FINITE $K(\pi, 1)$'S FOR ARTIN GROUPS

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To Bill Browder for his sixtieth birthday

INTRODUCTION

It is not so widely known that there is a beautiful and simple description of a certain finite CW-complex Z which is a $K(\pi, 1)$ for the braid group on $n+1$ strands. The complex Z is obtained by identifying certain faces of an n-dimensional convex polytope called a "permutohedron". The resulting CW complex has exactly one k-cell for each k-element subset of $\{1, \ldots, n\}$. We are not sure who first discovered this complex, but we first heard it described by C. Squirer in the mid 1980's and later by K. Tatsuoka [T]. It has also been known to J. Milgram for some time (a closely related construction appears already in [Mi]). Other references include [FSV], [P], and [S1]. The purpose of this paper is to give the details of the construction of this complex and its generalizations to other Artin groups.

It is a well-known fact that an Eilenberg-MacLane space for the pure braid group on $n+1$ strands is the set of points in $\mathbb{C}^{n+1}$ with all coordinates distinct. This space is the complement of an arrangement of hyperplanes in $\mathbb{C}^{n+1}$ associated to the action of the symmetric group on $n+1$ letters.

In [S], Salvetti showed how any hyperplane complement, obtained by complexifying a real arrangement, was homotopy equivalent to a certain cell complex. In the case at hand, this complex is a union of n-dimensional permutahedra. The symmetric group acts naturally and freely on it. Taking the quotient by the symmetric group, we obtain the complex Z.

There are two ingredients in the above program. The first is the fact that the hyperplane complement in $\mathbb{C}^{n+1}$ is a $K(\pi, 1)$. The second is Salvetti's identification of the hyperplane complement with a certain cell complex.

Suppose that W is a Coxeter group with fundamental generating set S. In the same way as the braid groups on $n+1$ strands is associated to the symmetric group on $n+1$ letters, one can associate to (W, S) an "Artin group" (or "generalized braid group") A_W. We would like to carry out a similar program to obtain a nice $K(\pi, 1)$-complex for A_W.

As in [V] one can always find a representation of W as a linear reflection group on a real vector space V so that W acts properly on a certain nonempty, W-stable,

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convex, open set I (the interior of the "Tits cone"). When W is finite, $I = V$. Inside $V \otimes \mathbb{C}$ one has the domain $V + iI$. Consider the hyperplane complement

$$M = [(V + iI) - \cup \text{ reflection hyperplanes}]/W\,.$$

As for the first ingredient in the program, it is known that $\pi_1(M) = A_W$ (see [CD1; Corollary 3.2.4] or [L]). In the case where W is finite, Deligne proved in [D] that M is a $K(A_W, 1)$. In [CD1] the authors proved that the same result holds for many infinite Coxeter groups. (A precise statement can be found in §1.4, below.) Thus, this paper is a continuation of [CD1].

As explained in §1 and §2, the second ingredient (Salvetti's complex) works for all W. Specifically, we show in Theorem 1.4.1 and Corollary 2.2.3, below, that M is homotopy equivalent to a certain finite CW-complex Z_W. The complex Z_W has one cell of dimension k for each k-element subset T of S such that the subgroup generated by T is finite. From this it is easy to compute the cohomological dimension of A_W (Corollary 1.4.2) and its Euler characteristic (Corollary 2.2.5) at least in the cases where M is known to be a $K(\pi, 1)$-space.

In §3, we consider the case where W is "right-angled", that is, the product of any two generators has order either 2 or ∞. The first ingredient holds in this case. In Theorem 3.1.1, we show that Z_W with its natural piecewise Euclidean metric is nonpositively curved. Thus, in the right-angled case, A_W is a "semihyperbolic group" in the sense of [AB]. (This was previously proved by Hermiller and Meier in [HM] using combinatorial methods.) In this case, it is an easy matter to calculate the cohomology ring of A_W and we do so in Theorem 3.2.4.

After completing this paper we learned of Salvetti's recent paper [S2], essentially on the same topic as this paper, at least in the case where W is finite.

1. Basic Definitions and Constructions

1.1. Coxeter groups and Artin groups.

A *Coxeter matrix* on a finite set S is an S by S symmetric matrix $(m_{ss'})$ such that each entry is a positive integer or ∞ and such that $m_{ss'} = 1$ if $s = s'$ and $m_{ss'} \geq 2$ if $s \neq s'$. Associated to $(m_{ss'})$ there is a group W defined by the presentation:

$$W = \langle S \mid (ss')^{m_{ss'}} = 1\rangle\,.$$

The natural map $S \to W$ is an injection ([B,Ch. V, §4.3, Prop. 4]) and henceforth, we identify S with its image in W. The pair (W, S) is a *Coxeter system* and W is a *Coxeter group.* If T is any subset of S then the subgroup W_T, generated by T, is called a *special subgroup.* It is known ([B; ch. IV, §1.8, Théoréme 2(i)]) that (W_T, T) is the Coxeter system associated to the restriction of the Coxeter matrix to T.

For each $s \in S$ introduce a symbol x_s and let $\mathcal{X} = \{x_s\}_{s\in S}$. Associated to a Coxeter matrix $(m_{ss'})$ there is also an *Artin group*, denoted A_W, and defined by the presentation

$$A_W = \langle \mathcal{X} \mid \text{prod}\,(x_s, x_{s'}, m_{ss'}) = \text{prod}\,(x_{s'}, x_s; m_{ss'})\rangle$$

where prod $(x,y;m)$ denotes the word $xyx\dots$ of length m. We say that A_W is of *finite type* if W is a finite group.

