

# A Geometric Rational Form for Artin Groups of FC Type

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## 1 Introduction

A Coxeter system  $(W, S)$  consists of a finite set  $S = \{s_1, \dots, s_n\}$  and a group  $W$  with a presentation of the form

$$W = \langle s_1, \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle$$

where  $m_{ij} \in \{2, 3, \dots, \infty\}$  and  $m_{ij} = m_{ji}$ . Associated to any Coxeter system is an *Artin group*

$$A = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j}_{m_{ij} \text{ terms}} \rangle$$

which projects naturally onto  $W$ . The Artin group associated to the symmetric group on  $n$  letters is the braid group on  $n$  strands, thus Artin groups are sometimes called generalized braid groups.

If the Coxeter group  $W$  is finite, then the associated Artin group is said to be of *finite type*. Finite type Artin groups have nice normal forms and are well understood algebraically. (See [4], [3], [6].) Little is known, however, about infinite type Artin groups. These groups arise in a number of contexts. For example, they arise as fundamental groups of hyperplane complements. (See [5], [10], [6], and [2].) In [5] it is shown that the universal covers of these hyperplane complements are homotopy equivalent to a certain simplicial complex, the (modified) Deligne complex. It is also shown that for certain classes of Artin groups  $A$ , the Deligne complex carries a CAT(0)

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geometry (in the sense of Gromov), hence it is contractible and the hyperplane complement is a  $K(A, 1)$ -space. There is a cocompact action of the Artin group on its Deligne complex with finite type isotropy groups.

In this paper we use the geometry of the Deligne complex to obtain a rational normal form for a certain class of infinite type Artin groups. Groups in this class, called FC Artin groups (where FC stands for “flag complex”), have the property that the Deligne complex has a natural cubical CAT(0) geometry. The idea for defining this normal form comes from G. Niblo and L. Reeves [8] who show that a group acting properly discontinuously and cocompactly on a CAT(0) cube complex has a normal form which gives a biautomatic structure on the group (in the sense of [7]). The action of an FC Artin group on its Deligne complex is not properly discontinuous. (The isotropy groups are of finite type but not finite.) Nonetheless, it is shown here that the “normal cube paths” of Niblo and Reeves can be used to obtain a normal form for these groups. Moreover, the language consisting of these normal forms is a regular language and we describe an algorithm for determining the normal form from an arbitrary word.

It is conjectured that the Deligne complex of any infinite type Artin group can be given a CAT(0) geometry. Our hope is that these methods can be generalized to give nice normal forms for other Artin groups. D. Peifer has shown that Artin groups of extra large type ( $m_{ij} \geq 4$  for  $i \neq j$ ) are biautomatic [9]. It is interesting to note that the Deligne complex for an Artin group of extra large type is also known to have a CAT(0) geometry (though not a cubical one). However, the methods of Peifer are based on small cancellation theory and do not make use of the geometry of the Deligne complex. Another approach to finding normal forms for FC Artin groups is taken by the first author, J. Altobelli, in [1]. He uses the fact that FC Artin groups can be obtained from finite type Artin groups by iterated amalgamated products. Such product decompositions are not unique, and each decomposition gives rise to a different normal form. The normal form discussed here is based on the geometry of the Deligne complex and is uniquely determined by the group itself. It is analogous to the normal form for infinite Coxeter groups obtained by crossing each wall of the Coxeter complex as soon as possible.

## 2 The Deligne Complex

A *cube complex*  $X$  is a polyhedral cell complex such that each cell is a standard  $n$ -cube for some  $n$ , and any two cubes are either disjoint or their intersection is a single face. The faces of  $X$  will be called the *cubes* of  $X$ . A more combinatorial definition of cube complex (in terms of the partially ordered set of cubes of  $X$ ) is given in [8].

The *star* of a cube  $C$  in  $X$  is the subcomplex of all (closed) cubes having  $C$  as a face:

$$St(C) = \bigcup_{C' \supseteq C} C'.$$

Let  $C_1$  and  $C_2$  be subcubes of a cube  $C$  of  $X$ . The smallest cube  $K$  in  $X$  containing both  $C_1$  and  $C_2$  is called the *span* of  $C_1$  and  $C_2$  and is denoted by  $K = Span(C_1, C_2)$ .  $C_1$  and  $C_2$  are said to *span* the cube  $K$  in  $X$ .

A *cube path* between two vertices  $v_\sigma$  and  $v_\tau$  in  $X$  is a sequence of cubes  $C_1, C_2, \dots, C_n$  such that

- 1)  $v_\sigma \subseteq C_1, v_\tau \subseteq C_n$  and
- 2)  $C_i \cap C_{i+1} \neq \emptyset$ .

A cube path is *normal* if, in addition,

- 3)  $C_i \cap C_{i+1}$  is a vertex  $v_i$  for  $i = 1, 2, \dots, n - 1$ ,
- 4)  $C_i = Span(v_{i-1}, v_i)$  for  $i = 1, 2, \dots, n$ , where  $v_0 = v_\sigma$  and  $v_n = v_\tau$ ,  
and
- 5)  $St(C_i) \cap C_{i+1} = v_i$  for  $i = 1, 2, \dots, n - 1$ .

(The original definition of Niblo and Reeves requires a cube path to satisfy (1)–(4).) Although Niblo and Reeves only consider locally finite cube complexes, an analysis of their proofs shows that the following result holds in general. (Niblo and Reeves make use of the locally finite hypothesis later in their paper.)

