M') \), the sequence \( P \) is carried into the sequence \( P' \) (i.e., \( P' = g_xP \)).

**Proposition 1.1.** With the above ordering, \( \mathcal{D}_A \) is a directed class.

**Proof.** What we must show is that if \( P_1, P_2 \) are in \( \mathcal{D}_A \), then there is a \( P \) in \( \mathcal{D}_A \) such that \( P > P_1, P > P_2 \). To this end, denote by \( \Delta: A \rightarrow A + A \) the diagonal map (i.e., \( \Delta = i_1 + i_2 \) where \( i_1 \) and \( i_2 \) are the injections of \( A \) into \( A + A \)). Then for any \( C \) in \( A \), we have \( \Delta^*: \text{Ext}^i(A + A, C) \rightarrow \text{Ext}^i(A, C) \). In particular, let \( P_1 + P_2 \) be the exact sequence

\[
\begin{array}{ccc}
0 & \rightarrow & M_1 + M_2 \\
& \rightarrow & P_1 + P_2 \\
& \rightarrow & A + A \rightarrow 0
\end{array}
\]

and let \( P = \Delta^*(P_1 + P_2): 0 \rightarrow M_1 + M_2 \rightarrow P \rightarrow A \rightarrow 0 \). Then it is clear that \( P \) dominates both \( P_1 \) and \( P_2 \).

One can define \( P \) directly as follows. If \( 0 \rightarrow M_1 \rightarrow P_1 \xrightarrow{\beta_1} A \rightarrow 0 \), \( 0 \rightarrow M_2 \rightarrow P_2 \xrightarrow{\beta_2} A \rightarrow 0 \) represent \( P_1 \) and \( P_2 \) respectively, then we may obtain the exact sequence

\[
\begin{array}{ccc}
0 & \rightarrow & P \\
& \rightarrow & P_1 + P_2 \xrightarrow{\gamma} A \rightarrow 0
\end{array}
\]

where \( \gamma = \beta_1 p_1 - \beta_2 p_2 \) and \( p_1, p_2 \) are the projections of \( P_1 + P_2 \) onto \( P_1 \) and \( P_2 \) respectively. \( P \), then, is the kernel of \( \gamma \). In terms of modules, we see that \( P = \{ (u_1, u_2) | u_1 \in P_1, u_2 \in P_2 \text{ and } \beta_1 u_1 = \beta_2 u_2 \} \).

**Proposition 1.2.** If

\[
\begin{array}{ccc}
0 & \rightarrow & M \\
g & \downarrow & f \\
0 & \rightarrow & M' \rightarrow P' \rightarrow A \rightarrow 0
\end{array}
\]

is a commutative diagram with exact rows, then we obtain a commutative diagram

\[
\begin{array}{ccc}
T(P') & \rightarrow & T(M') \\
& \downarrow & \downarrow \phi_{P'} \\
T(P) & \rightarrow & T(M) \rightarrow F_{P'} \rightarrow 0
\end{array}
\]
and \( \theta^*_{\rho} \) is independent of the choice of \( g \) (and \( f \)).

The proof is left to the reader.

We therefore have for each \( P \) in \( \mathcal{P}_A \), an object \( F_P \) in \( \mathcal{B} \), and for each \( P' < P \) a uniquely defined map \( \theta^*_{\rho_{P'}} : F_{P'} \rightarrow F_P \) satisfying the usual transitivity conditions. Making logical adjustments that are necessary to reconcile sets with classes (for instance, assume that the objects of \( \mathcal{A} \) form a set), we obtain a direct system of objects \( \{F_{P'}; \theta^*_{\rho_{P'}}\} \) in \( \mathcal{B} \). Since we are assuming that direct limits exist in \( \mathcal{B} \), we may define

\[
S^iT(A) = \text{Dir lim}\{F_{P'}; \theta^*_{\rho_{P'}}\}.
\]

If \( f : A \rightarrow B \) is a map, we define a map \( S^iT(f) : S^iT(B) \rightarrow S^iT(A) \) in the following way: to each exact sequence \( P : 0 \rightarrow M \rightarrow P \rightarrow B \rightarrow 0 \) in \( \text{Ext}^i(B, M) \), we assign the exact sequence \( f^*(P) \) in \( \text{Ext}^i(A, M) \). We thus obtain an exact sequence \( 0 \rightarrow M \rightarrow P' \rightarrow A \rightarrow 0 \) in \( \mathcal{P}_A \) and a commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & M & \rightarrow & P' & \rightarrow & A & \rightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & P & \rightarrow & B & \rightarrow & 0.
\end{array}
\]

