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Metric Geometry: Connections with Combinatorics

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Abstract. First introduced by Alexsandrov in the 1950's, metric geometry allows one to apply techniques of differential geometry to a very general class of metric spaces. Recent work of Gromov explores the notion of "nonpositive curvature" in metric geometry. In this expository paper, we discuss these ideas and how they connect to problems in combinatorics involving the *cd*-index, the lower bound theorem, and arrangements of hyperplanes.

1. Introduction

In differential geometry, the fundamental objects of study are smooth manifolds equipped with Riemannian metrics. The goal of metric geometry is to apply techniques of differential geometry in the more general context of topological metric spaces. Metric geometry dates back to the work of Alexsandrov in the 1950's (see [**A**]). Recent work of Gromov ([**G1**], [**G2**]) applying metric geometry to problems in infinite group theory have led to an explosion of new activity in this area with applications to topology, group theory, and combinatorics. What follows in an expository paper explaining the fundamental ideas of metric geometry and their connections with combinatorics. For more about metric geometry and its applications to group theory and topology, the reader is referred to [**G1**], [**G2**], [**Gh**], [**P**], and [**BH**].

Another very interesting connection between differential manifolds and combinatorics may be found in the work of MacPherson, Gelfand, Gabrielov, and Losik ([**GGL**], [**GM**], [**Ma**]). They define a "combinatorial differential manifold" using matroids to encode the "differential" structure. While their work may be viewed as a combinatorial version of differential topology, metric geometry is more akin to a combinatorial version of differential geometry, involving

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FIGURE 1

notions such as curvature and geodesics. It would be interesting to know if the two approaches could be effectively combined.

A Riemannian metric on a smooth manifold M is given by an inner product on the tangent space of M. From the Riemannian metric, one obtains a distance function d (that is, a "metric" in the topological sense) defined by letting $d(x, y)$ be the infimum of the lengths of paths from x to y. One also defines notions of connections, curvature, and geodesics in terms of the Riemannian metric. These are the fundamental tools of Riemannian geometry.

Metric geometry, while borrowing many ideas from differential geometry, applies to a much more general class of metric spaces. The starting point of metric geometry is the observation that certain properties of a Riemannian manifold M can be characterized in terms of its topological metric d. Geodesics, for example, can be characterized as distance minimizing paths. Even curvature can, in a sense, be recognized by the topological metric. To see this, consider the standard models for constant curvature $+1$, 0 and -1, respectively, namely the sphere $\mathbb{S}^n$, Euclidean space $\mathbb{E}^n$, and hyperbolic space $\mathbb{H}^n$. If we draw triangles in $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ with given side lengths, we find that the distance across the triangle (say from a vertex to a point on the opposite side) is smaller in $\mathbb{E}^n$ than in $\mathbb{S}^n$ and smaller still in $\mathbb{H}^n$; that is, the smaller the curvature, the "thinner" the triangles (Figure 1).

Now suppose (X, d) is an arbitrary metric space. (For simplicity, we will assume that all our metrics are complete.) We wish to define a notion of "geodesics" and "curvature" in X. A *geodesic segment* in X is an isometric embedding, $p\colon [a, b] \to X$, of an interval into X; that is, for any $t_1, t_2 \in [a, b]$, $d(p(t_1), p(t_2)) = |t_2 - t_1|$. We say that (X, d) is a *geodesic metric space* (or "length space") if any two points in the same path component of X are connected by a geodesic segment.

EXAMPLE. Let S^1 be the unit circle in $\mathbb{R}^2$. If we give S^1 the induced metric from $\mathbb{R}^2$, it is *not* a geodesic space since, for example, there is no path of length 2 on the circle from $(1, 0)$ to $(-1, 0)$. To get a geodesic metric on S^1, we must define the distance between two points to be the length of the circular arc connecting

FIGURE 2

them.

From now on, we assume (X, d) to be a (complete) geodesic metric space. The notion of "curvature" for such a space is defined by comparing shapes of triangles in X with those in a Riemannian model space. For any $c \in \mathbb{R}$, denote by M_c^n the complete, simply connected Riemannian n-manifold of constant curvature c. Thus, M_c^n is $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$ (according to whether $c > 0$, $c = 0$, or $c < 0$) with the standard metric scaled to give curvature c. These are our model spaces.

