Some Remarks on Factorization in Power Series Rings

DAVID A. BUCHSBAUM[1]

Communicated by E. SNAPPER

1. Introduction. It has long been common knowledge that if R is a unique factorization domain, then so is the polynomial ring $R[x]$. However, until recently, it has been an open question as to whether the same can be said for the formal power series ring $R[[x]]$. Samuel has just produced an example of a unique factorization domain R of characteristic two such that $R[[x]]$ is not factorial (*i.e.* a unique factorization domain). This example still leaves the problem of what stronger hypotheses might be imposed on R in order to prove that $R[[x]]$ is factorial.

Recent investigations have shown that the problem of unique factorization can be studied using homological techniques. For example, in [3] it was proved that every regular local ring is factorial. It therefore seems worthwhile to try to attack the factorization problem in $R[[x]]$ from a homological point of view.

We will show that if R is a noetherian unique factorization domain, then $R[[x]]$ is also if and only if a certain ideal in R is principal. (In particular, one gets the result if R is a principal ideal domain.) We hope that the realization of this fact might provide an approach to the problem mentioned at the end of the first paragraph above.

We also show, using the result of [3], that if R is a regular ring (*i.e.*, a noetherian ring all of whose local rings are regular) which is factorial, then $R[[x]]$ is factorial. Since the regularity of R implies the regularity of $R[[x]]$, we obtain the result that if R is a regular factorial ring, then so is $R[[x_1, \cdots, x_n]]$. Since a principal ideal domain is regular, we see that any power series ring over such a ring is factorial.

2. The crucial ideal. It is well-known that an integral domain R is factorial if and only if the following two conditions hold:

i) the ascending chain condition for principal ideals;
ii) the intersection of two principal ideals is principal.

[1]This work was done with the partial support of NSF grant G-4232.

Journal of Mathematics and Mechanics, Vol. 10, No. 5 (1961).

Lemma 2.1. *If the ascending chain condition for principal ideals holds in the integral domain R, then it also holds in the formal power series ring $R[[x]]$.*

Proof. Let $(\phi_1) \subset (\phi_2) \subset \cdots$ be an increasing chain of principal ideals in $R[[x]]$, with $\phi_i = \sum a_{ij}x^j$. Then $(a_{1,0}) \subset (a_{2,0}) \subset \cdots$ is an increasing chain of principal ideals in R. Therefore for some integer n, we have $(a_{n+p,0}) = (a_{n,0})$ for all $p \geq 0$, and $a_{n,0} = u_p a_{n+p,0}$ where u_p is a unit in R. Since $\phi_n = \psi_p \phi_{n+p}$ for some $\psi_p \in R[[x]]$, we see that the constant term of ψ_p must be u_p. But the fact that u_p is a unit in R implies that ψ_p is a unit in $R[[x]]$ so that $(\phi_n) = (\phi_{n+p})$ for $p \geq 0$. (Although we have tacitly assumed here that not all the $a_{i,0}$ are zero, it is clear that one need only consider this case.)

As a result of the lemma above, we see that if R is a unique factorization domain, then $R[[x]]$ is factorial if and only if the intersection of any two principal ideals in $R[[x]]$ is principal.

Proposition 2.2. *If I is an invertible ideal in the unique factorization domain R, then I is principal.*

Proof. Since I is invertible, I is finitely generated. Let $I = (a_1, \cdots, a_n)$. If q is an element of the field of quotients Q of R such that $qI \subset R$ (*i.e.*, if q is in I^{-1}), then $q = r/d$ where d is the greatest common divisor of $a_1, \cdots, a_n$. For we see that $qI \subset R$ if and only if qa_i is in R for all i. Now if $q = u/v$ with u and v relatively prime, then $ua_i/v = c_i \in R$ or $ua_i = vc_i$. Since u and v have no common factor, v divides a_i so that v is a common divisor of $a_1, \cdots, a_n$, and thus divides the greatest common divisor d. Hence $d = sv$ and $q = su/d$.

Again using the fact that I is invertible, we have $q_1, \cdots, q_n$ in Q such that $\sum q_i a_i = 1$ and q_i in I^{-1}. Thus $q_i = r_i/d$ so that $d = \sum (dq_i)a_i = \sum r_i a_i$ and this shows that $(d) \subset I$. Since obviously $I \subset (d)$, we see that I is principal.

Proposition 2.3. *If R is an integral domain in which every invertible ideal is principal, then the same is true in $R[[x]]$.*

Proof. Let I be an invertible ideal in $R[[x]]$. Then I is generated by $\alpha_1, \cdots, \alpha_n$ in $R[[x]]$. If we let $\alpha_i = \sum a_{ij}x^j$, we may clearly assume that not all the a_{i0} are zero.

Therefore there exist $\phi_1, \cdots, \phi_n$ in $Q[[x]]$ ($Q =$ the field of quotients of R) with $\sum \phi_i \alpha_i = 1$ and ϕ_i in I^{-1}. We see, then, that the ideal I_0 in R generated by the $\{a_{i0}\}$ is invertible and by our assumption on R, I_0 is principal; $I_0 = (d_0)$.