Thus to each \( P \) in \( \mathcal{P}_B \), we have assigned an element \( f^*(P) = P' \) in \( \mathcal{P}_A \), and it is clear that \( f^* \) is order-preserving. Moreover, the commutative diagram above yields the commutative diagram

\[
\begin{array}{cccc}
T(P) & \rightarrow & T(M) & \rightarrow & F_P & \rightarrow & 0 \\
\downarrow & & \downarrow & \downarrow & \downarrow_{\rho_P} & & \\
T(P') & \rightarrow & T(M) & \rightarrow & G_{P'} & \rightarrow & 0.
\end{array}
\]

Furthermore, if \( P_1 < P_2 \), we have the commutative diagram

\[
\begin{array}{ccc}
F_{P_1} & \xrightarrow{\rho_{P_1}} & G_{f^*(P_1)} \\
\downarrow & & \downarrow \\
F_{P_2} & \xrightarrow{\rho_{P_2}} & G_{f^*(P_2)}
\end{array}
\]

so that the system of maps \( (f^*, \{\rho_P\}) \) is a map from the direct system defining \( S^iT(B) \) to that defining \( S^iT(A) \). We therefore obtain a unique map \( S^iT(f) : S^iT(B) \rightarrow S^iT(A) \), and it is clear that \( S^iT \) is a contravariant additive functor.

The functors \( S^nT \) are defined as follows:

\[
S^nT = S^i(S^{n-1}T)
\]

for \( n > 1 \).

So far the discussion has been devoted to contravariant functors and their right satellites. If \( T \) is a covariant functor, we may define \( S^nT \) in the following way.
Let $\mathcal{A}^*$ denote the dual category of $\mathcal{A}$ (i.e., the objects of $\mathcal{A}^*$ are the same as those of $\mathcal{A}$, and $H^*(A, B)$ in $\mathcal{A}^*$ is defined to be $H(B, A)$). Given the covariant functor $T: \mathcal{A} \to \mathcal{B}$, we may define the contravariant functor $T^*: \mathcal{A}^* \to \mathcal{B}$ by setting $T^*(A) = T(A)$. We then define $S^n T: \mathcal{A} \to \mathcal{B}$ by setting $S^n T(A) = S^n T^*(A)$.

If $T: \mathcal{A} \to \mathcal{B}$ is a functor, and $\mathcal{B}$ has inverse limits, then $\mathcal{B}^*$ (the dual category of $\mathcal{B}$) has direct limits and the right satellites $S^n U: \mathcal{A} \to \mathcal{B}^*$ are defined, where $U: \mathcal{A} \to \mathcal{B}^*$ is defined by $U(A) = T(A)$. We may therefore define the left satellites $S_n T: \mathcal{A} \to \mathcal{B}$ by setting $S_n T(A) = S^n U(A)$.

2. The exact sequence

If we have an exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, we will define a map $\delta^s: S^n T(A') \to S^{n+1} T(A'')$. It is obviously sufficient to do this for $\delta^s: T(A') \to S^1 T(A'')$. However, this is trivial for we have

$$
T(A) \Rightarrow T(A') \begin{array}{c} a \end{array} F_A \Rightarrow 0
$$

exact, and the canonical map $h_A: F_A \to S^1 T(A'')$ (since $A$, or $0 \to A' \to A \to A'' \to 0$ is in $\mathcal{D}_A$, and thus $F_A$ is part of the defining system for $S^1 T(A'')$).

We therefore define $\delta^s$ to be $h_A a: T(A') \to S^1 T(A'')$.

Before stating the next theorem, let us recall that a (contravariant) functor $T$ is called half-exact [3] if for every exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence $T(A') \to T(A) \to T(A')$ is exact.