If T is a geodesic triangle in X, a *comparison triangle* in M_c^2 is a geodesic triangle T' in M_c^2 whose side lengths are the same as those of T. If $c \leq 0$, then such a comparison triangle always exists and is unique up to isotopy; if $c > 0$, it exists and is unique up to isotopy if and only if the perimeter of T is at most $\frac{2\pi}{\sqrt{c}}$. We say that (X, d) satisfies the *$CAT(c)$-inequality* if for every geodesic triangle T in X having a comparison triangle T' in M_c^2, the distance in X between any two points a, b on T is less than or equal to the distance in M_c^2 between the corresponding points a', b' in T' (Figure 2).

(The term "CAT(c)" stands for Comparison of Alexsandrov and Toponogov and was introduced by Gromov in [**G1**].) We say (X, d) has *curvature* $\leq c$ if it satisfies the CAT(c)-inequality locally; i.e., every point in X has a neighborhood satisfying CAT(c).

EXAMPLES.

(i) A Riemannian manifold M satisfies CAT(c) locally if and only if all of its sectional curvatures (in the Riemannian sense) are $\leq c$. Thus, in this setting, the two notions agree.

(ii) Let X be a tree, i.e, a graph with no circuits, and let d be a geodesic metric on X. Then (X, d) satisfies the CAT(c)-inequality for *all* $c \in \mathbb{R}$. To see this, note that all triangles in X degenerate to tripods (Figure 3) and are thus "thinner" than comparison triangles in any hyperbolic space, regardless of scaling.

(iii) For a (locally finite) graph X which is not a tree, X satisfies CAT(c) locally for all c, but it satisfies CAT(c) globally if and only if $c > 0$ and every circuit

FIGURE 3

in X has length at least $\frac{2\pi}{\sqrt{c}}$.

2. Non-positive Curvature

This notion of "curvature", though seemingly very general and rather coarse, is surprisingly powerful, especially in the case of nonpositive curvature (i.e., curvature ≤ 0). We will mention only briefly some of the topological and group theoretic implications of nonpositive curvature and refer the reader to [**G1**], [**Gh**], and [**ABr**] for more details.

It is easy to see that if (X,d) satisfies CAT(0), then geodesics in X are unique and vary continuously with endpoints. (If two geodesics have the same endpoints, they form a degenerate triangle – called a *digon* – with one side length 0. This triangle must be as thin as the comparison triangle in $\mathbb{E}^2$ which is a single line segment, hence the two geodesics must be identical. A similar argument shows that geodesics with nearby endpoints stay close throughout their length.) Choosing a basepoint x_0, one can thus contract X along geodesics back to x_0. Hence, any CAT(0) space is contractible; any space of curvature ≤ 0 is locally contractible.

The following fundamental theorem about nonpositive curvature was stated by Gromov in [**G1**] and proved by Alexander-Bishop [**AB**], Ballman [**B**], and Bridson [**Br**], under slightly varying hypotheses.

THEOREM 2.1. *If (X,d) is a simply connected geodesic metric space of curvature $\leq c \leq 0$, then (X,d) satisfies CAT(c) globally.*

A topological space is said to be *aspherical* if its universal covering space is contractible. Combining Theorem 2.1 with the discussion above gives the following corollary.

COROLLARY 2.2. *A geodesic metric space of nonpositive curvature is aspherical.*

The homotopy type of an aspherical space is completely determined by its fundamental group. Fundamental groups of compact spaces of nonpositive curvature satisfy a host of interesting group theoretic properties. For example, they have solvable word and conjugacy problems, they satisfy quadratic isoperimetric inequalities, and they are homologically of type FP_∞ (see [**Abr**]). Fundamental groups of compact spaces of strictly negative curvature (i.e., curvature $\leq \varepsilon < 0$) are "word hyperbolic" (in the sense of [**G1**]) and "biautomatic" (in the sense of [**E**]) and hence satisfy still stronger group theoretic properties. (This seems to have been one of Gromov's main motivations in studying these spaces.)