If W is the symmetric group on n letters, then A_W is the braid groups on n strands. Thus, Artin groups are sometimes called "generalized braid groups".

1.2. Posets and simplicial complexes associated to a Coxeter system. Let (W,S) be a Coxeter system. Associated to (W,S) there are the following three posets:

$$\begin{aligned}&\mathcal{S}^f = \{T \subseteq S \mid W_T \text{ is finite}\}\\ &W\mathcal{S}^f = \{wW_T \mid w \in W, T \in \mathcal{S}^f\}\,,\\ &W \times \mathcal{S}^f\,.\end{aligned}$$

The sets $\mathcal{S}^f$ and $W\mathcal{S}^f$ are partially ordered by inclusion. The partial order on $W\mathcal{S}^f$ is given explicitly as follows. If wW_T and $w'W_{T'}$ are elements of $W\mathcal{S}^f$, then $wW_T < w'W_{T'}$ if and only if the following two conditions hold:

(1) $T < T'$,
(2) $w^{-1}w' \in W_{T'}$.

The partial ordering on $W \times \mathcal{S}^f$ is defined as follows. If (w,T) and (w',T') are elements of $W \times \mathcal{S}^f$, then $(w,T) < (w',T')$ if and only if conditions (1), (2) and the following hold:

(3) for all $t \in T$, $\ell(w^{-1}w') < \ell(tw^{-1}w')$.

(Here ℓ denotes word length with respect to the generating set S.) The geometric meaning of (3) will be given in §1. The natural map $W \times \mathcal{S}^f \to W\mathcal{S}^f$ defined by $(w,T) \to wW_T$ is obviously order-preserving.

The posets $\mathcal{S}^f$ and $W\mathcal{S}^f$ appear in [CD1]. The poset $W \times \mathcal{S}^f$ is one of the main objects of study in this paper.

For the moment, suppose that $\mathcal{P}$ is an arbitrary poset. Given $p \in \mathcal{P}$, define a sub-poset, $\mathcal{P}_{\geq p} = \{x \in \mathcal{P} \mid x \geq p\}$. The sub-posets $\mathcal{P}_{\leq p}$, $\mathcal{P}_{>p}$ and $\mathcal{P}_{<p}$ are defined similarly. The poset $\mathcal{P}^{op}$, called the *dual* of $\mathcal{P}$, is equal to $\mathcal{P}$ as a set but with the order relations reversed. The *derived complex* of $\mathcal{P}$, denoted by $\mathcal{P}'$, is the set of finite chains in $\mathcal{P}$, partially ordered by inclusion. It is an abstract simplicial complex.

The geometric realizations of $(\mathcal{S}^f)'$, $(W\mathcal{S}^f)'$, and $(W \times \mathcal{S}^f)'$ are denoted, respectively, by

$$\begin{aligned}K_W &= |(\mathcal{S}^f)'|\\ \Sigma_W &= |(W\mathcal{S}^f)'|\\ \widetilde{\Sigma}_W &= |(W \times \mathcal{S}^f)'|\end{aligned}$$

When there is no ambiguity, we shall omit the subscript "W" from our notation. Following [CD1] we call Σ the *modified Coxeter complex* of W. Here $\widetilde{\Sigma}$ will be called the *Salvetti complex* of W. We note that W acts naturally and simplicially on Σ and on $\widetilde{\Sigma}$.

There is a projection map $\pi\colon W\mathcal{S}^f \to \mathcal{S}^f$ defined by $wW_T \to T$ and an embedding $i\colon \mathcal{S}^f \to W\mathcal{S}^f$ defined by $T \to W_T$. These induce simplicial maps $\Sigma \to K$ and

$K \to \Sigma$ which we again denote by π and i. We can identify K with its image $i(K)$. The map π is then a retraction; moreover, it induces a simplicial isomorphism

$$\Sigma/W \cong K\,.$$

The group W acts freely on $\widetilde{\Sigma}$. The orbit space will be denoted by Z_W (or simply by Z), i.e.,

$$\widetilde{\Sigma}/W = Z_W\,.$$

It is our primary object of study.

1.3. Sectors.

Let R denote the set of conjugates of S in W. An element of R is called a reflection. Given $r \in R$, its fixed points set on Σ is denoted by Σ^r and called a *wall* of Σ. The space $\Sigma - \Sigma^r$ has two connected components, which are interchanged by r. Each such component is called a *half-space*. The half-space containing the interior of K (where K is regarded as a subcomplex of Σ) is called *positive* and denoted H^r_+.

Let T be a subset of S. Put

$$R_T = R \cap W_T\,.$$

The components of

$$\Sigma - \bigcup_{r \in R_T} \Sigma^r$$

are permuted freely and transitively by W_T. Each such component is called a W_T*-sector* of Σ. Such a sector is an intersection of half-spaces. The W_T-sector containing the interior of K is called *positive*. It is the intersection of the positive half-spaces H^t_+, $t \in T$.

If Λ is a W_T-sector then we shall also want to call its translate $w\Lambda$ by an element w of W a "sector". We note that $w\Lambda$ is a component of

$$\Sigma - \bigcup_{r \in wR_Tw^{-1}} \Sigma^r\,.$$

We shall call it a wW_Tw^{-1}*-sector*.

