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ON THE LITTLEWOOD-RICHARDSON RULE FOR ALMOST SKEW-SHAPES

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Abstract. We describe combinatorially the coefficients occurring in the irreducible decomposition of the Weyl module associated with an almost skew-shape belonging to the family J.

The proof uses the fundamental exact sequence for almost skew-shapes to initiate an inductive procedure which ultimately reduces to the classical Littlewood-Richardson rule for skew partitions.

1. Introduction

The Weyl module, $K_{\lambda/\mu}(F)$, associated with a skew-partition λ/μ is a representation of the general linear group $GL(F)$, where F stands for a finite free R-module. Hence, if R is a field of characteristic zero, $K_{\lambda/\mu}(F)$ is isomorphic to a direct sum of Weyl modules, $K_\nu(F)$, associated with ordinary partitions ν. For under that assumption on R, it is well known (see, for instance, [3], Chapter I) that every finite dimensional representation, such as $K_{\lambda/\mu}(F)$, is completely reducible, and a complete set of irreducibles is given by the modules $K_\nu(F)$. The classical Littlewood-Richardson rule describes the partitions ν occurring in the isomorphism:

$$K_{\lambda/\mu}(F) \cong \bigoplus_{\nu} g(\lambda/\mu;\nu) K_\nu F.$$

Namely,

Theorem 1.1 (Cf. [3], Chapter I; in particular, Section 9)**.** $g(\lambda/\mu;\nu)$ *is the number of ways (possibly zero) in which we can fill the diagram of the skew-partition* λ/ν *with all the elements of the set of integers*

$$\left\{ \underbrace{1,\dots,1}_{\mu_1}; \underbrace{2,\dots,2}_{\mu_2}; \dots; \underbrace{r,\dots,r}_{\mu_r} \right\},$$

$r = \mathrm{length}(\mu)$, *so that*

(a) *the resulting tableau* T *is (Weyl-) standard, that is, each row of* T *is non-decreasing and each column is strictly increasing,*

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(b) *the word associated with T, $as(T)$, obtained by listing all entries of T from right to left on each row, starting from the top row, is a lattice permutation.*

Now let λ/μ stand for an **almost skew-shape**, where $\lambda = (\lambda_1, ..., \lambda_n)$ is a partition of length n and μ is a sequence of integers $(\mu_1, ..., \mu_n)$ such that

$$\lambda_i \geq \mu_i \ \forall i = 1, ..., n, \quad \mu_1 \geq ... \geq \mu_{n-1} \quad \text{and} \quad 0 \leq \mu_n \leq \mu_1.$$

In other words, the almost skew-shape is a skew-partition but for its last row, which, rather than projecting beyond (or flush with) the penultimate row, may not make it that far on the left.

We define the type, τ, of the given almost skew-shape as the integer $n-(i+1)$, where i is the largest index different from n such that $\mu_n \leq \mu_i$. Thus $\tau = 0$ ($i = n-1$) means that the almost skew-shape is in fact a skew-partition, while $\tau > 0$ ($i \leq n-2$) means that the last row is actually indented on the left from the penultimate row.

Since different pairs (λ, μ) may yield the same almost skew-shape, in order to have a canonical description of our almost skew-shapes of positive type, *from now on we assume* that $\mu_{n-1} = 0$ whenever $\tau > 0$. In particular, we have

$$\mu_1 \geq ... \geq \mu_i \geq \mu_n \gneqq \mu_{i+1} \geq \mu_{i+2} \geq ... \geq \mu_{n-2} \geq \mu_{n-1} = 0.$$

We call μ' the partition $(\mu_1, ..., \mu_i, \mu_n, \mu_{i+1}, \mu_{i+2}, ..., \mu_{n-2})$; it has length at most $n-1$.

For more details about almost skew-shapes, and Weyl modules associated with them, see [2], Chapter VI.