Now $a_{i0} = d_0 b_i$, and $d_0 = \sum r_i a_{i0}$. Let $\beta = \sum r_i \alpha_i$. Then β is in I and $\alpha_i - b_i\beta$ is in $I \cap (x)$. However, if λ is an element of $I \cap (x)$, then

$$\lambda = \lambda \cdot 1 = \sum (\lambda\phi_i)\alpha_i = \sum \frac{\lambda\phi_i}{x}(x\alpha_i),$$

and $\lambda\phi_i/x$ is in $R[[x]]$ (since ϕ_i is in $Q[[x]]$ and $\phi_i I \subset R[[x]]$). Thus $I \cap (x) \subset xI$ which implies that $xI = I \cap (x)$ and hence $I \subset (\beta) + xI$. Since x is in the radical of $R[[x]]$ and I is finitely generated we see that $I = (\beta)$.

The preceding propositions tell us that if we want to show that an ideal in $R[[x]]$ is principal, we need only show that it is invertible. However, it is shown in [4] that an ideal in an integral domain is invertible if and only if it is projective. We therefore have

Corollary 2.4. *$R[[x]]$ is factorial if and only if the intersection of any two principal ideals is projective. This condition is equivalent to the following: $hd_{R[[x]]}(u, v) \leqq 1$ for any two elements u, v in $R[[x]]$.*

Here $hd_{R[[x]]}$ denotes the homological dimension with respect to $R[[x]]$, and the proof of the equivalence of these two conditions is given in [2].

Lemma 2.5. *Let S be a noetherian ring, E a finitely generated S-module, and x an element in the radical of S. If x is not a zero divisor for E, then*

$$hd_S E/xE = 1 + hd_S E.$$

Proof. It is known that

$$hd_S E/xE = \sup_{\mathfrak{m}} hd_{\mathfrak{o}} E/xE \otimes_S \mathfrak{o},$$

where $\mathfrak{m}$ runs through the maximal ideals of S and $\mathfrak{o} = S_{\mathfrak{m}}$ [4]. Since x is in the radical of S, x is in the radical of the local ring $\mathfrak{o} = S_{\mathfrak{m}}$ for all $\mathfrak{m}$, and x is not a zero divisor for $E_{\mathfrak{m}} = E \otimes_S \mathfrak{o}$. Hence by [1], we know that

$$hd_{\mathfrak{o}} E_{\mathfrak{m}}/xE_{\mathfrak{m}} = 1 + hd_{\mathfrak{o}} E_{\mathfrak{m}},$$

so that we have the desired result.

The following lemma is well-known, but we state it for the sake of completeness.

Lemma 2.6. *If S is an integral domain, E a torsion-free S-module, and x an element of S, then*

$$hd_{S/x} E/xE \leqq hd_S E.$$

Corollary 2.7. *Let S be a noetherian integral domain, E a finitely generated torsion-free S-module, and x an element in the radical of S. Then*

$$hd_S E = hd_{S/x} E/xE.$$

Proof. By [4], we have

$$hd_S E/xE \leqq hd_{S/x} E/xE + hd_S S/x = hd_{S/x} E/xE + 1.$$

Combining this with the above lemmas, we have

$$1 + hd_S E = hd_S E/xE \leqq 1 + hd_{S/x} E/xE \leqq 1 + hd_S E.$$

Hence the inequalities are equalities and we have the result.

Before proceeding to power series rings, we will prove one more lemma.

Lemma 2.9. *Let R be a unique factorization domain, and E an R-module generated by two elements e_1, e_2 with e_1 torsion-free. If $r_1e_1 + r_2e_2 = 0$, let d be* *the greatest common divisor of* r_1 *and* r_2 . *Then* $hd_R E \leqq 1$ *if and only if the annihilator* I *of* $(r_1/d)e_1 + (r_2/d)e_2$ *is principal.*

Proof. Consider the exact sequence

$$0 \to K \to R \oplus R \xrightarrow{f} E \to 0,$$

where $f(1, 0) = e_1$, $f(0, 1) = e_2$. Then

$$K = \{(s_1 , s_2) \mid s_1e_1 + s_2e_2 = 0\};$$

and the pair of elements (r_1 , r_2) mentioned in the hypothesis belongs to K. If (s_1 , s_2) is any element of K, we have

$$r_2(s_1e_1 + s_2e_2) = (r_2s_1 - s_2r_1)e_1 = 0,$$

so that $r_2s_1 = s_2r_1$ (since e_1 is torsion-free). Letting d' be the greatest common divisor of s_1 and s_2 , we see that

$$\left(\frac{s_1}{d'}, \frac{s_2}{d'}\right) = u\left(\frac{r_1}{d}, \frac{r_2}{d}\right)$$

where u is a unit in R. Since

$$d'\left(\frac{s_1}{d'} e_1 + \frac{s_2}{d'} e_2\right) = 0,$$

we have

$$d'\left(\frac{r_1}{d} e_1 + \frac{r_2}{d} e_2\right) = 0.$$

We therefore obtain an isomorphism between K and the ideal I by assigning to each pair (s_1 , s_2) in K, the greatest common divisor d' of s_1 and s_2 . Now $hd_R E \leqq 1$ if and only if K is projective. This in turn is equivalent to I being invertible, hence principal.