**Theorem 2.1** ([8], **Prop. 3.4**) *Let  $X$  be a  $CAT(0)$  cube complex. Given two vertices  $u$  and  $v$  in  $X$ , there is a unique normal cube path from  $u$  to  $v$ .*

We now define the Deligne complex. See [5] for more details. Let  $A$  be the Artin group associated with the Coxeter system  $(W, S)$ .

If  $T \subseteq S$ , then the subgroup of  $W$  generated by  $T$ ,  $W_T$ , is called a *special subgroup of  $W$* . Likewise,  $A_T$ , the subgroup of  $A$  generated by  $T$  is called a *special subgroup of  $A$* . Any coset of a special subgroup is called a *special coset*. It was shown by van der Lek in [10] (see also [5]) that the intersection of special subgroups is a special subgroup

$$A_{T_1} \cap A_{T_2} = A_{T_1 \cap T_2}$$

and hence a nonempty intersection of special cosets is a special coset.

To define the Deligne complex, set

$$\mathcal{S}^f = \{T \subseteq S \mid W_T \text{ is finite}\},$$

and consider the partially ordered set of cosets

$$A\mathcal{S}^f = \{aA_T \mid a \in A, T \in \mathcal{S}^f\}$$

ordered by inclusion. For an ordered pair  $a_1A_{T_1} \subseteq a_2A_{T_2}$  in  $A\mathcal{S}^f$  we define the interval

$$(a_1A_{T_1}, a_2A_{T_2}) = \{aA_T \in A\mathcal{S}^f \mid a_1A_{T_1} \subseteq aA_T \subseteq a_2A_{T_2}\}.$$

Since  $a_1A_{T_1} \subseteq a_2A_{T_2}$  implies  $a_1 \in a_2A_{T_2}$ , we can always choose coset representatives so that  $a_1 = a_2$ . Thus, as posets,

$$(a_1A_{T_1}, a_2A_{T_2}) \cong (A_{T_1}, A_{T_2}) \cong \{T \in \mathcal{S}^f \mid T_1 \subseteq T \subseteq T_2\}.$$

An interval  $(a_3A_{T_3}, a_4A_{T_4})$  is called a subinterval of  $(a_1A_{T_1}, a_2A_{T_2})$  if

$$a_1A_{T_1} \subseteq a_3A_{T_3} \subseteq a_4A_{T_4} \subseteq a_2A_{T_2}.$$

The set of subintervals of  $(a_1A_{T_1}, a_2A_{T_2})$  is combinatorially equivalent to the set of faces of an  $n$ -cube where  $n = |T_2| - |T_1|$ . The vertices of the cube correspond to trivial subintervals  $(aA_T, aA_T)$ .

Thus we can define a cube complex  $\Phi_A$  whose vertex set is  $A\mathcal{S}^f$  and whose cubes are indexed by the intervals  $(a_1A_{T_1}, a_2A_{T_2})$ . This cube complex is the *Deligne complex* for  $A$ . (In [5] it is called the “modified Deligne complex”.)

**Theorem 2.2** ([5] **Theorem 4.5.3**) *Let  $A$  be the Artin group associated with the Coxeter system  $(W, S)$ . The Deligne complex  $\Phi_A$  is a  $CAT(0)$  cube complex if and only if  $(W, S)$  satisfies*

(FC) *If  $T \subseteq S$  and every pair of elements of  $T$  generates a finite subgroup of  $W$ , then  $T \in \mathcal{S}^f$ .*

### 3 Finite Type Artin Groups

Before embarking on our main theorems, we will need some facts about finite type Artin groups. For more details, see [3] and [4].

Let  $A$  be the Artin group associated to a finite Coxeter system  $(W, S)$ . There is a natural homomorphism  $A \rightarrow W$  which takes a generator  $s \in S$  of  $A$  to the generator of the same name in  $W$ . There is a set-theoretic section  $m : W \rightarrow A$  of this homomorphism defined as follows. Let  $F^+(S)$  denote the free monoid on  $S$ . Since  $s = s^{-1}$  in  $W$ , the natural map  $F^+(S) \rightarrow W$  is surjective. For  $g \in W$ , choose a word  $w$  in  $F^+(S)$  which is a representative of minimal length for  $g$ . Define  $m(g)$  to be the image of  $w$  in  $A$  under the canonical map  $F^+(S) \rightarrow A$ . It follows from Proposition 1.12 of [6] that this image is independent of the choice of the minimal length representative  $w$ . The elements  $m(g) \in A$  for  $g \neq 1$  are called *minimals*. Let  $M$  be the set of minimals in  $A$ . If  $A_T$  is a special subgroup of  $A$ , then  $a \in A_T$  is a minimal in  $A_T$  if and only if it is a minimal in  $A$ . Thus  $M_T$ , the set of minimals in  $A_T$ , is a subset of  $M$  and  $M_{T_1} \cap M_{T_2} = M_{T_1 \cap T_2}$ .

Clearly  $S \subseteq M$  and hence  $M$  is also a finite generating set for  $A$ . Let  $F(M)$  denote the free group on  $M$  and for  $w \in F(M)$ , let  $\bar{w}$  denote its image in  $A$ . In [4], a normal form for  $A$  in terms of this generating set is described. The normal form for  $a \in A$  is an element of  $F(M)$  representing  $a$ . It is called the *canonical minimal decomposition* or *cmd* for  $a$ . The cmd for  $a \in A$  will be denoted by  $cmd(a, A)$  or just  $cmd(a)$  when  $A$  need not be specified. We do not need the precise definition of the cmd here, but only the properties described in the following theorem.