If $K_{\lambda/\mu}(F)$ denotes the Weyl module associated with an almost skew-shape λ/μ of positive type, again we have in characteristic zero an isomorphism

$$K_{\lambda/\mu}(F) \cong \bigoplus_{\nu} h(\lambda/\mu; \nu) K_\nu F.$$

It is the purpose ot this note to describe for the first time the coefficients $h(\lambda/\mu; \nu)$. Namely,

Theorem 1.2. *Assume that $\lambda_{n-1} - \lambda_n \geq \tau\,(> 0)$. Then $h(\lambda/\mu; \nu)$ is the number of ways (possibly zero) in which we can fill the diagram of the skew-partition λ/ν with all the elements of the set of integers*

$$\left\{ \underbrace{1, ..., 1}_{\mu'_1}; \underbrace{2, ..., 2}_{\mu'_2}; ...; \underbrace{r, ..., r}_{\mu'_r} \right\},$$

$r = \text{length}(\mu')$, *so that*

(a) *the resulting tableau T is (Weyl-) standard,*

(b) *the word associated with T, $as(T)$, is a lattice permutation,*

(c) *if k is the largest index occurring in T ($k \geq i+1 = n-\tau$ for sure, since $\mu_n > 0$), then k only occurs on the n-th row of λ; furthermore, if $k > n-\tau$, then the number of times $n-\tau$ occurs on the first $n-1$ rows of λ equals the number of times k occurs on the n-th row.*

One should remark that

$$\left\{\underbrace{1,...,1}_{\mu'_1};\underbrace{2,...,2}_{\mu'_2};...;\underbrace{r,...,r}_{\mu'_r}\right\}$$

$$=\left\{\underbrace{1,...,1}_{\mu_1};...;\underbrace{i,...,i}_{\mu_i};\underbrace{i+1,...,i+1}_{\mu_n};\underbrace{i+2,...,i+2}_{\mu_{i+1}};\underbrace{i+3,...i+3}_{\mu_{i+2}};...;\underbrace{n-1,...,n-1}_{\mu_{n-2}}\right\}.$$

The rest of this paper is devoted to a proof of Theorem 1.2. As also indicated by some examples, we suspect that the assumption $\lambda_{n-1}-\lambda_n \geq \tau$ can be removed. But it is necessary for our inductive proof.

Consistent with [1], we will say that an almost skew-shape λ/μ of type τ "belongs to the family J" whenever $\lambda_{n-1}-\lambda_n \geq \tau$. In particular, all skew-partitions belong to the family J.

2. Proof of Theorem 1.2: Outline and preparations

Our inductive proof of Theorem 1.2 is based on the fundamental short exact sequence for almost skew-shapes, which is Theorem VII.1.2 of [2] (a theorem dealing with Weyl-Schur complexes, not just with modules). More precisely, thanks to the assumption $\lambda_{n-1}-\lambda_n \geq \tau$, we can recover our $K_{\lambda/\mu}(F)$ as the leftmost term of a suitable instance of that fundamental short exact sequence. Since the central and rightmost terms have lower type and still belong to the family J, their decompositions into irreducibles are known by induction and the exactness of the sequence yields the decomposition of $K_{\lambda/\mu}(F)$.

Proposition 2.1 (Special instance of the fundamental exact sequence)**.** *Notation as above and $\tau > 0$. Then there is a short exact sequence*

$$0 \to K_{\lambda/\mu}(F) \to K_{\widehat{\lambda}/\widehat{\mu}}(F) \to K_{\overline{\lambda}/\overline{\mu}}(F) \to 0,$$

where:
$\widehat{\lambda}=\lambda$,
$\widehat{\mu}=(\mu_1,...,\mu_i,\mu_n,\mu_{i+2},...,\mu_{n-2},\mu_{n-1}=0,\mu_{i+1})$
[*if i is maximal, i.e. $i+1=n-1$, then* $\widehat{\mu}=(\mu_1,...,\mu_i,\mu_n)$],
$\overline{\lambda}=(\lambda_1,...,\lambda_{n-1},\lambda_n+1)$,
$\overline{\mu}=(\mu_1,...,\mu_i,\mu_n,\mu_{i+2},...,\mu_{n-2},\mu_{n-1}=0,\mu_{i+1}+1)$
[*if i is maximal, then* $\overline{\mu}=(\mu_1,...,\mu_i,\mu_n,1)$].

We call $\widehat{\tau}$ and $\overline{\tau}$ the types of $\widehat{\lambda}/\widehat{\mu}$ and $\overline{\lambda}/\overline{\mu}$, respectively.

