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Dedicated to Paolo Valabrega on the occasion of his 60^{th} birthday

Abstract. A brief indication of the role of resolutions in diverse parts of algebra.

1. Introduction

First, I want to say how glad I am to be able to participate in this celebration of Paolo's 60^{th} birthday (although he is quick to tell me that he hasn't yet arrived at that venerable age: we must all wait until 29 May for that event to occur).

What I want to illustrate in this talk are the various ways it has helped to look for or at complexes that may be resolutions of significant modules. Another way to say this is that we're looking at resolutions or complexes with appropriate grade-sensitivity (in the case of commutative algebra) or which facilitate calculation of Ext modules (in the case of representation theory). First, let's look at some of the more classical or historical developments (from a personal point of view).

• The proof of the Hilbert Syzygy Theorem stimulated my interest in "resolutions". In this case a rather simple-minded construction (the Koszul complex) yielded a fairly powerful result. Hence the idea that a great deal of information can perhaps reside in complexes.

• A conversation with Emil Artin at the very beginning of my career made me aware of the need for a homological characterization of regular local rings. This conversation stimulated further interest in resolutions, and Gröbner's book on Algebraic Geometry [14] first led me to observe the fact that given an exact sequence

$$0 \to A \to B \to C \to 0,$$

the mapping cone of a map of a resolution of A into one of B, produces a resolution of C. It took a little work to show that if both given resolutions were minimal, and the map between them local, then so was the mapping cone. We'll see more on that theme later.

• Checking out the first step of the theorem:

$$\mathrm{pd}_R M + \mathrm{codim}\, M = \mathrm{codim}\, R,$$

involved the use of the minimal resolution of a module of projective dimension equal to 1.

• Proving that $\mathrm{gldim}(R) \geq \mathrm{edim}(R)$ (embedding dimension) involved explicit construction of the Koszul complex as a direct summand of any resolution of the residue field, k, of R.

So reasons for studying resolutions and special complexes abound; here are a few areas in which I've resolutely dug in:

(a) Generalized Koszul Complexes

(b) Resolutions of determinantal ideals

(c) Resolutions of Weyl modules

We'll take the above-mentioned items in turn.

2. Generalized Koszul Complexes

In [6], I introduced a family of complexes, the so-called Generalized Koszul Complexes, and in [9] Rim and I developed many of their properties. We also defined a more general notion of multiplicity; once we had a way of "resolving" an arbitrary finitely generated module of finite length, it seemed reasonable to attach a multiplicity to it and not just to a cyclic module of finite length.

The complexes defined in [6], however, were very fat; there were many redundant terms, and they were generally unwieldy. Around ten years later, Eisenbud and I [8] succeeded in writing down a corresponding family of "slimmed-down" complexes which, in a particular case, coincided with the very elegant Eagon-Northcott complex ([13]). Although neither Eisenbud nor I was familiar with representation theory, it turned out that our slimming-down involved certain modules which we learned a bit later were the Weyl modules corresponding to "hook" partitions, namely, representations of a very special type.

Although there now were two classes of "generalized Koszul complexes", no attempt was made to directly connect them until two years ago (as far as I know), when Boffi and I showed ([4]) that the fat complexes were homotopically equivalent to the slim ones. This permits us, among other things, to carry a certain 'homothety homotopy' that was used to prove the grade-sensitivity of the fat complexes, over to the slim ones. One possible explanation for there having been no direct comparison of these complexes way back in the 70's is that nobody working on them knew enough about representations to be able to make the comparison.

3. Resolutions of determinantal ideals

Shortly after Eisenbud and I had learned that we had used "hooks" in our complexes, Eisenbud was approached by A. Lascoux, a student, at the time, of Verdier in Paris (where Eisenbud was visiting for the year), who asked him to suggest some application of Lascoux' recently developed methods (involving combinatorics and representation theory) to commutative algebra. Eisenbud pointed out that certain representation modules had emerged in our work on complexes, that it had been long recognized that the terms in the resolutions of determinantal ideals should be representation modules, so that perhaps Lascoux could make a connection in that context.

Since the methods of Lascoux were tied to characteristic zero, that is the restriction he continued to work under. That restriction was understandable: one part of his method involved the use of Bott's theorem on vanishing of homology of vector bundles, which held only in characteristic zero, and another part of his technique involved the fact that the linear group is completely reductive over the rationals. Nevertheless, Lascoux succeeded in writing down explicitly the terms of the resolutions of the ideal, I_p, generated by the $p \times p$ minors of the generic $m \times n$ matrix over a commutative ring containing the field, $\mathbb{Q}$, of rational numbers. He also wrote down the terms of resolutions of Weyl and Schur modules in terms of their Jacobi-Trudi and Giambelli matrices (in the Grothendieck ring of representations of the general linear group).