1.4. The spaces Q_W and Y_W.

Put

$$Y_W = \Sigma \times \Sigma - \bigcup_{r \in R} \Sigma^r \times \Sigma^r$$

and

$$Q_W = Y_W/W\,.$$

As usual, we shall often omit the subscript W from our notation.

The following facts are proved in [CD1].

(i) Y is W-equivariantly homotopy equivalent to the complex hyperplane complement associated to any representation of W as a real linear reflection group ([CD1, Corollary 2.2.5]).
(ii) $\pi_1(Q) = A_W$ ([CD1, Corollary 3.2.4]).
(iii) $\pi_1(Y) = PA_W$, where PA_W denotes the kernel of the natural map $A_W \to W$. (PA_W is the "pure Artin group".)

The main conjecture of [CD1], then becomes the following.

Main Conjecture. *Q_W is an Eilenberg-MacLane space $K(A_W, 1)$.*

The poset $\mathcal{S}^f_{>\phi}$ is an abstract simplicial complex. Let K_0 denote its geometric realization. (Thus, K is the cone on the barycentric subdivision of K_0.)

A simplicial complex is said to be a *flag complex* if any set of vertices which are pairwise joined by edges span a simplex. In [CD1], it is proved that the Main Conjecture holds under either of the following two hypotheses:
(A) K_0 is a flag complex, or
(B) $\dim K_0 \leq 1$.

For example, (A) holds if W is finite.

The main result of this section is the following:

Theorem 1.4.1. *Q_W is homotopy equivalent to Z_W.*

Before giving a proof in the next subsection, let us deduce the corollary below.

Put $n_W = \max\{\operatorname{Card}(T) \mid T \in \mathcal{S}^f\}$. In other words, n_W is the common dimension of the simplicial complexes, K, Σ, $\widetilde{\Sigma}$, and Z.

Corollary 1.4.2. *Suppose the Main Conjecture holds for (W, S). Set $n = n_W$. Then the following statements are true.*
(i) A_W has a finite, n-dimensional $K(\pi,1)$-space, namely Z_W.
(ii) A_W is of type FP.
(iii) The cohomological dimension of A_W is n.

Proof. If the Main Conjecture holds, (i) follows immediately from Theorem 1.4.1 and (ii) follows immediately from (i).

Also by (i) the cohomological dimension of A_W is $\leq n$. Suppose $T \in \mathcal{S}^f$ has n elements. By [Br], $H^n(PA_{W_T}; \mathbb{Z}) \neq 0$; hence, the cohomological dimension of A_W is $\geq n$ so (iii) holds. □

1.5. An open cover of Y.

The vertices of K are naturally indexed by the elements of $\mathcal{S}^f$. For each $T \in \mathcal{S}^f$ let v_T denote the corresponding vertex of K. Similarly, the vertices of Σ are naturally indexed by $W\mathcal{S}^f$. For each $wW_T \in W\mathcal{S}^f$, let wv_T denote the corresponding vertex of Σ. Let $\operatorname{Star}(wv_T)$ denote the open star of wv_T in Σ. Consider the open cover of an arbitrary simplicial complex X by open stars of vertices. Of course, the nerve of this open cover is just X. Applying this fact in the case $X = \Sigma$ gives the following lemma.

Lemma 1.5.1. *Suppose $w_i W_{T_i}$, $0 \le i \le k$, are distinct elements in $W\mathcal{S}^f$. Then*

$$Star\,(w_0 v_{T_0}) \cap \cdots \cap Star\,(w_k v_{T_k}) \neq \emptyset$$

if and only if $\{w_0 W_{T_0}, \ldots, w_k W_{T_k}\}$ is a chain in $W\mathcal{S}^f$.

For each $(w, T) \in W \times \mathcal{S}^f$, let $\mathrm{Sec}\,(w, T)$ denotes the open wW_Tw^{-1}-sector of Σ which contains the interior of wK. Define a subset $U(w, T)$ of $\Sigma \times \Sigma$ by

$$U(w,T) = \mathrm{Sec}\,(w,T) \times \mathrm{Star}\,(wv_T)\,.$$

Clearly, $\Sigma^r \cap \mathrm{Star}\,(wv_T) \neq \emptyset$ if and only if $r \in wR_Tw^{-1}$. Also, for all $r \in wR_Tw^{-1}$, we have $\mathrm{Sec}\,(w,T) \cap \Sigma^r = \emptyset$. Hence, $U(w,T) \cap (\Sigma^r \times \Sigma^r) = \emptyset$ for all r in R. That is to say, $U(w,T)$ is an open subset of Y. Moreover, $\mathcal{U} = \{U(w,T)\}_{(w,T)\in W\times\mathcal{S}^f}$ covers Y.

Lemma 1.5.2.

(i) If $U(w,T) = U(w',T')$, then $(w,T) = (w',T')$.

(ii) Suppose (w_i, T_i), $0 \le i \le k$, are distinct elements in $W \times \mathcal{S}^f$. Then $U(w_0,T_0) \cap \cdots \cap U(w_k,T_k) \neq \emptyset$ if and only if $\{(w_0,T_0), \ldots (w_k,T_k)\}$ is a chain in $W \times \mathcal{S}^f$.

