GEODESIC AUTOMATION AND GROWTH FUNCTIONS FOR ARTIN GROUPS OF FINITE TYPE

Ruth Charney

Introduction

The theory of automatic groups was introduced by Epstein, Cannon, Holt, Levy, Paterson, and Thurston in [E], and further developed by Gersten and Short in [GS1], [GS2], and [GS3]. It concerns the study of infinite groups by means of finite state automata. Automatic groups are, in essence, groups which permit analysis by these finite state machines. Such groups have many nice properties. For example, they are finitely presented, have solvable word problems, and satisfy quadratic isoperimetric inequalities. A slightly stronger notion, biautomatic, gives still more nice properties, in particular, solvable conjugacy problem.

In this paper we study Artin groups of finite type. An **Artin group** is a group with presentation of the form

$$A = \langle s_1, \dots s_n | \text{ prod } (s_i, s_j; m_{ij}) = \text{ prod } (s_j, s_i; m_{ij}), i \neq j \rangle$$

where m_{ij} is an integer ≥ 2 or $m_{ij} = \infty$, and where

$$\text{prod } (s,t;m) = \underbrace{stst s \dots}_{m \text{ terms}}.$$

(If $m_{ij} = \infty$, then there is no relation between s_i and s_j). If we add the additional relations $s_i^2 = 1, i = 1, \dots, n$, then the relations $\text{prod}(s_i, s_j; m_{ij}) = \text{prod}(s_j, s_i; m_{ij})$ can be rewritten in the form $(s_i s_j)^{m_{ij}} = 1$. Thus we obtain a Coxeter group

$$W = \langle s_1, \dots s_n | s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle$$

as a natural quotient of A. We say A is of **finite type** if the associated Coxeter group, W, is finite. A well-known example of a finite type Artin group is the braid group B_n on n strands,

$$B_n = \langle s_1, \dots, s_{n-1} | s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i, |i - j| > 1 \rangle$$

whose associated Coxeter group is the symmetric group Σ_n on n letters.

In [E], Thurston showed that the braid groups are biautomatic. Vaguely, an automatic structure for a group G with respect to a finite generating set S consists of a set L of normal forms for elements of G as words in S (L is called a "language" for G) together with finite state automata designed to recognized and compare words

Partially supported by NSF grants DMS-9208071 and DMS-9100383

in L. A biautomatic structure is a pair of automatic structures for a language L and its inverse, L^{-1}, the language obtained by formally inverting the words in L. (See Section 1 for precise definitions). Thurston's proof of biautomaticity for the braid groups was based on a normal form for a braid as a product of "nonrepeating" braids, that is, braids in which any two strands cross at most once. In fact, an analogous normal form for all finite type Artin groups, based on "minimal" words, had previously been described by Deligne in [D] (long before the introduction of automatic groups), but was apparently unknown to Thurston. Combining Thurston's ideas with Deligne's normal form, the author showed in [C] that all finite type Artin groups are biautomatic.

In this paper we improve on our previous results by considering a slightly different language L for a finite type Artin group which has several advantages over the one used in [C]. This language reflects more closely the structure of the group itself in that it is one-to-one (i.e. normal forms are unique), it is geodesic (i.e. normal forms are words of minimal length), and it is symmetric (i.e. $L = L^{-1}$). Such a language was previously introduced in [E] for the braid groups.

Theorem 0.1. *An Artin group A of finite type has an automatic structure whose language L is symmetric, geodesic, and one-to-one.*

Of particular importance is the fact that the language is geodesic. (It should be noted, however, that this property is obtained at the expense of using a slightly larger generating set, namely the set of all "minimal" words. This set is in one-to-one correspondence with the non-trivial elements of the associated Coxeter group.) This property allows us to compute the growth series of these Artin groups. Recall that the growth series of a group G with respect to a generating set S is the power series

$$gr(t) = \sum_{n=1}^{\infty} a_n t^n$$

where a_n is the number of elements of G of length n (with respect to S). If G has a geodesic automatic structure, then $gr(t)$ is the power series of a rational function and that rational function can be computed explicitly from the automatic structure. In the last section of this paper, we apply these ideas to Artin groups of finite type.

The author would like to thank M. Shapiro for helpful conversations, and thank the Institute for Advanced Study for its hospitality during the writing of this paper.

§1. Definitions.

We begin with definitions of automatic groups and related concepts. For more details the reader is referred to [E].