3. Piecewise Euclidean Complexes

We are thus interested in constructing metric spaces of nonpositive curvature. Suppose X is a polyhedron (i.e., the geometric realization of a simplicial complex). Assign to each n-simplex σ in X the metric of a Euclidean n-simplex (i.e., the convex hull of $n+1$ linearly independent points in $\mathbb{R}^{n+1}$) in such a way that the metric on any face of σ is the restriction of the metric on σ. For example, one could assign every n-simplex the metric of the standard n-simplex in $\mathbb{R}^{n+1}$. There is an obvious concept of "path length" in X and for two points $x, y \in X$, we define

$$d(x,y) = \inf\{\text{length } (p) \mid p \text{ is a path from } x \text{ to } y\}\,.$$

Under some fairly mild hypotheses (e.g., if X is locally finite or if there are finitely many isometry types of simplicies in X) this infimum is realized by a path from x to y, and d is a complete, geodesic metric on X (see [**Br**] or [**P**]). In this case, we say (X, d) is a *piecewise Euclidean simplicial complex.* More generally, we can start with a cell complex X in which each cell is a convex polytope and do a similar construction to get a *piecewise Euclidean cell complex.* A particularly interesting example is the case in which the cells of X are all cubes (of varying dimension) and are assigned the metric of a regular Euclidean n-cube with all side lengths 1. In this case, we say (X, d) is a (regular) *Euclidean cubical complex.*

Remark. One can also assign each cell of X the metric of a simplex in $\mathbb{S}^n$ (resp. $\mathbb{H}^n$). Then X, together with the induced geodesic metric, is called a *piecewise spherical* (resp. *piecewise hyperbolic) cell complex.*

Suppose (X, d) is a piecewise Euclidean cell complex. While each cell clearly has curvature ≤ 0, the CAT(0)-inequality may fail at a point where several cells come together. To see this, let X be the surface of the 3-cube, and let v be a vertex of X. In any neighborhood of v we can find a small triangle T surrounding v as in the figure below. Let x be a vertex of T and let w be a point on the opposite side. Clearly, the shortest path on the surface of the cube from x to w is longer than the Euclidean distance from x to w. Thus, the CAT(0)-inequality fails in every neighborhood of v (Figure 4).

How, then, can we recognize when (X, d) has curvature $\leq c$? If X is a 2-dimensional cell complex, it is fairly easy to see that to get curvature ≤ 0, it suffices to check that whenever a collection of 2-cells form a circuit around a vertex v, the sum of the angles at v is at least 2π. (This precisely avoids the

FIGURE 4

sort of problem arising in the example of the cube above.) In particular, if X is a 2-dimensional cubical complex, any circuit of 2-cells around a vertex v, must contain at least four squares

These ideas can be generalized to higher dimensions by considering the links of vertices. Recall that the *link*, $Lk(v,X)$, of the vertex v in a cell complex X is the partially ordered set of cells of X which properly contain v. Assuming (for simplicity) that the cells of X are simple polytopes, $Lk(v,X)$ is a quasi-simplicial complex with one n-simplex, $Lk(v,\sigma)$, for each $(n+1)$-cell σ of X containing v. (By a *quasi-simplicial complex*, we mean a cell complex in which each cell is a simplex, but two simplicies may be identified along more than one face.) Moreover, identifying the n-simplex $Lk(v,\sigma)$ with the set of unit tangent vectors at v pointing inward toward σ, gives $Lk(v,\sigma)$ a natural metric of a simplex in $\mathbb{S}^n$. (Equivalently, intersect σ with an ε-sphere centered at v, then normalize to get a simplex on a sphere of radius 1.) Thus, $Lk(v,X)$ naturally has the structure of a piecewise spherical cell complex (Figure 5).

FIGURE 5

EXAMPLE. If X is a 2-dimensional cell complex, and v is a vertex of X, then $Lk(v,X)$ is a 1-dimensional complex. The length of a 1-cell $Lk(v,\sigma)$ in $Lk(v,X)$

is the angle of σ at v. Thus, in this case the condition that (X, d) has nonpositive curvature is equivalent to the condition that, for all v, $Lk(v, X)$ has no circuits of length $< 2\pi$. If X is a 2-dimensional Euclidean cubical complex, then every 1-cell in $Lk(v, X)$ has length $\frac{\pi}{2}$. In this case, the condition that (X, d) has nonpositive curvature reduces to a combinatorial one: that all circuits in $Lk(v, X)$ involve at least 4 edges.