Proof. (i) Suppose $U(w,T) = U(w',T')$. Then $\mathrm{Star}\,(wv_T) = \mathrm{Star}\,(w'v_{T'})$ and hence, $wW_T = w'W_{T'}$. It follows that $T = T'$ and that $w^{-1}w' \in W_T$. The condition that $\mathrm{Sec}\,(w,T) = \mathrm{Sec}\,(w',T)$ is equivalent to $\mathrm{Sec}\,(1,T) = \mathrm{Sec}\,(w^{-1}w',T)$. Thus, $w^{-1}w'$ lies in W_T and the interior of $w^{-1}w'K$ lies in the positive W_T-sector. Since W_T acts freely on the set of W_T-sectors, this forces, $w^{-1}w' = 1$, i.e., $w' = w$.

(ii) Suppose (w,T) and (w',T') are such that $wW_T < w'W_{T'}$. We claim that the following statements are then equivalent:

(a) $(w,T) < (w',T')$,
(b) for all $t \in T$, $\ell(w^{-1}w') < \ell(tw^{-1}w')$,
(c) $\mathrm{Sec}\,(w,T) \supset \mathrm{Sec}\,(w',T')$,
(d) $\mathrm{Sec}\,(w,T) \cap \mathrm{Sec}\,(w',T') \neq \emptyset$.

Condition (b) is just condition (3) of §1.2. Thus, (a) and (b) are equivalent. Given $r \in R$ and $u \in W$, the condition that $\ell(u) < \ell(ru)$ means that the interior of uK lies in the positive half-space for r. Thus, (b) means that the interior of $w^{-1}w'K$ lies in the positive W_T-sector. By hypothesis, $T < T'$ so every $W_{T'}$-sector is contained some W_T-sector. Since $w^{-1}w' \in W_{T'}$, we see that (b) is equivalent to the condition that $\mathrm{Sec}\,(1,T) \supset \mathrm{Sec}\,(w^{-1}w',T')$ which is equivalent to (c). Obviously (c)⇒(d). On the other hand, since we are assuming $T < T'$ and $w^{-1}w' \in W_{T'}$, we see that the intersection $\mathrm{Sec}\,(1,T) \cap \mathrm{Sec}\,(w^{-1}w',T')$ can be nonempty only if $\mathrm{Sec}\,(w^{-1}w',T') \subset \mathrm{Sec}\,(1,T)$. Thus, (d)⇒(c). Statement (ii) of the lemma now follows from the previous lemma together with the equivalence of (a) and (d).

Proof of Theorem 1.4.1. Consider the open cover $\mathcal{U} = \{U(w,T)\}$, of Y, indexed by the elements of $W \times \mathcal{S}^f$. Let $\sigma = \{(w_0,T_0), \ldots, (w_k,T_k)\}$ be a set of distinct elements of $W \times \mathcal{S}^f$, and put $U_\sigma = U(w_0,T_0) \cap \ldots U(w_k,T_k)$. By the previous lemma, U_σ is nonempty if and only if σ is a simplex in the derived complex $(W \times \mathcal{S}^f)'$.

Moreover, if this is the case, then after renumbering we may assume that $(w_0, T_0) < \cdots < (w_k, T_k)$. It follows from the proof of the previous lemma (the equivalence of (c) and (d)) that

$$U_\sigma = \mathrm{Sec}\,(w_k, T_k) \times \mathrm{Star}\,(p(\sigma))$$

where $p\colon \widetilde{\Sigma} \to \Sigma$ is the natural projection. It is shown in Lemma 2.2.6 of [CD1] that for $T \in \mathcal{S}^f$, each W_T-sector of Σ is homotopy equivalent to K_{W_T}. In particular, U_σ is contractible. Hence, the nerve of $\mathcal{U}$ is $\widetilde{\Sigma}$ and each nonempty intersection is contractible. It follows that the spaces $\widetilde{\Sigma}$ and Y are homotopy equivalent. The group W acts freely on both spaces. We leave it as an exercise for the reader to construct a W-equivariant embedding $\widetilde{\Sigma} \to Y$ such that Y equivariantly deformation retracts onto $\widetilde{\Sigma}$. Taking quotients by W we get the desired result: Z is homotopy equivalent to Q. $\square$

2. A cell structure on Z

2.1. Coxeter cells.

In this subsection only, the Coxeter group W will be required to be finite. Thus, $\mathcal{S}^f$ will be the poset of all subsets of S. We shall describe a certain convex polytope P_W, called a "Coxeter cell", such that the poset of faces of P_W is isomorphic to $W\mathcal{S}^f$. Some of this material is also described in [CD2; §6].

Associated to (W, S) there is an (essentially unique) representation of W on $\mathbb{R}^n$, $n = \mathrm{Card}\,(S)$, as an orthogonal linear reflection group ([B; Ch. V, §4]). Each element of S acts on $\mathbb{R}^n$ as an orthogonal reflection across a hyperplane. A "fundamental chamber" C is a simplicial cone bounded by the hyperplanes corresponding to the elements of S. To each function $x\colon S \to (0,\infty)$ there is a unique point, which we will also denote by x, in the interior of C such that the distance from x to the hyperplane fixed by s is $x(s)$. Explicitly, if u_s is the outward-pointing unit normal vector to the hyperplane fixed by s, then x is the point defined by: $x \cdot u_s = -x(s)$, $s \in S$.

Definition 2.1.1. The *Coxeter cell* P_W associated to (W, S) and x is the convex hull of the W-orbit of x.

Examples 2.1.2.. (i) If $W = \mathbb{Z}/2$, then P_W is an interval $[-x(s), x(s)]$.