This work of Lascoux stimulated me to develop, with K. Akin and J. Weyman ([3]), a characteristic-free representation theory of the general linear group* with the idea of constructing the Lascoux type of resolution in general. It soon became clear that, while it was possible to "universalize" the representation modules to arbitrary commutative ground rings, the direct transcription of the Lascoux terms was not going to yield the result we were looking for: a universal minimal resolution of the ideal, I_p, generated by the $p \times p$ minors of the generic $m \times n$ matrix over the integers, $\mathbb{Z}$. For one thing, even in the construction of the resolution of the ideal of submaximal minors of the generic matrix ([2]), Akin, Weyman and I ran across so-called $\mathbb{Z}$-forms of rational representations that had to be taken into account to ensure the acyclicity of these resolutions.

To explain loosely what is meant here by $\mathbb{Z}$-forms of rational representations, consider the rational representation of $GL(F \otimes \mathbb{Q})$, for some free abelian group, F, over $\mathbb{Z}$, denoted by $S_2(F \otimes \mathbb{Q})$, that is, the second symmetric power of $F \otimes \mathbb{Q}$. Over $\mathbb{Z}$, there are two distinct $GL(F)$-representations, $D_2(F)$ and $S_2(F)$, which yield equivalent representations, namely, $S_2(F \otimes \mathbb{Q})$, when tensored with $\mathbb{Q}$. Thus, $D_2(F)$ and $S_2(F)$ are called $\mathbb{Z}$-forms of the rational representation, S_2.

Another way to construct non-isomorphic $\mathbb{Z}$-forms is the following:

Consider the short exact sequence

$$0 \to D_{k+2} \to D_{k+1} \otimes D_1 \to K_{(k+1,1)} \to 0 \tag{1}$$

where $K_{(k+1,1)}$ is the Weyl module associated to the hook partition $(k+1, 1)$. (We are leaving out the module F, as that is understood throughout.)

If we take an integer, t, and multiply D_{k+2} by t, we get an induced exact se-

*It should be pointed out that both Carter and Lusztig ([12]) and Towber ([17]) had developed such theories a bit earlier. However, their constructions did not yield a category of representation modules large enough to do what we had in mind.

quence and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & D_{k+2} & \to & D_{k+1} \otimes D_1 & \to & K_{(k+1,1)} & \to & 0 \\ & & \downarrow t & & \downarrow & & \downarrow & & \\ 0 & \to & D_{k+2} & \to & E(t;k+1,1) & \to & K_{(k+1,1)} & \to & 0, \end{array}$$

where $E(t; k+1, 1)$ stands for the cofiber product of D_{k+2} and $D_{k+1} \otimes D_1$. Each of these modules is a $\mathbb{Z}$-form of $D_{k+1} \otimes D_1$, but for t_1 and t_2, two such are isomorphic if and only if $t_1 \equiv t_2 \mod k+2$ (see [1]). This says that $\mathrm{Ext}^1_A(K_{(k+1,1)}, D_{k+2}) \cong \mathbb{Z}/(k+2)$, where A is the Schur algebra of appropriate degree. (For those familiar with the Yoneda definition of Ext, it should be pointed out that the generator of $\mathrm{Ext}^1_A(K_{(k+1,1)}, D_{k+2})$ is the exact sequence (1).)

But let's return to the Lascoux resolutions of determinantal ideals and the question of transporting them to universal resolutions over the integers. It was known that the existence of such resolutions would imply that the Betti numbers of these ideals are independent of the characteristic of the ground ring. However, since these ideals don't exhibit pathology exhibited by some other classes of ideals (such as the projective plane), for example they are perfect in every characteristic, it was believed that this property must be true. But in [15] Hashimoto showed that the ideal of 2×2 minors of a generic 5×5 matrix has different Betti numbers depending on whether the ground ring is the rationals or a field of characteristic three. Clearly the problem of resolving determinantal ideals had to be rethought, and the arithmetic problems, such as $\mathbb{Z}$-forms, had to be addressed.

4. Resolutions of Weyl modules

While the results recounted above were a bit disappointing, it was heartening to see that, to produce the Hashimoto counterexample, it was necessary to have at hand the tools to deal with representations in all characteristics. It was also fascinating to come across the arithmetic problems presented by $\mathbb{Z}$-forms, and to see their connection with our familiar Ext groups. It wasn't long before it became apparent, at least to those of us who are resolution-oriented, that it was time to confront the problem of resolving classes of representation modules.

At first, Akin and I dealt with Schur modules, as Lascoux generally did, and the reasonable type of "resolution" for these modules consisted of direct sums of tensor products of exterior powers. For two-rowed skew-shapes, Akin and I had a complete description of their resolutions (see [1]). However, when we started to consider three-rowed shapes, the situation became so complicated that we weren't sure that such resolutions even existed; we had, after all, learned something from the determinantal ideal situation. In order to at least reassure ourselves of their existence, we considered Weyl modules and resolutions in terms of direct sums of tensor products of divided powers which are, modulo certain constraints, projective modules over the Schur algebra (the universal enveloping algebra for homogeneous polynomial representations of the general linear group of given degree). Using induction on the "combinatorial complexity" of the skew-shapes, and a mapping cone argument of the sort mentioned in the Introduction, we were able to prove the existence of resolutions of Weyl modules corresponding to all the shapes we were interested in. Then, by means of a “dualizing functor”, we could prove the existence of the resolutions of Schur modules corresponding to all these shapes. However, we were still far from explicitly describing these resolutions and, after a few years of steady but terribly slow progress, we turned our attention to other problems.