Definition: A (deterministic) **finite state automaton** (or **FSA**) is a quintuple $\mathcal{F} = (V, S, \mu, Y, v_0)$ where V is a finite set, called the **state set**, S is a finite set, called the **alphabet**, $\mu : V \times S \to V$ is a function, called the **transition function**, Y is a subset of V called the **accept states** and $v_0 \in V$ is called the **start state**.

The information contained in (V, S, μ) can be encoded into a directed, labelled graph Γ with vertices V and with an arrow labelled s from v to $\mu(v, s)$ for each $v \in V, s \in S$. We can, and usually will, specify $\mathcal{F}$ by $\mathcal{F} = (\Gamma, Y, v_0)$. We often wish to emphasize the alphabet S, in which case we call $\mathcal{F}$ a **finite state automaton over** S.

Let S be a finite set and let $F(S)^+$ denote the free monoid (with identity) on S. In other words, elements of $F(S)^+$ are finite sequences, or **strings**, of elements of S. The empty string is the identity element. A **language over** S is a subset of $F(S)^+$.

Definition: Let $\mathcal{F} = (\Gamma, Y, v_0)$ be an FSA over S. Define

$$L_{\mathcal{F}} = \{w \in F(S)^+ \Big| w \text{ labels a directed path in } \Gamma \text{ from } v_0 \text{ to an accept state } v \in Y\}$$

Then $L_{\mathcal{F}}$ is called the **language recognized by** $\mathcal{F}$. A subset $L \subset F(S)^+$ is a **regular language** if there exists an FSA $\mathcal{F}$ with $L = L_{\mathcal{F}}$.

Now let G be a group and suppose S generates G as a monoid; that is, the natural map $\pi : F(S)^+ \to G$ is a surjection. We will say a subset $L \subset F(S)^+$ is a **regular language for** G if L is regular and $\pi : L \to G$ is surjective.

The definition of automatic group in [E] involves the existence of a regular language L for G, together with FSA's which recognize when two words in L represent elements in G which are equal or differ by a single generator. However, in Theorem 2.3.5 of [E], a more geometric characterization of automatic is given, which better suits our purpose. We will take this characterization as definition. For this, we need the Cayley graph of G.

Let S be a finite generating set for a group G. The **Cayley graph**, $\mathcal{G} = \mathcal{G}(G, S)$, for G with respect to S is a directed, labelled graph. It has vertex set G and an edge labelled s from g to gs for each $s \in S, g \in G$. (If s and s^{-1} are both in S, we identify the edge $g \overset{s^{-1}}{\leftarrow} gs$ with the reverse of the edge $g \overset{s}{\rightarrow} gs$). We define a metric on $\mathcal{G}$, called the **word metric**, by identifying each edge with the unit interval, and declaring the distance between two points $x, y \in \mathcal{G}$ to be the length of the shortest path between them. Any path p from x to y of minimal length and parameterized by arc length is called a **geodesic** segment from x to y.

Let $F(S)$ denote the free group on S and $\pi : F(S) \to G$ the natural projection. For any $w \in F(S)$ there is a unique edge path in $\mathcal{G}$ from the identity vertex e to $\pi(w)$ whose edges are labelled by w. Let $a = d(e, \pi(w))$ where d is the word metric on $\mathcal{G}$. Denote by $\widehat{w} : [0, a] \to \mathcal{G}$ this edge path, parameterized by arc length. It is often convenient to extend $\widehat{w}$ over a larger interval, $\widehat{w} : [0, c] \to \mathcal{G}$, for some $c > a$. We do this by setting $\widehat{w}(t) = \widehat{w}(a)$ for all $t \geq a$. (In particular, when comparing two such paths in $\mathcal{G}$, we may assume without loss of generality that they are defined on the same interval).

Definition: Suppose (X, d) is a metric space, $p, q : [0, a] \to X$ are two paths in X, and k is a positive real number. We say that p and q are **k-fellow travellers** if $d(p(t), q(t)) \leq k$ for all t with $0 \leq t \leq a$.

Definition: A group G is **automatic** if there is a set S of monoid generators for G, a regular language $L \subset F(S)^+$ for G, and a constant $k > 0$ satisfying the following property. If $w_1, w_2 \in L$ are such that the paths $\widehat{w}_1$ and $\widehat{w}_2$ end distance at most one apart in $\mathcal{G}(G, S)$, then $\widehat{w}_1, \widehat{w}_2$ are k-fellow travellers.