It is an easy remark that $\widehat{\tau} \lneqq \tau$, and hence $\widehat{\lambda}/\widehat{\mu}$ still belongs to the family J. More precisely, let u be the positive integer such that

$$\mu_1 \geq ... \geq \mu_i \geq \mu_n \gneqq \mu_{i+1}=\mu_{i+2}=...=\mu_{i+u} \gneqq \mu_{i+u+1} \geq ... \geq \mu_{n-2} \geq \mu_{n-1};$$

then $\widehat{\tau}=\tau-u$.

As for $\overline{\tau}$, since

$$\mu_1 \geq ... \geq \mu_i \geq \mu_n \geq \mu_{i+1}+1 \gneqq \mu_{i+2} \geq ... \geq \mu_{n-2} \geq \mu_{n-1}=0,$$

we always get $\overline{\tau} = \tau - 1$. It also follows that $\overline{\lambda}/\overline{\mu}$ still belongs to the family J, because $\overline{\lambda}$ is obtained from λ by adding an extra box at the rightmost end of the last row.

Summarizing, the short exact sequence of Proposition 2.1 does not bring us outside of the family J, and Theorems 1.1 and 1.2 (the latter by induction hypothesis) apply to $\widehat{\lambda}/\widehat{\mu}$ and $\overline{\lambda}/\overline{\mu}$.

Three cases will have to be examined:

(1) $\widehat{\tau} = 0 = \overline{\tau}$ (which is equivalent to $\tau = 1$),
(2) $\widehat{\tau} = 0$ and $\overline{\tau} > 0$ (with $\tau = u = \overline{\tau} + 1$),
(3) $0 < \widehat{\tau} = \tau - u \leq \tau - 1 = \overline{\tau}$.

Before going into the details, we need some notation. Given a skew-partition λ/ν, and a tableau, T, of that shape, we denote by $C^T_\ell(s)$ (respectively, $C^T(s)$) the number of times the index s occurs in T on the ℓ-th row of λ (respectively, on all rows of λ). The letter C is meant to recall the word "content."

With this notation, condition (c) of Theorem 1.2 reads as follows: $C^T(k) = C^T_n(k)$ always, and also $C^T(n-\tau) - C^T_n(n-\tau) = C^T(k)$, whenever $k > n - \tau$.

3. PROOF OF THEOREM 1.2: DETAILS

3.1. Case 1: $\widehat{\tau} = 0 = \overline{\tau}$ (which is equivalent to $\tau = 1$).

The numbers occurring in the set

$$\left\{\underbrace{1,...,1}_{\mu'_1};\underbrace{2,...,2}_{\mu'_2};...;\underbrace{r,...,r}_{\mu'_r}\right\} =$$

$$\left\{\underbrace{1,...,1}_{\mu_1};...;\underbrace{i,...,i}_{\mu_i};\underbrace{i+1,...,i+1}_{\mu_n};\underbrace{i+2,...,i+2}_{\mu_{i+1}};\underbrace{i+3,...i+3}_{\mu_{i+2}};...;\underbrace{n-1,...,n-1}_{\mu_{n-2}}\right\}$$

of Theorem 1.2 can also be obtained by rearranging the parts of $\widehat{\mu}$ in order to get a partition, say $\widehat{\mu}'$, and then taking $\widehat{\mu}'_1$ copies of 1, $\widehat{\mu}'_2$ copies of 2, etc.

By Theorem 1.1, it follows that $g(\widehat{\lambda}/\widehat{\mu};\nu)$ is parametrized by the tableaux T of shape $\widehat{\lambda}/\nu = \lambda/\nu$ such that

- T is filled with the indices $\underbrace{1,...,1}_{\mu'_1};\underbrace{2,...,2}_{\mu'_2};...;\underbrace{r,...,r}_{\mu'_r}$,
- T is standard and $as(T)$ is a lattice permutation.