(ii) If W is the dihedral group of order $2m$, then P_W is a $2m$-gon.

(iii) If (W, S) is a direct product of two Coxeter systems, $(W, S) = (W_1 \times W_2, S_1 \coprod S_2)$, then P_W is isometric to $P_{W_1} \times P_{W_2}$.

(iv) In particular, if $W \cong (\mathbb{Z}/2)^n$ and x is the constant function, then P_W is an n-cube.

Lemma 2.1.3. *Suppose W is a finite Coxeter group. The poset of faces of the Coxeter cell P_W is isomorphic to $W\mathcal{S}^f$. In particular, for $T \subset S$, the convex hull of the W_T-orbit of x is a face of P_W and this face is isomorphic to P_{W_T}.*

Proof. Let e_s be a vector on the extremal ray of C which is opposite to the face fixed by s. Consider the linear form φ or $\mathbb{R}^n$ defined by $v \to v \cdot e_s$. Put $c = x \cdot e_s$. We claim that c is the maximum value of φ on P_W. Indeed, since C is a Dirichlet fundamental domain for the action of W on $\mathbb{R}^n$ any point of C is at least as close to x as it is to any other point in the orbit of x. In particular, $|wx - e_s|^2 \geq |x - e_s|^2$ for all $w \in W$. But this implies $wx \cdot e_s \leq x \cdot e_s$ for all $w \in W$ and hence, that the maximum value of φ is attained at x. Let $T = S - \{s\}$. The affine hyperplane $\varphi(v) = c$ contains the orbit of x under the subgroup W_T and is spanned by this orbit. It follows that $\varphi(v) = c$ is a supporting hyperplane of P_W and that the convex hull of $W_T x$ is a codimension one face of P_W.. Letting s vary over S we obtain all supporting hyperplanes containing the vertex x. Replacing x by wx and e_s by we_s we obtain in this way a description of all the supporting hyperplanes of P_W. The lemma follows easily. □

Remarks 2.1.4. (i) Associated to any convex polytope there is a dual polytope. The boundary complex of the dual polytope to P_W is called the *Coxeter complex* of W. The Coxeter complex is an $(n-1)$-dimensional simplicial complex. It is combinatorically isomorphic to the triangulation of S^{n-1} whose spherical $(n-1)$-simplices are the intersections $S^{n-1} \cap wC$, $w \in W$.

(ii) Since Σ_W is the geometric realization of $(W\mathcal{S}^f)'$, we see that Σ_W can be identified with the barycentric subdivision of P_W.

2.2. Cellulations of Σ and $\widetilde{\Sigma}$.

We return to the general situation where W can be infinite.

For each $wW_T \in W\mathcal{S}^f$ we have

$$(W\mathcal{S}^f)_{\leq wW_T} \cong W_T \mathcal{S}^f_{\leq T} .$$

Hence, the geometric realization of the derived complex of $(W\mathcal{S}^f)_{\leq wW_T}$ is a subcomplex of Σ_W isomorphic to Σ_{W_T}. By Lemma 2.1.3 and Remark 2.1.4(ii) we can identify this subcomplex with the Coxeter cell P_{W_T}. Thus, Σ_W is naturally cellulated by Coxeter cells. This gives Σ_W the structure of a convex cell complex: the associated poset of cells is $W\mathcal{S}^f$.

Similarly, for each $(w, T) \in W \times \mathcal{S}^f$, we have

$$\begin{aligned}(W \times \mathcal{S}^f)_{\leq (w,T)} &\cong (W_T \times \mathcal{S}^f_{\leq T})_{\leq (1,T)} \\ &\cong W_T \mathcal{S}^f_{\leq T}\end{aligned}$$

where the second isomorphism is basically the observation that the projection $W \times \mathcal{S}^f \to W\mathcal{S}^f$ restricts to an isomorphism

$$(W \times \mathcal{S}^f)_{\leq (w,T)} \cong (W\mathcal{S}^f)_{\leq wW_T} .$$

Hence, $\widetilde{\Sigma}_W$ also has a cellulation by Coxeter cells which projects to the cellulation on Σ_W described in the previous paragraph. We state this as the following lemma.

Lemma 2.2.1. *$\widetilde{\Sigma}_W$ is cellulated by Coxeter cells: there is one cell for each element of $W \times \mathcal{S}^f$.*

Remark 2.2.2. This cellulation of $\widetilde{\Sigma}$ does not give it the structure of a convex cell complex in the strictest sense: the intersection of two cells need not be a common face of both, rather it is a union of such faces. For example, if $W = \mathbb{Z}/2$, then $\widetilde{\Sigma}$ is a circle, cellulated into two intervals.

Since the W-action on $\widetilde{\Sigma}_W$ is obviously cellular, we get the following corollary, which can be considered the main result of this paper.

Corollary 2.2.3. *Z_W has the structure of a CW-complex: there is one cell of dimension $Card(T)$ for each $T \in \mathcal{S}^f$.*

Corollary 2.2.4. *The Euler characteristic, $\chi(Z_W)$ is given by the formula:*

$$\begin{aligned}\chi(Z_W) &= \sum_{T\in\mathcal{S}^f} (-1)^{\,\mathrm{Card}\,(T)} \\ &= 1 - \chi(K_0)\,.\end{aligned}$$

(Here K_0 is as in §1.4: it is the geometric realization of the simplicial complex $\mathcal{S}^f_{>\emptyset}$.)