**Theorem 2.1.** Let $T$ be a half-exact contravariant functor from $\mathcal{A}$ to $\mathcal{B}$, and let $0 \to A' \begin{array}{c} r \end{array} A \begin{array}{c} \sigma \end{array} A'' \to 0$ be exact. Then if $\mathcal{B} = \mathcal{M}_\Lambda$, i.e., if $\mathcal{B}$ is the category of (left) modules over some ring $\Lambda$ (or any category of objects with elements which preserves exact sequences under direct limits), the sequence

$$
T(A'') \to T(A) \to T(A') \to S^1 T(A'') \to S^1 T(A) \to \cdots
$$

$$
\to S^n T(A') \to S^{n+1} T(A'') \to S^{n+1} T(A) \to \cdots
$$

is exact.

**Proof.** We will merely sketch the proof, and occasionally use language as though $\mathcal{A}$ were a category of modules. All statements are valid for an arbitrary exact category.

First, we will show that $S^1 T$ is half-exact. From this it will follow that $S^n T$ is also.

Let $0 \to M \begin{array}{c} u \end{array} P \begin{array}{c} v \end{array} A \to 0$ be in $\mathcal{D}_A$. Then we make the following two observations:

---

$^2$ Grothendieck [4] has been able to handle functors whose range is a category satisfying his axiom AB5.
(a) we obtain an exact sequence $0 \longrightarrow P' \longrightarrow P \overset{\tau v}{\longrightarrow} A'' \longrightarrow 0$ in $\mathcal{D}_{A''}$, and the set of all such elements in $\mathcal{D}_{A''}$, i.e., those obtained in this way from $\mathcal{D}_A$, is cofinal in $\mathcal{D}_{A''}$;

(b) $P'$ is the counter-image of $A'$ in $P$ and the exact sequence $0 \longrightarrow M \longrightarrow P' \longrightarrow A' \longrightarrow 0$ is the one induced by $0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0$ under the map $\gamma: A' \to A$ (i.e., $P' = \gamma^*(P)$).

We can thus consider the commutative diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & M & N \longrightarrow P' \longrightarrow 0 \\
\downarrow^s & \downarrow^h & \downarrow^{u'} \\
0 & P' \rightarrow P' + P \rightarrow P \longrightarrow 0 \\
\downarrow^t & \downarrow^g & \downarrow^{\tau v} \\
0 & A' \overset{\gamma}{\longrightarrow} A \overset{\tau}{\longrightarrow} A'' \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

where $g(p', p) = \gamma t(p') + v(p)$. Not only are all the rows and columns exact, but the top two rows split (the middle one by construction). We therefore obtain the commutative diagram

\[
\begin{array}{ccc}
0 & T(P) & T(P' + P) \longrightarrow T(P') \longrightarrow 0 \\
\downarrow^{T(u')} & \downarrow^{T(h)} & \downarrow^{T(s)} \\
0 & T(P') & T(N) \longrightarrow T(M) \longrightarrow 0
\end{array}
\]

with split exact rows, and hence an exact sequence $F''_p \longrightarrow F'_{p'} + F'_{p'} \longrightarrow 0$ where $F''_p = \text{Coker } T(u'), F'_{p'} + F'_{p'} = \text{Coker } T(h), F'_{p'} = \text{Coker } T(s)$.

It can be shown without too much difficulty that the set of exact sequences $0 \longrightarrow N \longrightarrow P' + P \longrightarrow A \longrightarrow 0$ obtained as above from exact sequences $0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0$ is cofinal in $\mathcal{D}_A$. Thus, by taking direct limits of the exact sequences $F''_p \longrightarrow F'_{p'} + F'_{p'} \longrightarrow 0$, we obtain an exact sequence

$S^1T(A'') \longrightarrow S^1T(A) \longrightarrow \lim_{\rightarrow} \{F'_{p'}\}.$

The map $S^1T(A'') \longrightarrow S^1T(A)$ so obtained is easily seen to be $S^1T(\tau)$. Moreover, the composition $S^1T(A) \longrightarrow \lim_{\rightarrow} \{F'_{p'}\} \longrightarrow S^1T(A')$ is seen to be $S^1T(\gamma)$. Hence, if we show that the map $\lim_{\rightarrow} \{F'_{p'}\} \longrightarrow S^1T(A)$ is a monomorphism, we will have the half-exactness of $S^1T$.  