In the summer of 1990, G.-C. Rota and I met in Rome, and discovered that we were both working on related problems. Consequently, we decided to work together, bringing to bear on our common problems methods of combinatorics and homological algebra. As a first test of the efficacy of this type of collaboration, we chose the resolutions of Weyl modules.

It soon became clear that the maps that Akin, Weyman and I had called the “Weyl maps” and the “box maps” in [3] were, once one translated the more algebraic setup into letter-place terminology, certain compositions of so-called place polarizations. And for place polarizations, there are classical identities (the Capelli identities) which, with the more contemporary methods of multilinear algebra, could be generalized to higher powers to give:

Capelli Identities:

If $k \neq i$

$$\partial_{ij}^{(r)}\partial_{jk}^{(s)} = \sum_{\alpha\geq 0}\partial_{jk}^{(s-\alpha)}\partial_{ij}^{(r-\alpha)}\partial_{ik}^{(\alpha)}$$

$$\partial_{jk}^{(s)}\partial_{ij}^{(r)} = \sum_{\alpha\geq 0}(-1)^{\alpha}\partial_{ij}^{(r-\alpha)}\partial_{jk}^{(s-\alpha)}\partial_{ik}^{(\alpha)},$$

where $\partial_{ij}^{(r)}$ stands for the r^{th} divided power of the place polarization sending place j to place i.

If $i \neq k$ and $j \neq l$, then

$$\partial_{jk}^{(s)}\partial_{il}^{(r)} = \partial_{il}^{(r)}\partial_{jk}^{(s)}.$$

Using these Capelli identities, it was possible to simplify many of the very complicated calculations that had proved so impenetrable earlier on. While there is still a great deal of work left to do in connection with this problem of resolving Weyl modules, the terms of the resolutions have been completely written down (see [11]). The maps of the resolutions are not yet explicitly described; their existence is guaranteed, however, by the fact that for Weyl modules the resolutions being looked for are projective. The main trick in determining these resolutions goes back to the one mentioned in the beginning of this article, namely, if

$$0 \to A \to B \to C \to 0,$$

is a short exact sequence, the mapping cone of a map of a resolution of A into one of B, produces a resolution of C. In our current situation, the module, C, is a Weyl module, and using a fundamental theorem on short exact sequences (again, see [11]), we have Weyl modules A and B which are combinatorially less complex than C. Thus an inductive argument (on degree of complexity) says that we have resolutions of A and B of the desired type, and projectivity of the resolutions assures that there is a map between them. The mapping cone, then, gives us our resolution of C. (The case of Schur modules is taken care of by means of a duality functor between Weyl and Schur modules introduced in [1].)

5. Concluding remarks

I do not know of any recent work on generalized Koszul complexes per se, beyond that of [4]. There are still a number of mathematicians working on the generalized multiplicity mentioned earlier, and one would hope that the simplifications provided by use of the slimmer complexes might help in that regard.

On the question of resolving generic determinantal ideals, that is, ideals generated by minors of a given order of a generic $m \times n$ matrix, M. Hashimoto ([16]) has used the notion of tilting modules to prove the existence of resolutions consisting of those kinds of modules. There are also other types of ideals which are of determinantal type, such as Gorenstein ideals of codimension 3 (they are Pfaffians). It has long been asked whether Gorenstein ideals of higher codimension can be described similarly. I have no answer to that question, but I would offer a suggestion: make use of representation theory if possible. By this I mean that, in the case of codimension 3, the Gorenstein ideals correspond to a particular "symmetry" in $F \otimes F$, namely, to $\Lambda^2(F) \subset F \otimes F$. In the case of higher codimension, there are more matrices to consider, and one might try to rephrase these in terms of higher tensor products of copies of free modules. These tensor products, in turn, decompose into "symmetries" (Schur or Weyl modules), and the Gorenstein case may correspond to the correct choice of "symmetry". This is admittedly vague, but if I could be more precise, I would be writing a paper on that topic as well as this one. In any event, the point that I would like to make is that representation theory has in large part been neglected as a tool in commutative algebra (certainly not totally), and it would be interesting to see if a more aggressive use of representation-theoretic methods would be productive.

On the question of resolving Weyl modules, the problems mentioned here have been described in full detail in [11], as well as in a forthcoming book [5]. In the latter, a thorough description of letter-place methods is given, along with polarizations and other combinatorial techniques. In [1] and [7], connections are made between resolutions of Weyl modules and intertwining numbers, a topic that is of interest in modular representation theory. Here, too, one can see the application of homological technique (Universal Coefficient Theorem) to certain areas of representation theory.

This, then, is the offering I make at this conference in honor of Paolo Valabrega. If it has not convinced him of the ubiquity of resolutions, I hope that it does convince him of my own personal resolve in relation to them.

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