**Theorem 3.1** *Given  $w \in F(M)$  with  $\bar{w} = a$ , there is an algorithm for finding  $cmd(a, A)$ . If  $A_T$  is a special subgroup of  $A$  and  $a \in A_T$  then  $cmd(a, A_T) = cmd(a, A)$ ; in particular,  $cmd(a, A) \in F(M_T)$ .*

**Remark 3.2** *By choosing a representative in  $F(S)$  for each minimal in  $M$ , we could get our normal forms to lie in  $F(S)$ . The advantage of taking  $M$  as the generating set is that the normal form is more canonical. Moreover, it is shown in [4] that the cmd is a minimal length word for  $a$  with respect to  $M$  (but not with respect to  $S$ ).*

A system of coset representatives for special cosets of  $A$  is described in [1]. The representative in this system for the coset  $aA_T$  in  $A$  is

called the *minimal coset representative* for  $aA_T$  in  $A$ . Its  $\text{cmd}$  is denoted by  $\text{mrep}(aA_T, A)$  or just by  $\text{mrep}(aA_T)$  when  $A$  need not be specified. In particular,  $\text{mrep}(aA_\emptyset) = \text{cmd}(a)$ . As with the  $\text{cmd}$ , the definition of the minimal coset representatives is not given here; the properties given by the following theorem will suffice.

**Theorem 3.3** *Given  $w \in F(M)$  with  $\bar{w} = a$ , there is an algorithm for finding  $\text{mrep}(aA_T, A)$ . If  $aA_T \cap A_U \neq \emptyset$  then  $\text{mrep}(aA_T, A) \in F(M_U)$ .*

Theorems 3.1 and 3.3 imply that if  $aA_T \subseteq A_U$  then  $\text{mrep}(aA_T, A_U) = \text{mrep}(aA_T, A)$ . Thus, by van der Lek's intersection property, if  $A$  is any Artin group (not necessarily of finite type), and if  $A_U$  and  $A_V$  are finite type special subgroups with  $aA_T \subseteq A_U \cap A_V$ , then  $\text{mrep}(aA_T, A_U) = \text{mrep}(aA_T, A_V)$ . The algorithms described below will accept as input words representing group elements or cosets and use these words to produce other words. For this reason, it is convenient to overload the notation for  $\text{cmd}$  and  $\text{mrep}$  as follows. If  $w \in F(M)$  then  $\text{cmd}(w)$  will denote  $\text{cmd}(\bar{w})$  and  $\text{mrep}(wA_T)$  will denote  $\text{mrep}(\bar{w}A_T)$ .

## 4 Normal Forms

Now consider the case of an infinite type Artin group  $A$ . In this case, we define

$$M = \bigcup_{T \in \mathcal{S}^f} M_T$$

where  $M_T$  denotes the minimals in  $A_T$ . This is, again, a finite generating set for  $A$ . Our goal is to define a normal form for  $A$  in terms of these generators. For an element of a finite type subgroup  $A_T$ , it will be the usual  $\text{cmd}$ .

From now on, we assume that  $A$  is an infinite type Artin group associated to an (FC) Coxeter system  $(W, S)$ . Thus, by Theorem 2.2, the Deligne complex  $\Phi = \Phi_A$  is a  $\text{CAT}(0)$  cube complex. Recall that a cube in  $\Phi$  can be represented by an ordered pair  $(a_1A_{T_1}, a_2A_{T_2})$  in  $AS^f \times AS^f$  such that  $a_1A_{T_1} \subseteq a_2A_{T_2}$ . Note that  $T_1$  and  $T_2$  are uniquely determined, but  $a_1$  and  $a_2$  are not. Note also that we can always choose  $a_2 = a_1$ . Define a *labelled cube*  $L$  to be a triple  $L = (w, R, T)$ , where  $w \in F(M)$  and  $R \subseteq T \in \mathcal{S}^f$ . Associated to

a labelled cube  $L$  is its underlying cube  $\bar{L} = (\bar{w}A_R, \bar{w}A_T)$  in  $\Phi$ . A labelled vertex is a labelled cube with  $R = T$ .

**Lemma 4.1** *Let  $K_1 = (a_1A_{R_1}, a_1A_{T_1})$  and  $K_2 = (a_2A_{R_2}, a_2A_{T_2})$  be two cubes in  $\Phi$ . Then*

- 1)  $K_1 \cap K_2 \neq \emptyset$  if and only if  $R_1 \cup R_2 \subseteq T_1 \cap T_2$  and  $a_1^{-1}a_2 \in A_{T_1 \cap T_2}$ .  
In this case,

$$K_1 \cap K_2 = (a_1A_R, a_1A_{T_1 \cap T_2}),$$

where  $R$  is the smallest subset of  $S$  satisfying  $R_1 \cup R_2 \subseteq R \subseteq T_1 \cap T_2$  and  $a_1^{-1}a_2 \in A_R$ .