We suppose that $0 \rightarrow M \xrightarrow{a} P' \xrightarrow{t} A' \rightarrow 0$ is obtained from $0 \rightarrow M \xrightarrow{u} P \xrightarrow{v} A \rightarrow 0$, and let

$$
0 \rightarrow C \xrightarrow{u} Q \xrightarrow{v} A' \rightarrow 0
$$

be commutative ($Q$ in $\mathcal{D}_{A'}$). We construct the exact sequence

$$
0 \rightarrow D \xrightarrow{k} P + Q \rightarrow A \rightarrow 0
$$

where $k(p, q) = v(p) + \gamma \bar{v}(q)$, and observe that the sequence induced over $A'$ by this exact sequence is simply $0 \rightarrow D \rightarrow P' + Q \rightarrow A' \rightarrow 0$. The commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{u} & C \xrightarrow{\bar{v}} Q \xrightarrow{v} A' \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & D \xrightarrow{\gamma} P' + Q \xrightarrow{\bar{v}} A' \rightarrow 0 \\
\downarrow & & \downarrow \\
P' & = & P' \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}
$$

with exact rows and columns, yields

$$
\begin{array}{cccc}
0 & \rightarrow & T(P') & \rightarrow T(P' + Q) & \rightarrow T(Q) & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
T(P') & \rightarrow & T(D) & \rightarrow & T(C) & \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & F' & \rightarrow & F'_q & \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & \\
\end{array}
$$

with rows and columns exact. Since we have a commutative diagram
0 \longrightarrow D \longrightarrow P' + Q \longrightarrow A' \longrightarrow 0
\downarrow \quad \downarrow \quad \parallel
0 \longrightarrow M \longrightarrow P' \longrightarrow A' \longrightarrow 0,

we see that the map \( F'_{p'} \rightarrow F'_q \) is factorable: \( F'_{p'} \rightarrow F' \rightarrow F'_q \). Thus, if \( f' \) is in \( \text{Ker}(F'_{p'} \rightarrow F'_q) \), it is in \( \text{Ker}(F'_{p'} \rightarrow F') \). This implies that the map \( \lim \{ F'_{p'} \} \rightarrow S'T(A') \) is a monomorphism.

The proofs of exactness of \( T(A) \longrightarrow T(A') \overset{\beta}{\longrightarrow} S'T(A'') \) and \( T(A') \overset{\beta}{\longrightarrow} S'T(A'') \longrightarrow S'T(A) \) are analogous to those in [3, Chap III, section 3] with minor modifications to allow for direct limits.

The above theorem holds, of course, when \( T \) is covariant, since no modifications of the range category \( \mathcal{B} \) are necessary in order to define the right satellites of a covariant functor.

The situation for left satellites seems to be a bit mysterious. All we can say so far is that if \( T: \mathcal{A} \rightarrow \mathcal{B} \) is a half-exact functor, then the exactness of the left-satellite sequence

\[
\cdots \longrightarrow S_n T(A') \longrightarrow S_{n-1} T(A'') \longrightarrow \cdots \longrightarrow T(A) \longrightarrow T(A')
\]

is equivalent to the exactness of the right-satellite sequence

\[
U(A') \rightarrow U(A) \rightarrow U(A'') \rightarrow S^1 U(A') \rightarrow \cdots \rightarrow S^n U(A'') \rightarrow S^{n+1} U(A') \rightarrow \cdots
\]

where \( U: \mathcal{A} \rightarrow \mathcal{B}^* \) is defined by \( U(A) = T(A) \).

3. The relative satellites

In this section, we assume that \( \mathcal{J} \) is an h.f. class of monomorphisms of \( \mathcal{A} \) (see [1]). We then let \( \mathcal{I}D_A \) be the class of all exact sequences \( \{0 \rightarrow M \rightarrow P \rightarrow 0\} \) where the monomorphism \( M \rightarrow P \) is in \( \mathcal{J} \). It is clear that \( \mathcal{I}D_A \) is a subclass of \( \mathcal{D}_A \), and with the induced ordering is a directed subclass. We may therefore define \( \mathcal{J}S'T(A) \) to be the limit of the direct system in \( \mathcal{B} \) taken over \( \mathcal{I}D_A \). \( \mathcal{J}S^A T(A) \) is defined to be \( \mathcal{J}S'(\mathcal{J}S^{A-1} T(A)) \).