If, in addition, the same fellow traveller property holds for the language L^{-1} obtained by formally inverting the elements of L, then G is said to be **biautomatic**. A particularly strong, and convenient form of biautomaticity is the case in which the language L is **symmetric**, namely $L = L^{-1}$. In this case we say G is **fully**

automatic. If for every $w \in L$, the path $\widehat{w}$ is a geodesic segment in $\mathcal{G}$, then L is a **geodesic language** and G is said to be **geodesically automatic**. If L consists of *all* of the geodesics in $\mathcal{G}$, then G is **strongly geodesically automatic**. (Note that in this case, L is necessarily symmetric). For example, the "word hyperbolic" groups of Gromov [Gr] are strongly geodesically automatic.

§2. Normal Forms for A.

We first recall some facts and definitions from [C]. (Most of these ideas are due to Garside [Ga] and Deligne [D], but the terminology and notation used here is that of [C]).

Let A be an Artin group of finite type with presentation

$$\begin{aligned} A &= \langle S | R \rangle \\ &= \langle s_1, \dots, s_n | \text{ prod } (s_i, s_j; m_{ij}) = \text{ prod}(s_j, s_i; m_{ij}), i \neq j \rangle \end{aligned}$$

and let

$$W = \langle s_1, \dots, s_n | (s_i s_j)^{m_{ij}} = s_i^2 = 1 \rangle$$

be the associated Coxeter group. Set

$$\begin{aligned} F(S) &= \text{ free group on } S \\ F(S)^+ &= \text{ free monoid on } S \\ A^+ &= F(S)^+/\text{equivalence relation generated by } R \end{aligned}$$

From Deligne [D], we know that A^+ injects into A. In particular, left and right cancellation laws hold in A^+. Note also that the equivalence relation generated by R preserves lengths of words in $F(S)^+$, hence elements of A^+ have a well-defined word length.

Since $s_i = s_i^{-1}$ in W, the natural map $F(S)^+ \to W$ is surjective and clearly factors through A^+. A non-empty word $w \in F(S)^+$ is called **minimal** if it is a minimal length representative for its image in W; or equivalently, if it represents a geodesic in the Cayley graph of $\mathcal{G}(W, S)$. An element a of A^+ is **minimal** if it is an equivalence class of minimal words in $F(S)^+$.

Define a partial ordering on A^+ by

$$a \succ b \text{ if } a = cb \text{ for some } c \in A^+.$$

Then for any $a \in A^+$, the set

$$\{\mu \in A^+ | \mu \text{ is minimal and } a \succ \mu\}$$

has a unique maximal element [C, Lemma 2.2] which we denote by **maxmin(a)**. We can decompose any $a \in A^+$ as a product of minimals as follows. Set $\mu_1 = \max\min(a)$. Then by definition $a \succ \mu_1$ so $a = a_1\mu_2$ for some $a_1 \in A^+$. Let $\mu_2 = \max\min(a_1)$. Then $a_1 = a_2\mu_1$ for some $a_2 \in A^+$, so $a = a_2\mu_2\mu_1$. Continue this process. Since the word length of a_{i+1} is strictly less than that of a_i, this process must end and we obtain a decomposition

$$a = \mu_k \dots \mu_2\mu_1.$$

Let M be the set of minimal elements of A^+. Clearly M generates A^+ as a monoid (in fact, $S \subset M$). The decomposition of $a \in A^+$ into minimals described above gives a preferred expression for a as a word in $F(M)^+$. We call this the **normal form** for a (in [C] it is called the canonical minimal decomposition, or cmd for a).

The following is a restatement of Corollary 2.5 of [C].

Proposition 2.1. *Let $\mu_k \dots \mu_1$ be a word in $F(M)^+$ and let a be its image in A^+. Then $\mu_k \dots \mu_1$ is the normal form for a if and only if $\mu_i = \max\min(\mu_{i+1}\mu_i)$ for $i = 1, i-1$. In other words, $\mu_k \dots \mu_1$ is in normal form if and only if $\mu_{i+1}\mu_i$ is in normal form for each i.*

The natural surjection $A^+ \to W$ gives a bijective correspondence between minimals in A^+ and nonidentity elements of W. Let $\Delta \in A^+$ be the minimal corresponding to the unique longest element of W. By Lemma 2.3 of [C], we have the following properties of Δ.