To see how all this applies to power series rings, we observe first of all that we must merely show that if $u = u_0 + xu'$, $v = v_0 + xv'$ with $u_0 \neq 0$, then $hd_{R[[x]]}(u, v) \leqq 1$. (If $u_0 = 0, v_0 = 0$, we would have $(u, v) = x(u', v') \approx (u', v')$. Hence, we may assume that $u_0 \neq 0$.) Since $hd_{R[[x]]}(u, v) \leqq 1$ if and only if $hd_{R[[x]]}u{:}v = 0$, and since $u{:}v = (u{:}x){:}v = u{:}xv$ (since $u_0 \neq 0$, $u{:}x = u$), we see that $hd_{R[[x]]}(u, v) \leqq 1$ if and only is $hd_{R[[x]]}(u, xv) \leqq 1$. However, by the preceding lemmas, if R is noetherian we have that $hd_{R[[x]]}(u, xv) \leqq 1$ if and only if $hd_R(u, xv)/x(u, xv) \leqq 1$. Since this latter module is generated by two elements, one of which is torsion-free, and since u_0xv is in $x(u, xv)$, we see that $hd_{R[[x]]}(u, xv) \leqq 1$ if and only if $\{r \in R/rv \in (u, xv)\}$ is principal. Therefore we have the following theorem. This result could be obtained without using Lemma 2.9, but 2.9 is of some technical interest in other connections.

Theorem 2.10. *Let* R *be a noetherian unique factorization domain. Then* $R[[x]]$ *is a unique factorization domain if and only if, for every pair of elements*

$u = u_0 + xu'$ ($u_0 \neq 0$) and v in $R[[x]]$, the ideal in R consisting of the constant terms of the elements of $u{:}v$ is principal.

Proof. All that we must show is that this ideal coincides with the ideal in the discussion immediately preceding this theorem. Clearly, if $rv = \alpha u + \beta xv$, then $(r - \beta x)v = \alpha u$, so that $r - \beta x$ is in $u{:}v$ and r is the constant term of the series.

3. Regular rings. In [1], a ring R is defined to be *regular* if it is noetherian, and every local ring $R_{\mathfrak{p}}$ is regular.

Lemma 3.1. *If R is a regular ring, then so is $R[[x_1, \cdots, x_n]]$.*

Proof. We may first of all assume that $n = 1$. In the proof of Theorem 1.12 of [2], we showed that if $\mathfrak{m}$ is a maximal ideal of $R[[x]]$, and $\mathfrak{m}'$ is the image of $\mathfrak{m}$ under the obvious map $R[[x]] \to R$, then gldim $R[[x]]_{\mathfrak{m}} = 1 +$ gldim $R_{\mathfrak{m}'}$. Since R is regular, $R_{\mathfrak{m}'}$ is a regular local ring, so that gldim $R_{\mathfrak{m}'}$ is finite. Hence, by the homological characterization of regular local rings in [1], we see that $R[[x]]_{\mathfrak{m}}$ is regular. Therefore $R[[x]]_{\mathfrak{m}}$ is regular for every maximal ideal $\mathfrak{m}$, and this suffices to show that $R[[x]]$ is regular.

Theorem 3.2. *Let R be a regular unique factorization domain. Then $R[[x_1, \cdots, x_n]]$ is a regular unique factorization domain.*

Proof. Again we may assume that $n = 1$. Since $R[[x]]$ is noetherian and integrally closed, it suffices to show that every minimal prime $\mathfrak{p}$ of $R[[x]]$ is principal. However, by 2.3, we need only show that $\mathfrak{p}$ is invertible or, equivalently, that $\mathfrak{p}$ is projective.

Now

$$hd_{R[[x]]}\mathfrak{p} = \sup_{\mathfrak{m}} hd_{\mathfrak{o}}\, \mathfrak{p}_{\mathfrak{o}} = 0,$$

where $\mathfrak{o} = R[[x]]_{\mathfrak{m}}$ since $R[[x]]_{\mathfrak{m}}$ is regular for every maximal ideal $\mathfrak{m}$, and every regular local ring is a unique factorization domain [3]. Hence $hd_{R[[x]]}\mathfrak{p} = 0$ so that $\mathfrak{p}$ is projective.

Corollary 3.3. *If R is a principal ideal domain, then $R[[x_1, \cdots, x_n]]$ is a unique factorization domain.*

Bibliography

[1] M. Auslander & D. A. Buchsbaum, Homological dimension in local rings, *Trans. Am. Math. Soc.*, **85** (1957) 390–405.

[2] ———, Homological dimension in noetherian rings, II, *Trans. Am. Math. Soc.*, **88** (1958) 194–206.

[3] ———, Unique factorization in regular local rings, *Proc. Nat. Acad. Sci.*, **45** (1959) 733–734.

[4] H. Cartan & S. Eilenberg, *Homological Algebra*, Princeton, 1956.

Brown University
Providence, Rhode Island