- 2)  $K_1$  and  $K_2$  span a cube in  $\Phi$  if and only if  $a_1A_{R_1} \cap a_2A_{R_2} \neq \emptyset$  and  $T_1 \cup T_2 \in \mathcal{S}^f$ . In this case,

$$\text{Span}(K_1, K_2) = (aA_{R_1 \cap R_2}, aA_{T_1 \cup T_2}),$$

where  $a$  is any element of  $a_1A_{R_1} \cap a_2A_{R_2}$ . Moreover,  $a_1^{-1}a_2 \in A_{R_1 \cup R_2}$ .

*Proof:* Exercise.  $\square$

We now define the intersection and span of labelled cubes.

**Definition 4.2** *Suppose  $L_1 = (w_1, R_1, T_1)$  and  $L_2 = (w_2, R_2, T_2)$  are two labelled cubes.*

- 1) Assume  $\bar{L}_1 \cap \bar{L}_2 \neq \emptyset$ . Then

$$L_1 \cap L_2 = (w_1, R, T_1 \cap T_2),$$

where  $R$  is the smallest subset of  $S$  satisfying  $R_1 \cup R_2 \subseteq R \subseteq T_1 \cap T_2$  and  $\bar{w}_1^{-1}\bar{w}_2 \in A_R$ .

- 2) Assume  $\bar{L}_1$  and  $\bar{L}_2$  span a cube. Let  $u = \text{mrep}(\bar{w}_1^{-1}\bar{w}_2A_{R_2}, A_{T_1 \cup T_2})$ . Define

$$\text{Span}(L_1, L_2) = (w_1u, R_1 \cap R_2, T_1 \cup T_2).$$

**Remark 4.3** *i) If  $\bar{L}_1$  and  $\bar{L}_2$  span a cube, then  $A_{R_1} \cap \bar{w}_1^{-1}\bar{w}_2A_{R_2} \neq \emptyset$ , and it follows from Theorem 3.3 that  $u$  lies in this intersection. Hence  $\bar{w}_1\bar{u} \in \bar{w}_1A_{R_1} \cap \bar{w}_2A_{R_2}$  as required. ii) The definitions of  $L_1 \cap L_2$  and  $\text{Span}(L_1, L_2)$  are not symmetrical and, in fact, only depend on  $L_1$  and  $\bar{L}_2$ . Thus, it makes sense to write  $L_1 \cap \bar{L}_2$  or  $\text{Span}(L_1, \bar{L}_2)$ .*

Let  $v_\sigma$  and  $v_\tau$  be labelled vertices. A *labelled cube path* from  $v_\sigma$  to  $v_\tau$  is a sequence of labelled cubes  $C_1, C_2, \dots, C_n$  such that

- 1)  $\bar{v}_\sigma \subseteq \bar{C}_1, \bar{v}_\tau \subseteq \bar{C}_n$
- 2)  $\bar{C}_i \cap \bar{C}_{i+1} \neq \emptyset$

A labelled cube path is *normal* if, in addition,

- 3)  $C_i \cap C_{i+1}$  is a labelled vertex  $v_i$  for  $i = 1, 2, \dots, n - 1$
- 4)  $C_i = \text{Span}(v_{i-1}, v_i)$  for  $i = 1, 2, \dots, n$ , where  $v_0 = v_\sigma$  and  $v_n = v_\tau$
- 5)  $St(\bar{C}_i) \cap \bar{C}_{i+1} = \bar{v}_i$  for  $i = 1, 2, \dots, n - 1$ .

A normal labelled cube path will be referred to as an *NLC path*. Conditions 3, 4, and 5 imply that the underlying cube path  $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_n$  is a normal cube path in the sense of Niblo and Reeves. Suppose  $C_1, C_2, \dots, C_m$  is an NLC path. If  $v_{i-1} = v_i = C_i$ , we say that  $C_i$  is a *degenerate cube*. Two NLC paths  $C_1, \dots, C_m$  and  $C'_1, \dots, C'_n$  are *equivalent* if they can be obtained from each other by adding and/or removing degenerate cubes. Clearly, every NLC path is equivalent to a unique NLC path with no degenerate cubes. However, to avoid constant renumbering, it will be convenient to allow for degenerate cubes.

**Theorem 4.4** *Let  $v_\sigma$  be a labelled vertex. Then for any labelled vertex  $v_\tau$ , there is a unique (up to equivalence) NLC path from  $v_\sigma$  to  $v_\tau$  which depends only on the underlying vertex  $\bar{v}_\tau$ .*

*Proof:* By Reeves' Proposition 3.4, there is a unique normal cube path  $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_m$  from  $\bar{v}_\sigma$  to  $\bar{v}_\tau$  in  $\Phi$ . Let  $\bar{v}_i = \bar{C}_{i-1} \cap \bar{C}_i$  and let  $v_0 = v_\sigma$ . We determine the labelled cubes  $v_i$  and  $C_i$  inductively by the formulas

$$\begin{aligned} C_i &= \text{Span}(v_{i-1}, \bar{v}_i) \\ v_i &= C_i \cap \bar{C}_{i+1}. \end{aligned}$$

These are well-defined in light of Remark 4.3(ii).  $\square$

We are now ready to define normal forms for special cosets. For a special coset  $aA_T$ ,  $T \in S^f$ , set  $v_\sigma = (1, \emptyset, \emptyset)$ , and  $v_\tau = (w_\tau, T, T)$ , where  $w_\tau$  is any word representing  $a$  (so that  $\bar{v}_\tau = aA_T$ ), and let  $\mathcal{C}$  be the unique NLC path from  $v_\sigma$  to  $v_\tau$ . Suppose  $C_n = (w_n, R_n, T_n)$ .