(i) $\mu \in A^+$ is minimal if and only if $\Delta \succ \mu$.
(ii) $\Delta^2 a = a\Delta^2$ for all $a \in A^+$.
(iii) The involution $a \mapsto \bar{a} = \Delta a \Delta^{-1}$ on A preserves A^+ and takes minimals to minimals.
(iv) Given a minimal $\mu \neq \Delta, \exists$ a minimal μ^* such that $\Delta = \mu^*\mu = \mu\bar{\mu}^* = \bar{\mu}\mu^*$. In particular, $(\mu^*)^* = \bar{\mu}$. (By convention, we also set $\Delta^* = 1$).

It follows from these properties that any element g in A can be written in the form $g = a\Delta^{-m}$ where $a \in A^+$. Then writing a in normal form, we have

$$g = \mu_k \dots \mu_1 \Delta^{-m}.$$

If we require that $\mu_1 \neq \Delta$ (i.e., do the obvious cancellations), then this decomposition is uniquely determined. In [C] we proved that the language consisting of these decompositions gives a biautomatic structure for A. However, this language is neither symmetric, nor geodesic. To construct a language with these properties we will need several technical lemmas.

Lemma 2.2. *Suppose $\mu, \eta \in A^+$ are minimal. Then $\mu \succ \eta$ if and only if $\eta(\bar{\mu}^*)$ is minimal.*

Proof. Assume $\mu \succ \eta$. By definition, this means that $\mu = \alpha\eta$ for some $\alpha \in A^+$. Hence $\Delta = \mu\bar{\mu}^* = \alpha\eta\bar{\mu}^*$, so $\Delta \succ \eta\bar{\mu}^*$.

Conversely, suppose $\alpha = \eta\bar{\mu}^*$ is minimal. Then $\mu\bar{\mu}^* = \Delta = \alpha^*\alpha = \alpha^*\eta\bar{\mu}^*$. Cancelling $\bar{\mu}^*$ gives $\mu = \alpha^*\eta$, so $\mu \succ \eta$.

Definition: Let $a, b \in A^+$. We say a is **orthogonal** to b, denoted $a \perp b$, if the only element $c \in A^+$ satisfying $a \succ c$ and $b \succ c$ is the identity element of A^+.

Lemma 2.3. *Suppose $a, b \in A^+$. Let $\mu = \max\min(a), \eta = \max\min(b)$. Then the following are equivalent:*

(i) $a \perp b$
(ii) $\mu \perp \eta$
(iii) $\mu\bar{\eta}^*$ *is in normal form*
(iv) $\eta\bar{\mu}^*$ *is in normal form*

Proof. (i). $\Leftrightarrow$ (ii). Clearly (i) implies (ii). For the converse, suppose $a = a_1c$ and $b = b_1c$. Let $\rho = \max\min(c)$. Then $a \succ c \succ \rho$, so $\mu = \max\min(a) \succ \rho$, and similarly $\eta \succ \rho$. If $\mu \perp \eta$, then $\rho = 1$ and hence $c = 1$.

(iii) $\Rightarrow$ (ii). Assume $\bar{\eta}^* = \max\min(\mu\bar{\eta}^*)$. Suppose $\mu \succ c$ and $\eta \succ c$. Then $\mu\bar{\eta}^* \succ c\bar{\eta}^*$ and by Lemma 2.2, the latter is minimal. Thus $\bar{\eta}^* = \max\min(\mu\bar{\eta}^*) \succ c\bar{\eta}^*$, and we conclude that $c = 1$.

(ii) $\Rightarrow$ (iii). Suppose $\max\min(\mu\bar{\eta}^*) = \rho \neq \bar{\eta}^*$. Then $\rho = c\bar{\eta}^*$ for some nontrivial $c \in M$. Since $c\bar{\eta}^*$ is minimal, $\eta \succ c$ by Lemma 2.2. On the other hand $\mu\bar{\eta}^* = \alpha\rho = \alpha c\bar{\eta}^*$ for some $\alpha \in A^+$. Cancelling $\bar{\eta}^*$ we see that $\mu \succ c$, so μ is not orthogonal to η.

(ii) $\Leftrightarrow$ (iv). The relation $\perp$ is symmetric, so this follows immediately from the equivalence of (ii) and (iii).