Hence $h(\lambda/\mu;\nu) = g(\widehat{\lambda}/\widehat{\mu};\nu) - g(\overline{\lambda}/\overline{\mu};\nu)$ will be parametrized by the previous tableaux T which *cannot* be put in one-to-one correspondence with the tableaux $\overline{T}$ parametrizing $g(\overline{\lambda}/\overline{\mu};\nu)$. These tableaux $\overline{T}$ have the following properties:

- the shape is that of T, but with the addition of an extra box at the rightmost end of the n-th row of λ,
- the indices filling $\overline{T}$ are the same as T, but with the addition of a further index n,
- $\overline{T}$ is standard and $as(\overline{T})$ is a lattice permutation.

Since $\overline{T}$ is standard, its index n must occur on the n-th row of λ, at the rightmost end. Thus if we remove from $\overline{T}$ the box containing n, a tableau T related to $g(\widehat{\lambda}/\widehat{\mu};\nu)$ is obtained. Conversely, taking a tableau T related to $g(\widehat{\lambda}/\widehat{\mu};\nu)$ and adding an extra box, containing n, to the rightmost end of the n-th row of λ, we always get a tableau $\overline{T}$ related to $g(\overline{\lambda}/\overline{\mu};\nu)$, provided $as(\overline{T})$ stays a lattice permutation. But $as(\overline{T})$ fails to be a lattice permutation precisely when $C^T(n-1) = C_n^T(n-1)$. Hence the tableaux T related to $h(\lambda/\mu;\nu)$ have the following properties:

- T is of shape λ/ν,
- T is filled with the indices $\underbrace{1,...,1}_{\mu'_1};\underbrace{2,...,2}_{\mu'_2};...;\underbrace{r,...,r}_{\mu'_r},$
- T is standard and $as(T)$ is a lattice permutation,
- the largest index occurring in T only occurs on the n-th row of λ.

Since the largest index is $n-1 = n-\tau$, condition (c) of Theorem 1.2 is fully satisfied and Case 1 is completely proved.

Remark 3.1. In the above, the skew-partition $\overline{\lambda}/\overline{\mu}$ has been represented in a rather unusual form, since $\overline{\mu}_n = 1$.

3.2. **Case 2:** $\widehat{\tau} = 0$ **and** $\overline{\tau} > 0$ **(with** $\tau = u = \overline{\tau}+1$**).**

As in Case 1, $g(\widehat{\lambda}/\widehat{\mu};\nu)$ is parametrized by the tableaux T of shape $\widehat{\lambda}/\nu = \lambda/\nu$ such that

- T is filled with the indices $\underbrace{1,...,1}_{\mu'_1};\underbrace{2,...,2}_{\mu'_2};...;\underbrace{r,...,r}_{\mu'_r},$
- T is standard and $as(T)$ is a lattice permutation.

We point out to the reader that the indices occurring in T never exceed $n-\tau$, since $\tau = u$.

By Theorem 1.2, the tableaux $\overline{T}$ related to $h(\overline{\lambda}/\overline{\mu};\nu)$ have the following properties:

- the shape is that of T, but with the addition of an extra box at the rightmost end of the n-th row of λ,
- the indices filling $\overline{T}$ are the same as T, but with the addition of a further index $n-\tau+1$,
- $\overline{T}$ is standard and $as(\overline{T})$ is a lattice permutation,
- $n-\tau+1$ occurs on the n-th row of $\overline{\lambda}$.

Clearly, taking a tableau T related to $g(\widehat{\lambda}/\widehat{\mu};\nu)$ and adding an extra box (filled with $n-\tau+1$) to the rightmost end of the n-th row of λ, one gets a tableau $\overline{T}$ related to $h(\overline{\lambda}/\overline{\mu};\nu)$, provided $as(\overline{T})$ stays a lattice permutation. This fails to happen when $C^T(n-\tau) = C_n^T(n-\tau)$. Hence the tableaux T related to $h(\lambda/\mu;\nu)$ have the following properties:

- T is of shape λ/ν,
- T is filled with the indices $\underbrace{1,...,1}_{\mu'_1};\underbrace{2,...,2}_{\mu'_2};...;\underbrace{r,...,r}_{\mu'_r},$
- T is standard and $as(T)$ is a lattice permutation,
- the largest index occurring in T only occurs on the n-th row of λ.

Since the largest index is $n-\tau$, condition (c) of Theorem 1.2 is fully satisfied and Case 2 is completely proved.