Since  $\bar{v}_\tau$  is a vertex of  $\bar{C}_n$ ,  $\bar{w}_n A_{R_n} \subseteq aA_T \subseteq \bar{w}_n A_{T_n}$ . In particular,  $w_n$  is a word representing the coset  $aA_T$ . We select  $\bar{w}_n$  as the distinguished representative of the coset  $aA_T$  and define the normal form of the coset to be  $w_n$ . This also gives normal forms for elements  $a \in A$ ; namely, the normal form of  $a$  is the normal form of the coset  $aA_\emptyset$ .

**Remark 4.5** *If  $aA_U \subset A_V$  and  $V \in S^f$ , then the NLC path from  $v_\sigma = (1, \emptyset, \emptyset)$  to  $\bar{v}_\tau = aA_U$  is given by  $C_1 = (1, \emptyset, V')$ ,  $C_2 = (b, U, V')$  where  $V' \in S^f$  is the smallest set such that  $aA_U \subseteq A_{V'}$  and  $b$  is the minimal coset representative of  $aA_U$  in  $A_{V'}$ . Thus, the normal form of  $aA_U$  is  $mrep(aA_U)$  in the finite type case.*

Since our goal is to solve the word problem for  $A$ , we will need an algorithm for finding NLC paths. This algorithm will make heavy use of the intersection and span operations, for which the following lemma provides algorithms.

**Lemma 4.6** *Let  $C_1 = (w_1, R_1, T_1)$ ,  $C_2 = (w_2, R_2, T_2)$  and let  $r$  be a finite type word such that  $\bar{w}_2 = \bar{w}_1 \bar{r}$ . Then there are algorithms for determining  $C_1 \cap C_2$  and  $Span(C_1, C_2)$ .*

*Proof:* 1) We first determine whether  $C_1 \cap C_2 \neq \emptyset$ . By Lemma 4.1,  $C_1 \cap C_2 \neq \emptyset$  if and only if  $R_1 \cup R_2 \subseteq T_1 \cap T_2$  and  $\bar{r} = \bar{w}_1^{-1} \bar{w}_2 \in A_{T_1 \cap T_2}$ . Since  $r$  is a finite type word, we may assume it is in normal form. Then  $\bar{r} \in A_{T_1 \cap T_2}$  if and only if  $r \in F(M_{T_1 \cap T_2})$ .

If  $C_1 \cap C_2 \neq \emptyset$ , let  $P$  be the smallest subset of  $T_1 \cap T_2$  such that  $r \in F(M_P)$  and set  $R = P \cup R_1 \cup R_2$ . Then  $C_1 \cap C_2 = (w_1, R, T_1 \cap T_2)$ .

2) Assume  $T_1 \cup T_2 \in S^f$ . (If not, then  $Span(C_1, C_2)$  is empty.) By Lemma 4.1,  $Span(C_1, C_2) \neq \emptyset$  if and only if  $A_{R_1} \cap \bar{r}A_{R_2} \neq \emptyset$ . Assuming, as above, that  $r$  is in normal form, this holds if and only if  $u = mrep(\bar{r}A_{R_2}, A_{T_1 \cup T_2})$  lies in  $F(M_{R_1})$ . In this case,  $Span(C_1, C_2) = (w_1 u, R_1 \cap R_2, T_1 \cup T_2)$ .  $\square$

**Theorem 4.7** *Let  $v_\sigma = (w_\sigma, T_\sigma, T_\sigma)$  and  $v_\tau = (w_\tau, T_\tau, T_\tau)$  be labelled vertices and let  $\mathcal{C} = C_1, C_2, \dots, C_n$  with  $C_i = (w_i, R_i, T_i)$  be a labelled cube path from  $v_\sigma$  to  $v_\tau$ . Assume we are given finite type words  $r_0, r_1, \dots, r_n$  such that  $w_{i+1} = w_i r_i$  for  $i = 0, 1, \dots, n-1$  (where  $w_0 = w_\sigma$ ) and  $\bar{w}_\tau = \bar{r}_0 \bar{r}_1 \cdots \bar{r}_n$ . Then there is an algorithm for finding the NLC path  $\mathcal{C}' = C'_1, C'_2, \dots, C'_n$  (with  $C'_i = (w'_i, R'_i, T'_i)$ ) from  $v_\sigma$  to  $v_\tau$  and finite type words  $r'_0, \dots, r'_n$  such that  $w'_{i+1} = w'_i r'_i$ , (where  $w'_0 = w_\sigma$ ) and  $\bar{w}_\tau = \bar{r}'_0 \bar{r}'_1 \cdots \bar{r}'_n$ . Moreover,  $\bar{C}'_n \subseteq \bar{C}_n$ .*

Note that if  $v_\sigma = (1, \emptyset, \emptyset)$ , then  $w'_n$  is the normal form of the coset  $\overline{w}_\tau A_{T_\tau}$ . Before proving the theorem we give the following corollary.