Proof. The first equation is immediate from the previous corollary. To see the second, note that the dimension of the simplex of K_0 corresponding to $T \in \mathcal{S}^f_{>\emptyset}$ is Card $(T) - 1$. Hence

$$\chi(K_0) = -\sum_{T\in\mathcal{S}^f_{>\emptyset}} (-1)^{\,\mathrm{Card}\,(T)}\,. \qquad \square$$

Corollary 2.2.5. *If the Main Conjecture holds for (W, S), then the Euler characteristic of the Artin group A_W is given by the same formula:*

$$\chi(A_W) = 1 - \chi(K_0)\,.$$

Lemma 2.2.6. *Suppose that (W, S) is the direct product of two Coxeter systems: $(W,S) = (W_1 \times W_2, S_1 \coprod S_2)$. Then*

(i) $A_W = A_{W_1} \times A_{W_2}$,
(ii) $\Sigma_W = \Sigma_{W_1} \times \Sigma_{W_2}$,
(iii) $\widetilde{\Sigma}_W = \widetilde{\Sigma}_{W_1} \times \widetilde{\Sigma}_{W_2}$,
(iv) $Z_W = Z_{W_1} \times Z_{W_2}$.

Proof. Clear.

Corollary 2.2.7. *Suppose $W = (\mathbb{Z}/2)^n$. Then*

(i) $A_W \cong \mathbb{Z}^n$,
(ii) Σ_W *($= P_W$) is an n-cube,*
(iii) $\widetilde{\Sigma}_W$ *is an n torus (cellulated by 2^n n-cubes), and*
(iv) Z_W *is an n-torus (formed by identifying opposite faces of a single n-cube in the standard fashion).*

2.3. Links.

We have just explained how the space $\widetilde{\Sigma}$ is cellulated by Coxeter cells. The barycentric subdivision of this cell structure gives $\widetilde{\Sigma}$ its natural simplicial structure discussed in §1.

Let $(w, T) \in W \times \mathcal{S}^f$. Then (w, T) corresponds to a vertex v in the simplicial structure on $\widetilde{\Sigma}$. This vertex is the barycenter of a unique Coxeter cell σ (of dimension $\operatorname{Card}(T)$). Any top-dimensional simplex in the barycentric subdivision of σ has v as its maximal vertex. The link of such a simplex in the simplicial structure on $\widetilde{\Sigma}$ is the geometric realization of the derived complex of $(W \times \mathcal{S}^f)_{>(w,T)}$. This can also be thought of as the barycentric subdivision of the link of σ in $\widetilde{\Sigma}$ which we denote $Lk(\sigma, \widetilde{\Sigma})$. The underlying poset of cells in $Lk(\sigma, \widetilde{\Sigma})$ is $(W \times \mathcal{S}^f)_{>(w,T)}$.

Each Coxeter cell is a "simple" polytope. This means that for each pair (σ, τ) where τ is a Coxeter cell and σ is a face of τ, that $Lk(\sigma, \tau)$ is a simplex. It follows that $Lk(\sigma, \widetilde{\Sigma})$ is a "simplicial cell complex", in the sense that all its cells are simplices. (We use this term even though it is not a convex cell complex in the strict sense of Remark 2.2.2. We reserve the term "simplicial complex" when the strict property of Remark 2.2.2 is satisfied.)

Example 2.3.1. Let v be the 0-cell in $\widetilde{\Sigma}_W$ corresponding to the element $(1, \emptyset)$ in $W \times \mathcal{S}^f$. We compute $Lk(v)$ $(= Lk(v, \widetilde{\Sigma}))$ in some simple cases.

(i) If $W = \mathbb{Z}/2$, then $\widetilde{\Sigma}_W$ is a circle and $Lk(v) = S^0$.

(ii) If $W = (\mathbb{Z}/2)^n$, then $\widetilde{\Sigma}_W$ is a Cartesian product of n circles and $Lk(v)$ is the n-fold join $S^0 * \cdots * S^0$. In other words, $Lk(v)$ is the boundary complex of an n-dimensional octahedron.

(iii) Suppose that W is a dihedral group of order $2m$ and that $S = \{s_1, s_2\}$. Thus, $\widetilde{\Sigma}_W$ is a 2-complex cellulated by $2m$ $2m$-gons. The complex $Lk(v)$ is 1-dimensional. Its 0-cells correspond to the elements of

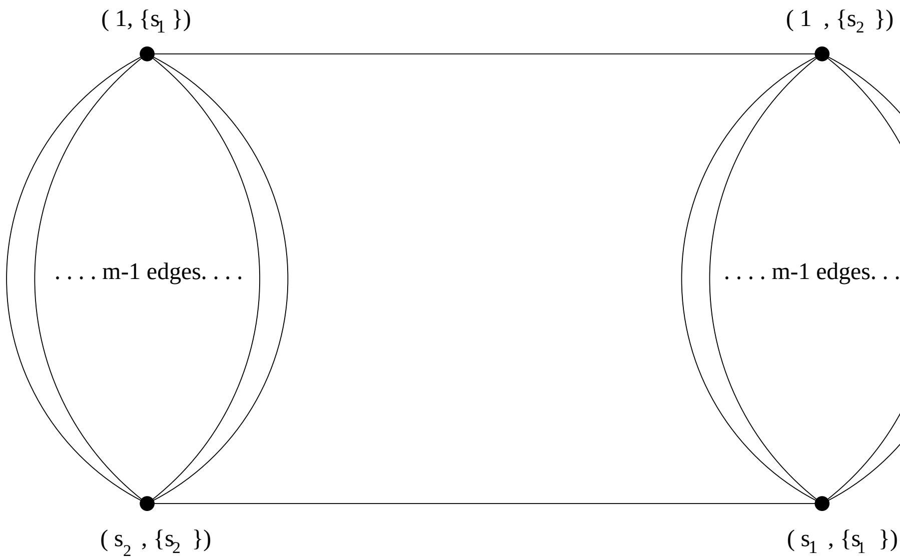
$$(W_{\{s_1\}} \times \{s_1\}) \cup (W_{\{s_2\}} \times \{s_2\}) .$$