3.3. **Case 3:** $0 < \widehat{\tau} = \tau - u \leq \tau - 1 = \overline{\tau}$.

By Theorem 1.2, $h(\widehat{\lambda}/\widehat{\mu};\nu)$ is parametrized by the tableaux T of shape $\widehat{\lambda}/\nu = \lambda/\nu$ such that

- T is filled with the indices $\underbrace{1,...,1}_{\mu'_1};\underbrace{2,...,2}_{\mu'_2};...;\underbrace{r,...,r}_{\mu'_r}$,
- T is standard and $as(T)$ is a lattice permutation,
- the largest index $k = i+1+u = n-\tau+u = n-\widehat{\tau}$ only occurs on the n-th row of λ.

The tableaux $\overline{T}$ related to $h(\overline{\lambda}/\overline{\mu};\nu)$ have the following properties:

- the shape is that of T, but with the addition of an extra box at the rightmost end of the n-th row of λ,
- the indices filling $\overline{T}$ are the same as T, but with the addition of a further index $n-\tau+1 = n-\overline{\tau}$,
- $\overline{T}$ is standard and $as(\overline{T})$ is a lattice permutation,
- the largest index $k = n-\tau+u$ only occurs on the n-th row of $\overline{\lambda}$ and, if $u \geq 2$,

$$(*) \qquad C^{\overline{T}}(n-\tau+1) - C^{\overline{T}}_n(n-\tau+1) = C^{\overline{T}}(n-\tau+u).$$

If $u = 1$, it is obvious that if we remove from $\overline{T}$ the rightmost box of the last row (a box filled with the largest index $n-\tau+1$), we obtain a tableau T related to $h(\widehat{\lambda}/\widehat{\mu};\nu)$. If $u \geq 2$, then condition $(*)$ says that there is a copy of $n-\tau+1$ (no longer the largest index) on the n-th row of $\overline{\lambda}$. Call $\overline{\overline{T}}$ the tableau obtained from $\overline{T}$ by bringing that $n-\tau+1$ into the rightmost box of the same row. If we remove from $\overline{\overline{T}}$ that rightmost box, again we get a tableau T related to $h(\widehat{\lambda}/\widehat{\mu};\nu)$.

Conversely, taking a tableau T related to $h(\widehat{\lambda}/\widehat{\mu};\nu)$ and adding to the right end of the n-th row of λ a new box containing $n-\tau+1$, we obtain a tableau $\overline{T}$ which may or may not be related to $h(\overline{\lambda}/\overline{\mu};\nu)$. However, since

$$\mu'_{i+2} = ... = \mu'_{i+1+u},$$

if we call $\overline{\overline{T}}$ the tableau obtained by rearranging the last row of $\overline{T}$ in increasing order, the new index $n-\tau+1$ does not have larger indices on top of it in $\overline{\overline{T}}$, and the latter is therefore standard. As for $as(\overline{\overline{T}})$ being a lattice permutation, this amounts to asking whether the following condition held for T:

$$C^T(n-\tau) - C^T_n(n-\tau) \gneqq C^T(n-\tau+1) - C^T_n(n-\tau+1).$$

It follows from the above that a tableau T related to $h(\widehat{\lambda}/\widehat{\mu};\nu)$ is in fact related to $h(\lambda/\mu;\nu)$ if, in addition, it satisfies

$$(**) \qquad C^T(n-\tau) - C^T_n(n-\tau) = C^T(n-\tau+1) - C^T_n(n-\tau+1).$$

Recall, though, that $\mu'_{i+2} = ... = \mu'_{i+1+u}$ means

$$(\#) \qquad C^T(n-\tau+1) = ... = C^T(n-\tau+u),$$

so that $C^T_n(n-\tau+1) = 0$ because $as(T)$ is a lattice permutation and $n-\tau+u$ only occurs on the n-th row of λ. Hence $(**)$ translates into

$$C^T(n-\tau) - C^T_n(n-\tau) = C^T(n-\tau+1),$$

that is (by (#)), into

$$C^T(n-\tau) - C_n^T(n-\tau) = C^T(n-\tau+u),$$

as required.

This completes the proof of Case 3, as well as of Theorem 1.2.

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