**Corollary 4.8** *Given a word  $w \in F(M)$  and a set  $T \in \mathcal{S}^f$ , there is an algorithm for finding the normal form  $w'$  for  $\overline{w}A_T$ , and a word  $r \in F(M_T)$  such that  $\overline{w} = \overline{w}'\overline{r}$ . In particular, taking  $T = \emptyset$ , this algorithm gives a solution to the word problem for  $A$ .*

*Proof:* Decompose  $w$  as a product of finite type words  $w = r_1 r_2 \cdots r_n$  and let  $T_i$  be the smallest subset of  $T$  such that  $r_i \in F(M_{T_i})$ . Let  $w_i = r_1 \cdots r_i$ . Define a labelled cube path  $\mathcal{C}$  from  $v_\sigma = (1, \emptyset, \emptyset)$  to  $v_\tau = (w, T, T)$  as follows:

$$\begin{aligned} C_1 &= (1, \emptyset, T_1) \\ C_2 &= (w_1, \emptyset, T_1) \\ C_3 &= (w_1, \emptyset, T_2) \\ &\vdots \\ C_{2i} &= (w_i, \emptyset, T_i) \\ C_{2i+1} &= (w_i, \emptyset, T_{i+1}) \\ &\vdots \\ C_{2n} &= (w_n, \emptyset, T_n) \end{aligned}$$

By Theorem 4.7, there is an algorithm to find the NLC path  $C'_1, \dots, C'_{2n}$  from  $v_\sigma$  to  $v_\tau$  and finite type words  $r'_i$  such that  $w'_{i+1} = w'_i r'_i$ . Since  $\overline{w} = \overline{w}'_{2n+1}$  and the normal form for  $\overline{w}A_T$  is  $w'_{2n}$ , this completes the proof.  $\square$

*Proof of Theorem 4.7:* It will be necessary to keep track of the finite type words  $r_i$  along with the labelled cube paths throughout the proof. We will call this data a “fully labelled cube path” and denote it by a sequence

$$v_0, (r_0), C_1, (r_1), \dots, C_n, (r_n), v_n.$$

where  $v_0 = v_\sigma$  and  $v_n = v_\tau$ . The condition  $\overline{C}_i \cap \overline{C}_{i+1} \neq \emptyset$  implies that  $\overline{r}_i \in A_{T_i \cap T_{i+1}}$ . We will always assume that  $r_i$  is in normal form, that is,  $r_i = cmd(\overline{r}_i)$ , and hence  $r_i \in F(M_{T_i \cap T_{i+1}})$ . (This also holds for  $i = 0$  and  $i = n$  if we take  $T_0 = T_\sigma$  and  $T_{n+1} = T_\tau$ .)

The proof proceeds by induction on  $n$ . For  $n = 1$ ,  $\overline{w}_\sigma^{-1} \overline{w}_\tau = \overline{r}_0 \overline{r}_1$  and  $r_0 r_1 \in F(M_T)$ . Since  $v_\sigma$  and  $v_\tau$  span a cube,  $r = cmd(r_0 r_1)$



Since  $r_2 \in F(M_{T_\tau})$ , we can find a finite type representative for  $(\bar{w}_\sigma \bar{r}'_0)^{-1} \bar{w}_\tau = \bar{r}'_0^{-1} \bar{r}_0 \bar{r}_1 \bar{r}_2$ , namely,  $cmd(sr_2, A_{T \cup T_\tau})$ . By Lemma 4.6, we can find

$$C'_2 = Span(v_1, v_\tau) = (w_\sigma r'_0 r'_1, T \cap T_\tau, T \cup T_\tau),$$

where  $r'_1 = mrep(sr_2 A_{T_\tau}, A_{T \cup T_\tau})$ . By Theorem 3.1, we can calculate  $r'_2 = cmd(r'^{-1}_1 sr_2)$ . Clearly  $\bar{C}'_2 \subseteq \bar{C}_2$  since both  $\bar{v}_1$  and  $\bar{v}_\tau$  lie in  $\bar{C}_2$ . We need to verify that  $St(\bar{C}'_1) \cap \bar{C}'_2 = \bar{v}_1$ . Suppose  $\bar{v}$  is a vertex in  $St(\bar{C}'_1) \cap \bar{C}'_2$ . Then  $\bar{v}$  and  $\bar{v}_\sigma$  span a cube and  $\bar{v} \in \bar{C}_2$  so  $\bar{v} \in B$ . The fact that  $\bar{v}_1$  is the nearest vertex in  $B$  to  $\bar{v}_\tau$  implies that  $B \cap \bar{C}'_2 = \bar{v}_1$ , so we conclude that  $\bar{v}_1 \subseteq St(\bar{C}'_1) \cap \bar{C}'_2 \subseteq B \cap \bar{C}'_2 = \bar{v}_1$ . Thus,

$$v_\sigma, (r'_0), C'_1, (r'_1), C'_2, (r'_2), v_\tau$$

is a normal fully labelled cube path. This completes the case  $n = 2$ .

Now assume by induction that the theorem holds for paths of length less than  $n$ . Choose a vertex  $v_{n-2}$  in  $C_{n-2} \cap C_{n-1}$ .

Step 1: Apply the case  $n = 2$  to the path

$$v_{n-2}, (r_{n-2}), C_{n-1}, (r_{n-1}), C_n, (r_n), v_n (= v_\tau)$$

to get an NLC path

$$v_{n-2}, (r'_{n-2}), C'_{n-1}, (r'_{n-1}), C'_n, (r'_n), v_n$$

with  $\bar{C}'_n \subseteq \bar{C}_n$ . Note that  $r'_n \in F(M_{T'_n \cap T_\tau})$ .