Returning now to the general case, we have the following theorem, first stated by Gromov [**G1**] and subsequently proved by Bridson [**Br**] and Ballman [**B**].

THEOREM 3.1. *Let (X, d) be a piecewise Euclidean cell complex. Assume X is locally finite or has finitely many isometry types of simplicies. Then (X, d) has curvature ≤ 0 if and only if, for all vertices v, $Lk(v, X)$, with its natural piecewise spherical metric, satisfies the $CAT(1)$-inequality.*

This condition is used, for example, in [**CD1**] to determine when (certain types of) branched covers of nonpositively curved Riemannian 3- and 4-manifolds are still nonpositively curved in the sense of metric geometry.

In general, verifying directly that links satisfy CAT(1) is not feasible. However, in the case of cubical complexes, the metric on $Lk(v, X)$ is particularly simple. Namely, each n-simplex in $Lk(v, X)$ is isometric to

$$\sigma^n = \mathbb{S}^n \cap \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \ \forall i\},$$

the unique (up to isometry) n-simplex in $\mathbb{S}^n$ with all edge lengths $= \frac{\pi}{2}$.

A piecewise spherical structure on a quasi-simplicial complex L with all edge lengths $\frac{\pi}{2}$ is called an *all right* structure. In this situation, the condition that L satisfies CAT(1) reduces to a purely combinatorial condition.

DEFINITION. A quasi-simplicial complex L is a *flag complex* if it is a genuine simplicial complex and any collection of vertices $\{v_1, \ldots, v_n\}$ which are pairwise joined by edges span a simplex in L.

THEOREM 3.2 (GROMOV [**G1**]). *Let L be a quasi-simplicial complex with an all right piecewise spherical structure. Then L satisfies the $CAT(1)$-inequality if and only if L is a flag complex.*

COROLLARY 3.3. *A Euclidean cubical complex X has nonpositive curvature if and only if $Lk(v, X)$ is a flag complex for every vertex v in X.*

EXAMPLE. A 1-dimensional complex L is a flag complex if and only if every circuit in L involves at least 4 edges. Thus, for a 2-dimensional Euclidean cubical complex X, the theorem states (as already observed above) that X has curvature ≤ 0 if and only if at least 4 faces are involved in any circuit about a vertex.

An important generalization of this theorem was proved by Moussong in his 1988 Ph.D. thesis [**M**]. A piecewise spherical metric on a quasi-simplicial complex L is said to be *nonacute* if all edge lengths are $\geq \frac{\pi}{2}$.

Definition. A quasi-simplicial complex L with a piecewise spherical metric is a *metric flag complex* if whenever $\{v_0, \dots, v_k\}$ is a set of vertices which are pairwise joined by edges and the lengths of these edges can be realized by some k-simplex in $\mathbb{S}^k$, then $\{v_0, \dots, v_k\}$ span a unique simplex in L.

Theorem 3.4 (Moussong [**M**]). *Let L be a quasi-simplicial complex with a nonacute piecewise spherical metric. Then L satisfies the $CAT(1)$-inequality if and only if L is a metric flag complex.*

Corollary 3.5. *Suppose X is a piecewise Euclidean complex made up of simple polytopes with all face angles $\geq \frac{\pi}{2}$. Then X has nonpositive curvature if and only if $Lk(v, X)$ is a metric flag complex for every vertex v of X.*

In the remainder of the paper, we will discuss connections of these ideas with several problems in combinatorics which have arisen in joint work of the author and Michael Davis ([**CD1-5**]).

4. Euler characteristics and the *cd*-index

An old conjecture of H. Hopf states that if M is a $2n$-dimensional, closed, Riemannian manifold of nonpositive curvature then its Euler characteristic should satisfy

$$(-1)^n \chi(M) \geq 0.$$

Let us consider this conjecture in the situation where M is not a Riemannian manifold, but a topological manifold with a metric of nonpositive curvature in the metric geometry sense. In particular, suppose the metric on M comes from a Euclidean cubical structure. In this case, by Corollary 3.3, M has nonpositive curvature if and only if $L_v = Lk(v, M)$ is a flag complex for every vertex v. Since M is a $2n$-manifold, L_v is a triangulated $(2n-1)$-sphere.