In order to prove a theorem analogous to 2.1, we must impose the following additional condition on the h.f. class \( \mathcal{J} \):

\((*)\) If

\[
\begin{array}{c}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & A' & A' \\
\rho' \downarrow & \rho' \downarrow & \rho' \\
0 & B' & B' \\
\downarrow & \downarrow & \downarrow \\
0 & C' & C' \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]
is a commutative diagram with exact rows and columns such that $\alpha'$, $\beta'$, $\gamma'$, $\rho'$, $\rho''$ are in $\mathcal{I}$, then $\rho$ is also in $\mathcal{I}$.

We now obtain

**Theorem 3.1.** Let $\mathcal{I}$ be an h.f. class satisfying the additional condition $(\ast)$, let $0 \rightarrow A' \overset{i}{\rightarrow} A \overset{r}{\rightarrow} A'' \rightarrow 0$ be an exact sequence with $\gamma$ in $\mathcal{I}$, and let $T$ be a half-exact functor $T: \mathcal{A} \rightarrow \mathcal{M}_\Lambda$ (where $\mathcal{M}_\Lambda$ is as in 2.1). Then the sequence

$$T(A'') \rightarrow T(A) \rightarrow T(A') \rightarrow \mathcal{I} \cdot S^1 T(A'') \rightarrow \mathcal{I} \cdot S^1 T(A) \rightarrow \cdots$$

$$\rightarrow \mathcal{I} \cdot S^n T(A') \rightarrow \mathcal{I} \cdot S^{n+1} T(A'') \rightarrow \mathcal{I} \cdot S^{n+1} T(A) \rightarrow \cdots$$

is exact.

The proof of this theorem follows from that of 2.1 if one observes that all the monomorphisms constructed in the proof of 2.1 are in $\mathcal{I}$.

4. Universal functors

We recall that a connected sequence of functors (contravariant) [3], is a sequence $\{T^n\}_{n \geq 0}$ of (contravariant) functors together with maps $\delta^n_\epsilon: T^n(A') \rightarrow T^{n+1}(A'')$ for every short exact sequence $E: 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ such that

(a) for every exact sequence $E: 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$, the sequence

$$T(E): \ T^n(A'') \rightarrow T^n(A) \rightarrow T^n(A') \rightarrow T^{n+1}(A') \rightarrow T^{n+1}(A) \rightarrow \cdots$$

$$\rightarrow T^n(A') \rightarrow T^{n+1}(A'') \rightarrow T^{n+1}(A) \rightarrow \cdots$$

is of order two;

(b) for every commutative diagram of exact sequences

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

$$0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0,$$

the diagram

$$T^n(B') \rightarrow T^{n+1}(B'')$$

$$\downarrow \quad \downarrow$$

$$T^n(A') \rightarrow T^{n+1}(A'')$$

commutes.

The connected sequence of functors is called **exact** if for every exact sequence $E$, the sequence $T(E)$ is exact. It is called **universal** [4] if for any connected sequence $\{U^n\}$ and any natural transformation $\lambda_0: T^0 \rightarrow U^0$,
there is a unique extension \( \lambda = \{ \lambda^n: T^n \to U^n \} \) such that for every exact sequence \( E \), the diagram

\[
\begin{array}{ccc}
T^n(A') & \longrightarrow & T^{n+1}(A'') \\
\downarrow & & \downarrow \\
U^n(A') & \longrightarrow & U^{n+1}(A'')
\end{array}
\]

commutes.

**Proposition 4.1.** Let \( T: \mathcal{A} \to \mathcal{B} \) be a contravariant functor, and let \( \mathcal{B} \) have direct limits. Then the connected (not necessarily exact) sequence of functors \( \{ S^nT \} \), with \( S^0T = T \), is universal.

The construction of \( S^nT \) as direct limits forces the universality of the sequence of functors. In fact, one might have used the motivation of constructing a universal connected sequence of functors beginning with \( T \) as a means of obtaining the satellites. We will therefore not give a proof of this proposition.