Step2: Let  $v'_{n-1} = C'_n \cap C'_{n-1}$  so that the label on  $v'_{n-1}$  agrees with that of  $C'_n$ . Apply the inductive hypothesis to the path

$$v_0, (r_0), C_1, (r_1), \dots, C_{n-2}, (r'_{n-2}), C'_{n-1}, (r'_{n-1}), v'_{n-1}$$

to get a fully labelled normal cube path

$$v_0, (r'_0), C'_1, (r'_1), \dots, C'_{n-2}, (r''_{n-2}), C''_{n-1}, (r''_{n-1}), v'_{n-1}$$

with  $\bar{C}''_{n-1} \subseteq \bar{C}'_{n-1}$ . Note that  $r''_{n-1} \in F(M_{T''_{n-1} \cap T'_n})$ .

Moreover,  $\bar{C}''_{n-1} \cap \bar{C}'_n \subseteq \bar{C}'_{n-1} \cap \bar{C}'_n = \bar{v}'_{n-1}$ . Let  $v_{n-1} = C''_{n-1} \cap C'_n$  so  $v_{n-1}$  is labelled by  $w''_{n-1}$ . Thus, the underlying vertices  $\bar{v}_{n-1}$  and  $\bar{v}'_{n-1}$  are the same, but  $v_{n-1}$  is labelled by  $w''_{n-1}$  while  $v'_{n-1}$  is labelled by  $w'_{n-1}$ . Set

$$C''_n = Span(v_{n-1}, v_n).$$

Then the underlying cubes  $\overline{C}'_n$  and  $\overline{C}''_n$  agree since both are spanned by  $\overline{v}_{n-1}$  and  $\overline{v}_n$ . Thus  $C''_n = (w''_n, R'_n, T'_n)$  with  $w''_n = w''_{n-1}u$ ,  $u \in F(M_{T'_n})$ . Set

$$r''_n = cmd(u^{-1}r''_{n-1}r'_n).$$

This makes sense since  $u$ ,  $r''_{n-1}$ , and  $r'_n$  all lie in  $F(M_{T'_n})$ . This gives

$$\overline{w}_\tau = \overline{w}'_n \overline{r}'_n = \overline{w}''_{n-1} \overline{r}''_{n-1} \overline{r}'_n = \overline{w}''_{n-1} \overline{w}''_n = \overline{w}''_n \overline{r}''_n.$$

Thus we obtain a fully labelled cube path

$$\mathcal{C}' = v_0, (r'_0), C'_1, (r'_1), \dots, C'_{n-2}, (r''_{n-2}), C''_{n-1}, (r''_{n-1}), C''_n, (r''_n), v_n.$$

This path satisfies all conditions for an NLC path except possibly the condition

$$(*) \quad St(\overline{C}''_{n-1}) \cap \overline{C}''_n = \overline{v}_{n-1}.$$

If this condition holds, we are done. If not, replace the original cube path  $\mathcal{C}$  by  $\mathcal{C}'$  and return to Step 1. This process must eventually terminate. To see this, note that  $C''_n = Span(v_{n-1}, v_n)$  means that  $v_{n-1}$  is the vertex of  $C''_n$  at greatest distance from  $v_n$ . If the condition  $(*)$  fails, then there is some other vertex in  $St(\overline{C}''_{n-1}) \cap \overline{C}''_n$  which is necessarily closer to  $v_n$ . In this case, Step 1 will strictly reduce the dimension of  $C''_n$ . In other words, each time we return to Step 1, the dimension of the last cube becomes strictly smaller, thus the algorithm must terminate in finitely many steps.  $\square$

## 5 Rationality

Let  $\mathcal{L}$  be the language of normal forms for elements of  $A$ . In this section we prove that  $\mathcal{L}$  is a regular language.

A *labelled edge path* in the Deligne complex is a sequence of labelled vertices  $v_0, \dots, v_n$  such that, as cosets, either  $v_{i-1} \subseteq v_i$  or  $v_{i-1} \supseteq v_i$  for  $i = 1, \dots, n$ . (Note that  $v_{i-1}$  and  $v_i$  span a cube, but are not necessarily connected by an edge.) For a labelled edge path beginning at  $v_0 = (1, \emptyset, \emptyset)$ , we necessarily have  $v_0 \subseteq v_1$ . We will say such a path is *alternating* if  $v_{2i-1} \supseteq v_{2i} \subseteq v_{2i+1}$  for all  $i$ . For any labelled edge path  $v_0, \dots, v_n$ , we have an associated labelled cube path  $\mathcal{C} = C_1, \dots, C_n$  defined by  $C_i = Span(v_{i-1}, v_i)$ .

**Lemma 5.1** *Let  $\mathcal{C} = C_1, C_2, \dots, C_n$  be the (nondegenerate) NLC path from  $v_0 = (1, \emptyset, \emptyset)$  to a vertex  $v_n$  and let  $v_i = C_i \cap C_{i+1}$ . Then*

$v_0, \dots, v_n$  is an alternating labelled edge path and  $\mathcal{C}$  is its associated cube path.