Thus, in this situation, the hypothesis of Hopf's conjecture can be stated as a local condition at the vertices. We would like to state the conclusion of Hopf's conjecture as a local condition as well. For this, we need to break $\chi(M)$ into a sum of local contributions at each vertex. Using the decomposition of M into cubes to compute the Euler characteristic, each i-cube contributes $(-1)^i$ to $\chi(M)$. Since an i-cube has 2^i vertices, it must contribute $\left(-\frac{1}{2}\right)^i$ at each vertex. Hence, we define the "local Euler characteristic" at the vertex v to be

$$\kappa_v = 1 - \frac{1}{2} f_0 + \frac{1}{4} f_1 - \cdots + \left(-\frac{1}{2}\right)^{2n} f_{2n-1}$$

where

$$\begin{aligned} f_i &= \#(i+1) - \text{cubes containing } v \\ &= \#i - \text{simplices in } L_v\,. \end{aligned}$$

The vector $(f_0, \ldots, f_{2n-1})$ is known as the *f-vector* for the link L_v. We have defined κ_v so that

$$\chi(M) = \sum_{\text{vertices}} \kappa_v \, .$$

More generally, if L is a cell complex of dimension n, and $(f_0, \ldots, f_m)$ is its f-vector, let

$$\kappa(L) = 1 - \frac{1}{2} f_0 + \frac{1}{4} f_1 - \cdots + \left(-\frac{1}{2}\right)^{m+1} f_m \, .$$

Combinatorial Hopf Conjecture ([**CD2**]). *If L is a $(2n-1)$-sphere triangulated as a flag complex, then*

$$(-1)^n \kappa(L) \geq 0 \, .$$

Clearly, the Combinatorial Hopf Conjecture implies the original Hopf conjecture for manifolds with a Euclidean cubical structure of nonpositive curvature. In fact, it is not difficult to show that they are equivalent in this context.

The Combinatorial Hopf Conjecture is still open; however, a number of special cases are known. The most important of these was observed by Babson, using work of Stanley on the "*cd*-index". The *cd-index*, $\Phi_A(c, d)$, defined for any "Eulerian poset" A, is a certain non-commutative polynomial which encodes the f-vector of A (see [**S1**] for an excellent survey). In particular, if $\dim A = 2n - 1$, and A' is the derived complex of A, one can easily check that

$$\kappa(A') = 2^{-2n} \Phi_A(0, -2) = (-1)^n (\text{coefficient of } d^n) \, .$$

Stanley [**S2**], showed that if A is the poset of faces on the boundary on a convex m-cell, then all the coefficients of Φ_A are nonnegative; hence, the conjecture holds for the derived complex A' of such a poset.

It is also shown in [**CD2**] that certain types of subdivisions preserve the sign of κ. This gives rise to other classes of triangulations for which the conjecture can be shown to hold.

Remark. One can carry out a similar discussion for a manifold with the structure of a piecewise Euclidean cell complex (not necessarily cubical) of nonpositive curvature. In this more general situation, however, the "local Euler characteristic", $\kappa^{\text{geom}}(L_v)$, at a vertex depends on a certain "volume", $a^*(\sigma)$, of simplicies in the link, so the corresponding "Local Hopf Conjecture" is not purely combinatorial. It states,

Local Hopf Conjecture. *If L is a $(2n-1)$-sphere with a piecewise spherical metric and L satisfies CAT(1), then $(-1)^n \kappa^{\text{geom}}(L) \geq 0$.*

If all edge lengths in L are $\geq \frac{\pi}{2}$ (i.e., L is nonacute), then by Moussong's Theorem (Theorem 3.4), L satisfies CAT(1) if and only if it is a metric flag complex. It is shown in [**CD2**], that the Local Hopf Conjecture for nonacute L is actually implied by the Combinatorial Hopf conjecture. A situation in which the Local Hopf Conjecture is known to hold is discussed at the end of the next section.