Hence, there are four 0-cells. There are $2m$ edges corresponding to the elements of W. In fact, $Lk(v)$ is as pictured below.

Remark 2.3.2. Suppose that W is the dihedral group of order $2m$ and that $m > 2$. Then the link pictured above has cycles consisting of two edges. Thus, there is no way to assign lengths in $(0, \pi)$ to these edges so that this link is large (that is, so it does not contain cycles of length $< 2\pi$). It follows from this that any piecewise Euclidean structure on $\widetilde{\Sigma}_W$ which is compatible with the given cell structure and with the W-action cannot satisfy CAT(0) (see [G] for the definition of the "CAT" inequalities). On the other hand, if $m = 2$, then $\widetilde{\Sigma}$ is a flat 2-torus.

3. The right-angled case

A Coxeter system (W, S) is *right-angled* if given any two distinct elements s and s' of S, the order of ss' is either 2 or ∞ (i.e., if each off-diagonal entry of the Coxeter matrix is either 2 or ∞).



Artin groups associated to right-angled Coxeter groups are known as *graph groups*; they are "graph products" of groups where each vertex groups is infinite cyclic. (See [C], [HM].)

We will say that a simplicial cell complex is a *flag complex* if it is a genuine simplicial complex and if given any set of vertices which are pairwise joined by edges, then this set actually spans a simplex. If W is right-angled, then it follows immediately from the definitions that the geometric realization K_0 of $\mathcal{S}^f_{>\emptyset}$ is a flag complex. Hence, as explained in §1.4, the Main Conjecture holds in this case.

3.1. Nonpositive curvature.

If W is finite and right-angled, then $W \cong (\mathbb{Z}/2)^n$ for some n and the Coxeter cell P_W can be taken to be isometric to a regular Euclidean n-cube.

It then follows from §2.2 that if W is an arbitrary right-angled Coxeter group, then both the Salvetti complex $\widetilde{\Sigma}_W$ and its quotient Z_W naturally have the structure of piecewise Euclidean complexes in which each cell is a regular Euclidean cube.

Such a piecewise Euclidean complex is *nonpositively curved* if, locally, all geodesic triangles satisfy Gromov's CAT(0)-inequalities (cf. [G, p. 119]). Moreover, if each cell in the complex is a regular cube, then nonpositive curvature is equivalent to the condition that the link of each vertex is a flag complex (cf. [G, p. 122] and [M]).

The main result of this subsection is the following.

Theorem 3.1.1. *Suppose W is right-angled. Then Z_W with its natural piecewise Euclidean, cubical structure is nonpositively curved.*

Remark 3.1.2. Any nonpositively curved, piecewise Euclidean, finite complex is an Eilenberg-MacLane space ([G, p. 119]). Hence, in the case where W is right-angled, the above theorem gives an alternate proof of the main result of [CD1]. The fact that Z_W is aspherical is claimed as Theorem 10 in [KR]; the proof given there is incorrect as it is based on a false lemma ([KR, Lemma 9]). However, there is a simple and direct argument for this along the lines indicated in the remark following Lemma 4.3.7 of [CD1].

According to the remarks preceding the theorem, the statement that Z_W is nonpositively curved is equivalent to the statement that the link of its 0-cell is a flag complex. Equivalently, we can consider a 0-cell in $\widetilde{\Sigma}_W$. Thus, the theorem is an immediate consequence of the following.

Lemma 3.1.3. *Suppose W is right-angled and that v is the 0-cell of $\widetilde{\Sigma}_W$ corresponding to $(1, \emptyset) \in W \times \mathcal{S}^f$. Then $Lk(v, \widetilde{\Sigma}_W)$ is a flag complex.*

Proof. We first show that $Lk(v, \widetilde{\Sigma}_W)$ is a simplicial complex. For this, we must show that $(W \times \mathcal{S}^f)_{>(1,\emptyset)}$ is isomorphic to the poset of simplices in a simplicial complex. Given $(w, T) \in (W \times \mathcal{S}^f)_{>(1,\emptyset)}$, let $(W \times \mathcal{S}^f)_{((1,\emptyset),(w,T)]}$ denote the "half-open interval" between $(1, \emptyset)$ and (w, T), i.e., it is

$$\{(w', T') \in W \times \mathcal{S}^f \mid (1, \emptyset) < (w', T') \leq (w, T)\}.$$

We must show that $(W \times \mathcal{S}^f)_{((1,\emptyset),(w,T)]}$ is isomorphic to the poset of all nonempty subsets of T. Since $(w, T) > (1, \emptyset)$, $w \in W_T$. Let $t_1, \dots, t_n$ be the elements of T. Since $W_T \cong (\mathbb{Z}/2)^n$, any $w \in W_T$ can be put in the form

$$(*) \qquad w = t_1^{\varepsilon_1} \dots t_n^{\varepsilon_n}$$

where each ε_i is either 0 or 1. Then $(w', T') < (w, T)$ if and only if $T' < T$ and the expression for w' is obtained from $(*)$ by deleting those t_i which lie in $T - T'$. In other words, given (w, T) and T' with $(1, \emptyset) < (w, T)$, and $T' \leq T$, there is a unique element w' such that $(1, \emptyset) < (w', T') \leq (w, T)$. Thus, $(W \times \mathcal{S}^f)_{>(1,\emptyset)}$ is an abstract simplicial complex as claimed.