**Proposition 4.2.** Let \( T = \{ T^n \} \) be an exact, connected sequence of (contravariant) functors with \( T^n: \mathcal{A} \to \mathcal{M}_\lambda \). If for each \( A \) in \( \mathcal{A} \), each positive integer \( n \), and each \( t \) in \( T^n(A) \) there is an exact sequence \( E_t: 0 \to M \to P \to A \to 0 \) such that \( t \) goes into zero under the map \( T^n(A) \to T^n(P) \), then \( T = \{ T^n \} \) is universal.

**Proof.** Let \( U = \{ U^n \} \) be any connected sequence of (contravariant) functors, \( U^n: \mathcal{A} \to \mathcal{M}_\lambda \), and let \( \lambda^0: T^0 \to U^0 \) be a natural transformation. We will define the extension \( \lambda = \{ \lambda^n: T^n \to U^n \} \), but will not verify all the things one must usually verify in proving such a theorem.

Suppose we have defined \( \lambda^0, \ldots, \lambda^{n-1} \), and we want to define \( \lambda^n(A): T^n(A) \to U^n(A) \). Let \( t \) be in \( T^n(A) \). Then, by our hypothesis, we have the exact sequence \( E_t: 0 \to M \to P \to A \to 0 \), and thus the diagram

\[
\begin{array}{ccc}
T^{n-1}(M) & \delta^{n-1}_E \longrightarrow & T^n(A) \longrightarrow & T^n(P) \\
\downarrow \lambda^{n-1}(M) & & \downarrow & \\
U^{n-1}(M) & \delta^{n-1}_E \longrightarrow & U^n(A)
\end{array}
\]

with top row exact. Since \( t \to 0 \), \( t = \delta^{n-1}_E(m) \) for some \( m \) in \( T^{n-1}(M) \). Define \( \lambda^n(A)(t) = \delta^{n-1}_E \lambda^{n-1}(M)(m) \).

To show that this definition is independent of the choice of \( m \) in \( T^{n-1}(M) \) is trivial. To show that it is independent of the choice of \( E_t \), one makes use of the fact that any two exact sequences in \( \mathcal{D}_\mathcal{A} \) are dominated by a third. The same device shows that \( \lambda^n \) is additive. All commutativity properties are equally easy to show.
PROPOSITION 4.3. Let \( \mathcal{A} \) be an exact category, let \( C \) be a fixed object of \( \mathcal{A} \), and let \( T(A) = \text{Hom}(A, C) \). Then the connected sequences of functors \( \{S^nT\} \) and \( \{\text{Ext}^n(A, C)\} \) are naturally equivalent (for definition of \( \text{Ext}^n(A, C) \), see [1]).

PROOF. We have already seen that \( \{S^nT\} \) is universal. Since \( \{\text{Ext}^n(A, C)\} \) is an exact connected sequence of functors, we will show that it is universal if we can show that the hypotheses of 4.2 apply.

Let \( t \) in \( \text{Ext}^n(A, C) \) be represented by
\[
0 \rightarrow C \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow A \rightarrow 0.
\]
Let \( E_t \) be the sequence \( 0 \rightarrow M \rightarrow X_1 \rightarrow A \rightarrow 0 \), where \( M = \text{Ker} \tau \). Then clearly \( t \rightarrow 0 \) under the map \( \text{Ext}^n(A, C) \rightarrow \text{Ext}^n(X_1, C) \). Thus \( \{\text{Ext}^n(A, C)\} \) is universal and \( \text{Ext}^n(A, C) = \text{Hom}(A, C) = T(A) \). Hence we have our result (since two universal sequences of functors with the same initial functor are naturally equivalent).

We should remark that if we hold \( A \) fixed, and consider the functor \( U(C) = \text{Hom}(A, C) \), then the connected sequences of covariant functors \( \{S^nU\}, \{\text{Ext}^n(A, C)\} \) are naturally equivalent.

5. Applications

Let \( \mathcal{X} \) be a topological space, and let \( \mathcal{A} \) be the category of sheaves (of abelian groups) over \( \mathcal{X} \). Let \( \Phi \) be a family of paracompact supports as defined in [2] and let \( \Gamma_\Phi(F) \) be the functor which assigns to each sheaf \( F \) the group of sections of \( F \) with supports in \( \Phi \). Then we have

THEOREM 5.1. The connected sequences of (covariant) functors \( \{S^n\Gamma_\Phi(F)\} \) and \( \{H^\Phi_\mathcal{X}(F, F)\} \) are naturally isomorphic.