5. The Lower Bound Theorem

Let X be an n-dimensional simple polytope and let $(f_1, \ldots, f_n)$ be its f-vector, i.e., $f_i = \#i$-faces of X. The lower bound theorem, proved by Barnette in [**Ba**], gives lower bounds on the f_i's in terms of f_{n-1}. For example, one inequality states,

$$f_{n-2} \geq n f_{n-1} - \frac{n(n+1)}{2} .$$

Metric geometry can be used to give a geometric proof of this inequality.

Let X^* denote the simplicial complex dual to the boundary complex of X, so the k-simplicies of X^* correspond to the codimension $k+1$ faces of X. In [**CD3**], it is shown that any hyperbolic structure on X (i.e., a realization of X as a convex polytope in $\mathbb{H}^n$) gives rise to a natural piecewise spherical structure on X^* satisfying the CAT(1)-inequality. (This was first shown for $n = 3$ by Rivin and Hodgson in [**RH**]. They show, in fact, that for $n = 3$, *every* piecewise spherical structure on X^* satisfying CAT(1) arises in this way.) Moreover, two hyperbolic structures on X give rise to the same piecewise spherical structure on X^* if and only if they differ by an isometry of $\mathbb{H}^n$. It follows that there is an embedding of the space $\mathcal{H}(X)$ of hyperbolic structures on X (modulo isometry) into the space $\mathcal{S}(X^*)$ of piecewise spherical structures on X^*. In particular, $\dim \mathcal{H}(X) \leq \dim \mathcal{S}(X^*)$.

The dimensions of these spaces are easily computed. Using the hyperboloid model for $\mathbb{H}^n$,

$$\mathbb{H}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\,,\ x_{n+1} > 0\}\,,$$

a realization of X as a convex polytope in $\mathbb{H}^n$ is determined by unit normal vectors to the codimension 1 faces of X. There are f_{n-1} of these normal vectors and they lie in the n-dimensional "pseudo-sphere"

$$\mathbb{S}_1^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\}\,.$$

Thus, the space of all such hyperbolic structures has dimension $n f_{n-1}$. The dimension of the isometry group of $\mathbb{H}^n$ is $\frac{n(n+1)}{2}$, so we obtain

$$\dim \mathcal{H}(X) = n f_{n-1} - \frac{n(n+1)}{2} .$$

To compute the dimension of $\mathcal{S}(X^*)$, note that a spherical simplex is uniquely determined by the length of its edges, hence

$$\begin{aligned} \dim \mathcal{S}(X^*) &= \#\text{ edges of } X^* \\ &= \#\operatorname{codim} 2 \text{ faces of } X \\ &= f_{n-2}\,. \end{aligned}$$

This gives the desired inequality. It would be interesting to know whether other such inequalities can be obtained by similar geometric arguments.

REMARK. X^* is topologically an $(n-1)$-sphere. It follows from a formula of Hopf [**H**], that for $n = 2m$, any piecewise spherical metric on X^* arising as the dual of a hyperbolic structure on X satisfies the Local Hopf Conjecture discussed in the previous section. Indeed, Hopf's formula shows that

$$(-1)^m 2 \operatorname{vol}(X) = \kappa^{\text{geom}}(X^*)$$

where $\operatorname{vol}(X)$ is the volume of the hyperbolic structure on X.

6. Arrangements of Hyperplanes

There are many interesting combinatorial and topological questions concerning arrangements of hyperplanes in a vector space (see [**OT**]). One question which has been widely studied is when the space Y, obtained by removing a collection of complex hyperplanes from $\mathbb{C}^n$, is aspherical. This is known as the $K(\pi,1)$-problem. (An aspherical space is also known as a $K(\pi,1)$-space.) A particularly interesting example is the case of an arrangement of hyperplanes associated to a reflection group acting on $\mathbb{R}^n$. We discuss this case below.