*Proof:* First note that for a cube  $C = \text{Span}(v, w)$ , if  $v$  is the minimal (resp. maximal) vertex of  $C$ , then  $w$  is the maximal (resp. minimal) vertex of  $C$  (where the ordering is given by the inclusion of cosets). Now consider the NLC path from  $v_0 = (1, \emptyset, \emptyset)$  to  $v_n$ . Since  $\bar{v}_0 = \{1\}$ ,  $v_0$  is clearly the minimal vertex in  $C_1$  and hence  $v_1$  is the maximal vertex in  $C_1$ . Suppose by induction on  $i$  that  $v_i$  is maximal (resp. minimal) in  $C_i$ . We claim that  $v_i$  is also maximal (resp. minimal) in  $C_{i+1}$ . For if not, then there is a vertex  $v \neq v_i$  in  $C_{i+1}$  such that  $\bar{v}$  lies in  $St(\bar{C}_i)$ , contradicting the assumption that the cube path is normal. Thus,  $v_i$  is maximal (resp. minimal) in  $C_{i+1} = \text{Span}(v_i, v_{i+1})$  and hence,  $v_{i+1}$  is minimal (resp. maximal) in  $C_{i+1}$ . It follows inductively that  $v_0 \subseteq v_1 \supseteq v_2 \subseteq v_3 \supseteq \dots$ .  $\square$

**Lemma 5.2** *Let  $\mathcal{C} = C_1, C_2, \dots, C_n$  be the labelled cube path associated to an alternating labelled edge path from  $v_0, \dots, v_n$  with  $v_0 = (1, \emptyset, \emptyset)$ ,  $v_i = (w_i, T_i, T_i)$ . Then  $\mathcal{C}$  is the NLC path from  $v_0$  to  $v_n$  if and only if for  $i = 1, \dots, \lfloor n/2 \rfloor$ ,*

- (a<sub>i</sub>)  $w_{2i} = w_{2i+1} = w_{2i-1}r_i$ ,  
where  $r_i = \text{mrep}(\bar{w}_{2i-1}^{-1}\bar{w}_{2i}A_{T_{2i}}, A_{T_{2i-1}})$  and  $w_1 = 1$ ,
- (b<sub>i</sub>)  $\forall R, T_{2i} \subset R \subseteq T_{2i+1} \implies T_{2i-1} \cup R \notin \mathcal{S}^f$ , and
- (c<sub>i</sub>)  $\forall R, T_{2i} \subseteq R \subset T_{2i-1} \implies A_R \cap \bar{r}_i A_{T_{2i-2}} = \emptyset$ .

*Proof:* The labelled cube path associated to an alternating edge path automatically satisfies condition (4) in the definition of NLC path. For condition (5), it is a straightforward exercise to verify that condition (b<sub>i</sub>) of the lemma is equivalent to

$$St(\bar{C}_{2i}) \cap \bar{C}_{2i+1} = \bar{v}_{2i},$$

and condition (c<sub>i</sub>) is equivalent to

$$St(\bar{C}_{2i-1}) \cap \bar{C}_{2i} = \bar{v}_{2i-1}.$$

Finally we note that  $St(\bar{C}_j) \cap \bar{C}_{j+1} = \bar{v}_j$ ,  $j = 2i - 1, 2i$ , implies  $\bar{C}_j \cap \bar{C}_{j+1} = \bar{v}_j$ . Since the label on  $C_{2i} = \text{Span}(v_{2i-1}, v_{2i})$  is  $w_{2i-1}r_i$ , where

$$r_i = \text{mrep}(\bar{w}_{2i-1}^{-1}\bar{w}_{2i}A_{T_{2i}}, A_{T_{2i-1}}),$$

and the label on  $C_{2i+1} = \text{Span}(v_{2i}, v_{2i+1})$  is  $w_{2i}$ , the NLC path condition (3) ( $C_j \cap C_{j+1} = v_j$ ) is equivalent to (a<sub>i</sub>).  $\square$

**Theorem 5.3**  $\mathcal{L}$  is a regular language.

*Proof:* We describe a finite state automaton recognizing  $\mathcal{L}$ . The states consist of a start state  $s_0 = s(\emptyset, \emptyset, T_3)$  and a state  $s(T_1, T_2, T_3)$  for each triple  $T_1 \supseteq T_2 \subseteq T_3$ ,  $T_i \in \mathcal{S}^f$  satisfying condition  $(b_1)$  of the lemma. There is a directed edge labelled  $r \in F(T_3)$  from  $s(T_1, T_2, T_3)$  to  $s(T_3, T_4, T_5)$  whenever

- (1)  $r = mrep(\bar{r}A_{T_4}, A_{T_3})$  and
- (2)  $T_4 \subseteq R \subset T_3 \implies A_R \cap \bar{r}A_{T_2} = \emptyset$ .

All states are accept states. To see that the language recognized by this FSA is  $\mathcal{L}$ , we note that a directed path

$$s(\emptyset, \emptyset, T_1) \xrightarrow{r_1} s(T_1, T_2, T_3) \xrightarrow{r_2} s(T_3, T_4, T_5) \xrightarrow{r_3} \dots$$

gives rise to an alternating labelled edge path

$$(1, \emptyset, \emptyset), (r_1, T_1, T_1), (r_1, T_2, T_2), (r_1 r_2, T_3, T_3), \dots$$

satisfying the conditions of the lemma and *vice versa*.  $\square$

The normal forms defined by Altobelli in [1] give rise to asynchronously automatic structures for these groups. It seems likely that the techniques used there could be adapted to our situation to prove that the language  $\mathcal{L}$  is also an asynchronously automatic structure.

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