PROOF. Since \( S^n\Gamma_\Phi(F) = \Gamma_\Phi(F) = H^\Phi_\mathcal{X}(\mathcal{X}, F) \), and since \( \{S^n\Gamma_\Phi(F)\} \) is universal, it suffices to prove that \( \{H^\Phi_\mathcal{X}(\mathcal{X}, F)\} \) is universal. To do this, we must show that \( \{H^\Phi_\mathcal{X}\} \) satisfies the (dual of the) hypotheses of 4.2. However, we can show even more. For if \( F \) is a sheaf, and \( S_0 \) is the sheaf of zero-dimensional Alexander-Spanier cochains over \( \mathcal{X} \), then \( F \rightarrow F \otimes S_0 \) is a monomorphism, \( F \otimes S_0 \) is fine, and thus \( H^\Phi_\mathcal{X}(\mathcal{X}, F) \otimes S_0 = 0 \) for all \( n > 0 \). Thus \( \{H^\Phi_\mathcal{X}\} \) is universal, and we have our result.

Now let \( V \) be a projective variety, and let \( \mathcal{A} \) be the category of coherent algebraic sheaves over \( V \). Let \( T(C) \) be the functor \( \text{Hom}(\mathcal{O}, C) = \Gamma(C) \) where \( \mathcal{O} \) is the sheaf of local rings of \( V \).

THEOREM 5.2. The connected sequences of functors \( \{S^n\Gamma(C)\}, \{H^n(V, C)\}, \) and \( \{\text{Ext}^n(\mathcal{O}, C)\} \) are all naturally isomorphic.

PROOF. We have already seen in section 4 that \( \{S^n\Gamma\} \) and \( \{\text{Ext}^n(\mathcal{O}, C)\} \) are naturally isomorphic. Hence it will suffice to prove that \( \{H^n(V, C)\} \)
is universal.\textsuperscript{3} Let $D$ be the divisor of hyperplane sections of $V$ and let $D(q)$ be the $q$th multiple of $D$ as defined in [5]. Let $C$ be any coherent algebraic sheaf, and let $S(q)$ be the sheaf associated with $D(q)$ ($S(q)$ is coherent). Then the map $C \rightarrow C \otimes S(q)$ is a monomorphism, and $H^n(V, C \otimes S(q))$ is zero for all positive $n$ if $q$ is sufficiently large. Hence \{\text{Ext}^n(V, C)\} is universal, and we are done.

We should stress here that Ext$^n(\mathcal{O}, C)$ is defined in the category of coherent algebraic sheaves. Hence an element of Ext$^n(\mathcal{O}, C)$(or $H^n(V, C)$) can be represented by an $n$-fold extension

$$0 \longrightarrow C \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \mathcal{O} \longrightarrow 0$$

of coherent sheaves.

We should also remark that if we let $U(A) = \text{Hom}(A, C)$, with $C$ a fixed coherent sheaf, then $S^n U(A) = \text{Ext}^n(A, C)$. In particular, $S^n U(\mathcal{O}) = \text{Ext}^n(\mathcal{O}, C) = H^n(V, C)$ In section 1 we saw that $S^n U(\mathcal{O})$ could be computed as a direct limit of \{\text{Ext}^n(P)\} where

$$S^{n-1} U(P) \longrightarrow S^{n-1} U(M) \longrightarrow F_p \longrightarrow 0$$

is exact, coming from an exact sequence of coherent sheaves:

$$0 \longrightarrow M \longrightarrow P \longrightarrow \mathcal{O} \longrightarrow 0.$$ 

However, if we restrict ourselves to locally free sheaves $P$, we obtain a cofinal subset of $\mathcal{P}_\mathcal{O}$. Thus $\mathcal{P}_\mathcal{O}' = \{0 \rightarrow M \rightarrow P \rightarrow \mathcal{O} \rightarrow 0 | P$ locally free\} is an indexing set for defining $S^n U(0)$.

If one could show for an abstract variety $V$ that $H^n(V, C)$ satisfies the relatively weak conditions of 4.2, we would again have $H^n(V, C) \cong \text{Ext}^n(\mathcal{O}, C)$.

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**Bibliography**


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