Reflection groups are closely related to Coxeter groups. A *Coxeter group* is a group W having a presentation of the form

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1\,,\ (s_i s_j)^{m_{ij}} = 1\,,\ 1 \le i\,, j \le n \rangle$$

where $m_{ij} \in \{2, 3, \ldots, \infty\}$. The group W, together with the generating set $S = \{s_1, \ldots, s_n\}$ is called a *Coxeter system.*

Associated to a Coxeter system (W, S) is another group, A_W, called the associated *Artin group*, defined by

$$A_W = \langle s_1, \ldots, s_n \mid \underbrace{s_i s_j s_i \ldots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \ldots}_{m_{ij} \text{ terms}}\,,\ 1 \le i\,, j \le n \rangle\,.$$

It is easy to see that the map taking the generator s_i of A_W to the generator s_i of W induces a surjective homomorphism $A_W \twoheadrightarrow W$.

Coxeter groups can be finite or infinite. If W is finite, we say that A_W is of *finite type.* The best known examples of finite Coxeter groups are the symmetric groups $W = \Sigma_n$ whose associated Artin groups are the braid groups $A_W = B_n$. Some familiar infinite Coxeter groups are the "right angled" Coxeter groups (those with all $m_{ij} \in \{2, \infty\}$) whose associated Artin groups are known as "graph groups".

Another well-known class of infinite Coxeter groups are the Euclidean and hyperbolic triangle groups — discrete groups of isometries of $\mathbb{E}^2$ or $\mathbb{H}^2$ generated by reflections in the sides of a triangle. Indeed, Tits [**T**] and Vinberg [**V**] showed that *every* Coxeter group can be realized as a "reflection group", that is a discrete group of linear transformations of $\mathbb{R}^n$ generated by reflections, and conversely, every reflection group is, abstractly, a Coxeter group.

Now suppose W is acting as a reflection group on $\mathbb{R}^n$. Let $\mathcal{H}$ be the set of hyperplanes fixed by some reflection in W. (Note that the reflections in W include not only the generators, but also the conjugates of the generators.) If we remove these hyperplanes from $\mathbb{R}^n$, the resulting space is not topologically interesting: it is a disjoint union of contractible spaces. We are interested, rather, in the complexification of this arrangement, namely the set of complex hyperplanes

$$\mathcal{H}_{\mathbb{C}} = \{H + iH \mid H \in \mathcal{H}\}$$

in $\mathbb{C}^n$ $(= \mathbb{R}^n + i\mathbb{R}^n)$. W acts on $\mathbb{R}^n + i\mathbb{R}^n$ diagonally, and the hyperplane complement

$$Y = \mathbb{C}^n - \bigcup_{H \in \mathcal{H}} (H + iH)$$

is precisely the non-singular set of this action, that is, the set of points with trivial isotropy group. Thus W acts freely on Y and $Y \to Y/W$ is a covering space. The spaces Y and Y/W have interesting topological and group theoretic properties, which we will discuss below. They also play an important role in singularity theory (which we will not discuss below).

In the early 70's, Brieskorn [**Bk**] and Deligne [**D**] proved that for a *finite* Coxeter group W, the hyperplane complement Y is aspherical (i.e., its universal covering space $\widetilde{Y}$ is contractible) and the fundamental group $\pi_1(Y)$ is the kernel of the homomorphism $A_W \to W$. It follows that Y/W is likewise aspherical with $\pi_1(Y/W) = A_W$. It was conjectured that the analogous statement should hold for infinite Coxeter groups.

In the case of an infinite Coxeter group W, one must modify Y slightly to state the "analogous" conjecture. Namely, one replaces $\mathbb{R}^n$ by a certain open, convex cone I in $\mathbb{R}^n$ which is preserved by the W-action and on which W acts properly (so isotropy groups are finite subgroups of W). Then $\mathbb{R}^n + iI$ is an open, convex set in $\mathbb{C}^n$ and Y is defined by

$$Y = (\mathbb{R}^n + iI) - \bigcup_{H \in \mathcal{H}} (H + iH).$$

As before, Y is the non-singular set of the (diagonal) W-action on $\mathbb{R}^n + iI$, and $Y \to Y/W$ is a covering space. With this definition, the conjecture states:

CONJECTURE 6.1. *For W an infinite Coxeter group, Y/W is aspherical with fundamental group A_W.*

In his Ph.D. thesis [**L**], van der Lek showed that $\pi_1(Y/W) = A_W$, as predicted by the conjecture, but the $K(\pi,1)$-problem remained open.