The vertices of $Lk(v, \widetilde{\Sigma}_W)$ correspond to those elements (w, T) in $(W \times \mathcal{S}^f)_{>(1,\emptyset)}$ such that T is a singleton. Hence, a vertex corresponds to an element $(w, \{t\})$ where $w = t$ or $w = 1$. Suppose $\{(w_0, \{t_0\}), \dots, (w_k, \{t_k\})\}$ corresponds to a set of distinct vertices which are pairwise joined by edges. Put $T = \{t_0, \dots, t_k\}$. The condition that $(w_i, \{t_i\})$ is joined by an edge to $(w_j, \{t_j\})$ means that $t_i t_j$ has order 2 and hence, that $W_T = (\mathbb{Z}/2)^{k+1}$. Thus, $T \in \mathcal{S}^f$. Let $w = w_1 w_2 \dots w_k \in W_T$. By the discussion above, (w, T) is a k-simplex of $(W \times \mathcal{S}^f)_{>(1,\emptyset)}$ whose vertices are $(w_0, \{t_0\}), \dots, (w_k, \{t_k\})$. Thus, $Lk(v, \widetilde{\Sigma}_W)$ is a flag complex. □

3.2. Cohomology.

Proposition 3.2.1. *Suppose W is right-angled. The CW structure on Z_W is "perfect" in the sense of Morse theory. That is to say, in the cellular chain complex, $C_*(Z_W)$, all boundary maps are 0.*

Proof. As pointed out in [KR, p. 180] the space Z_W can be identified with a subcomplex of the torus $(S^1)^S$ with its standard cubical cell structure: $Z_W = \{(x_s)_{s \in S} \in (S^1)^S \mid \text{if } s \text{ and } t \text{ do not commute, then either } x_s = 1 \text{ or } x_t = 1\}$. In other words, the i-cell corresponding to T, $T \subset S$ and $\operatorname{Card}(T) = i$, belongs to Z_W if and only if $T \in \mathcal{S}^f$. It follows that the cellular chain complex for Z_W injects into that of the torus. In the cellular chain complex of a torus, all boundary maps are 0; hence, the same is true in Z_W. □

Corollary 3.2.2. *Suppose W is right-angled. Then $H_k(A_W)$ $(= H_k(Z_W))$ is free abelian. Its rank is the number of elements T in $\mathcal{S}^f$ such that $Card(T) = k$.*

Remark 3.2.3. The homology of Z_W was calculated by Kim and Roush in [KR, Cor. 11].

Still supposing W is right-angled, we turn now to the calculation of the ring structure of $H^*(A_W)$. The k-cells of Z_W are in one-to-one correspondence with those subsets T of S such that $T \in \mathcal{S}^f$ and $\mathrm{Card}\,(T) = k$. Order the elements of S, $s_1, \ldots, s_n$. Let e_i denote the 1-cell corresponding to $\{s_i\}$. Choose an orientation for e_i. If $T = \{s_{i_1}, \ldots, s_{i_k}\}$, with $i_1 < \cdots < i_k$, is an element of $\mathcal{S}^f$, then the oriented k-cell e_T corresponding to T is the Cartesian product: $e_T = e_{i_1} \times \cdots \times e_{i_k}$. The group of cellular 1-chains $C_1(Z_W)$ is the free abelian group on $\{e_1, \ldots, e_n\}$. Let $z_1, \ldots, z_n$ be the dual basis for $C^1(Z_W)$, the 1-cochains. By Proposition 3.2.1 all boundary maps and coboundary maps are zero; hence we can safely blur the distinctions between cochains, cocycles and cohomology classes. If $T = \{s_{i_1}, \ldots, s_{i_k}\}$, then put

$$z_T = z_{i_1} \cup \cdots \cup z_{i_k}\,.$$

Then $z_T(e_T) = 1$ and $z_T(e_{T'}) = 0$ if $T' \neq T$. Thus, $\{e_T\}$ and $\{z_T\}$, $T \in \mathcal{S}^f$, $\mathrm{Card}\,(T) = k$, are dual bases for $C_k(Z_W)$ and $C^k(Z_W)$. From the above remarks we can easily deduce the following theorem.

Theorem 3.2.4. *Suppose (W, S) is right-angled. Let $n = Card(S)$ and let $\Lambda[y_1, \ldots, y_n]$ be the exterior algebra (over $\mathbb{Z}$) on indeterminates $y_1, \ldots, y_n$. Let I be the ideal generated by all products $y_i y_j$ such that $s_i s_j$ has infinite order in W. Then the map $y_i \to z_i$ defines an isomorphism of graded rings $\varphi\colon \Lambda[y_1, \ldots, y_n]/I \to H^*(Z_W)$. Thus,*

$$H^*(A_W) \cong \Lambda[y_1, \ldots, y_n]/I\,.$$

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