By Corollary 2.2, any space which supports a metric of nonpositive curvature is aspherical. This suggests an approach to proving the conjecture: try to find a metric of nonpositive curvature on Y/W, or equivalently, on its universal covering space $\widetilde{Y}$. (Since $Y \to Y/W$ is a covering space, they have the same universal covering space.)

Since the most effective technique available for constructing nonpositively curved metrics is by means of Euclidean cubical complexes, the first step is to find a simplicial complex Φ_W which is homotopy equivalent to Y and has a natural decomposition into combinatorial cubes. (Φ_W is called the "Deligne complex" since it is analogous to a complex constructed by Deligne in his proof of the $K(\pi, 1)$-problem for finite W.) The complex Φ_W can be described algebraically: it is the derived complex of the partially ordered set of cosets aA_T, where $a \in A_W$ and A_T is a finite type Artin subgroup of A_W generated by a subset $T \subset S$. (That only the finite type subgroups of A_W appear, is related to the fact that W acts properly on $\mathbb{R}^n + iI$.)

Φ_W has certain "building-like" properties. In particular, it can be written as a union of subcomplexes, each of which is naturally isomorphic to the "Coxeter complex", Σ_W, of W. (Here, Σ_W is defined analogously to Φ_W using cosets wW_T of finite Coxeter subgroups W_T of W.)

As mentioned above, Φ_W decomposes naturally into cubes. Giving each cube the metric of a regular Euclidean cube, gives a Euclidean cubical structure on Φ_W. By Corollary 3.3, this structure has curvature ≤ 0 if and only if the link of every vertex in Φ_W is a flag complex. The link of a vertex aA_T can be described in terms of cosets of the finite type Artin group A_T. Since the finite type Artin groups are thoroughly understood (see [**D**], [**C1**], and [**C2**]), the flag complex condition can be checked algebraically. One finds:

THEOREM 6.2 ([**CD4**]). *The link of every vertex in Φ_W is a flag complex if and only if the Coxeter system (W, S) satisfies*

> *(FC): If $T \subset S$ generates an infinite subgroup of W, then $m_{ij} = \infty$ for some $s_i, s_j \in T$.*

(Recall that m_{ij} = order of $s_i s_j$ in W.) The simplest examples of Coxeter groups satisfying (FC) are the "right-angled" Coxeter groups. More generally, if a Coxeter group is represented by its "Coxeter diagram", then condition (FC) excludes a certain list of subdiagrams.

COROLLARY 6.3. *The Euclidean cubical metric on Φ_W has curvature ≤ 0 if and only if (W, S) satisfies (FC).*

COROLLARY 6.4. *If (W, S) satisfies (FC), then Conjecture 6.1 holds for W.*

REMARK. There is an alternate proof for showing that Φ_W is contractible for (FC) groups by viewing these groups as iterated amalgamated free products, and using induction.

What about the cases in which (W, S) fails to satisfy (FC)? In this case, we need to look for some other candidate for a nonpositively curved metric on Φ_W. In his Ph.D. thesis [**M**], Moussong produced a piecewise Euclidean metric on the Coxeter complex Σ_W and proved (using Theorem 3.4) that this metric is nonpositively curved for every W. Since Φ_W can be described as a union of copies of Σ_W, there is a corresponding piecewise Euclidean metric on Φ_W.

CONJECTURE 6.5. *The Moussong metric on Φ_W has nonpositive curvature for all W.*

Conjecture 6.5 implies Conjecture 6.1.

The situation for Φ_W, however, is considerably more complicated than for Σ_W. (For example, a link which in Σ_W would be a single sphere, would be an non-locally-finite union of spheres in Φ_W.) Conjecture 6.5 remains open, except in the case that Φ_W is a 2-dimensional complex. This case is covered by the theorem below.

THEOREM 6.6 ([**CD4**]). *Suppose (W, S) satisfies the following condition: any three elements of S generate an infinite subgroup of W. Then the Moussong metric on Φ_W has nonpositive curvature and hence Conjecture 6.1 holds for W.*

The hypotheses of this theorem are satisfied, for example, by the Euclidean and hyperbolic triangle groups and by the Coxeter groups of "large type" (i.e., all m_{ij}'s are ≥ 3).

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