Remark 2.4 Setting $\sigma = \bar{\eta}^*$ in the above lemma, we have $\sigma^* = \eta$. So the lemma states that the following are equivalent: (i) $\mu \perp \sigma^*$, (ii) $\mu\sigma$ is in normal form, (iii) $\sigma^*\bar{\mu}^*$ is in normal form. We also note that the involution $a \mapsto \bar{a}$ preserves the ordering $\succ$, and hence preserves normal forms. Thus the conditions above are also equivalent to (iv) $\bar{\mu}\bar{\sigma}$ is in normal form, and (v) $\bar{\sigma}^*\mu^*$ is in normal form.

Now let a be an element of A^+ with normal form $\mu_k \dots \mu_1 \in F(M)^+$ and consider $a^{-1}\Delta^k \in A$. We claim that $a^{-1}\Delta^k$ lies in the image of A^+ in A. In fact,

$$a^{-1}\Delta^k = \mu_1^{-1} \dots \mu_k^{-1}\Delta^k = \begin{cases} \mu_1^{-1}\Delta\bar{\mu}_2^{-1}\Delta \dots \mu_k^{-1}\Delta & k \text{ odd} \\ \mu_1^{-1}\Delta\bar{\mu}_2^{-1}\Delta \dots \bar{\mu}_k^{-1}\Delta & k \text{ even} \end{cases}$$
$$= \begin{cases} \bar{\mu}_1^*\mu_2^* \dots \bar{\mu}_k^* & k \text{ odd} \\ \bar{\mu}_1^*\mu_2^* \dots \mu_k^* & k \text{ even.} \end{cases}$$

Recall that if $\mu_i = \Delta$, then $\mu_i^* = \bar{\mu}_i^* = 1$, so some of the terms in this product may be trivial. In the normal form for a, all Δ's must appear at the end, so all trivial μ_i^* appear at the beginning of this product. Say $\mu_i = \Delta$ for $i = 1, \dots, m-1$, and $\mu_i \neq \Delta$ for $i = m, \dots, k$. Let $(\mu_1, \dots \mu_k)^*$ denote the word in $F(M)^+$ given by

$$(\mu_k \dots \mu_1)^* = \hat{\mu}_m^* \dots \hat{\mu}_k^* \quad \text{where } \hat{\mu}_i = \begin{cases} \bar{\mu}_i & i \text{ odd} \\ \mu_i & i \text{ even} \end{cases}$$

Lemma 2.5. *If $\mu_k \dots \mu_1$ is the normal form for $a \in A^+$, then $(\mu_k \dots \mu_1)^*$ is the normal form for $a^{-1}\Delta^k$.*

Proof. By Proposition 2.1, we need only show that if $\mu_{i+1}\mu_i$ is in normal form, then $\hat{\mu}_i^*\hat{\mu}_{i+1}$ is in normal form. This follows immediately from Remark 2.4.

Theorem 2.6. *Let $g \in A$. Then there is a unique pair of elements a, b in A^+ such that $g = ab^{-1}$ and $a \perp b$.*

Proof. We have already observed that any g can be written in the form $g = p\Delta^{-n}$ where $p \in A^+$. Let c_0 be any maximal element of the set

$$\{c \in A^+ | p \succ c \text{ and } \Delta^n \succ c\}.$$

(This set is finite since the length of c as a word in $F(S)^+$ is bounded by that of p). Write $p = ac_0$ and $\Delta^n = bc_0$. Then $a \perp b$ and $g = p\Delta^{-n} = ab^{-1}$. This proves existence.

For uniqueness, first consider the special case of $g \in A^+$. If $g = ab^{-1}$, then $gb = a$ so $a \succ b$. This contradicts $a \perp b$ unless $b = 1$ and $g = a$. Similarly the theorem holds for strictly negative g. For the general case $g \in A$, suppose $g = ab^{-1} = cd^{-1}$ with $a \perp b, c \perp d$, and a, b, c, d nontrivial. Write a, b, c, d in normal form:

$$a = \alpha_n \dots \alpha_1 \qquad b = \beta_m \dots \beta_1$$
$$c = \alpha'_{n'} \dots \alpha'_1 \qquad d = \beta'_{m'} \dots \beta'_1.$$

Note that the orthogonality condition implies that there are no Δ factors in these products. Say $m \geq m'$. Then $g\Delta^m \in A^+$ and we have

$$g\Delta^m = ab^{-1}\Delta^m = \alpha_n \dots \alpha_1(\beta_m \dots \beta_1)^*$$
$$g\Delta^m = cd^{-1}\Delta^m = \alpha'_{n'} \dots \alpha'_1(\beta'_{m'} \dots \beta'_1)^*\Delta^{m-m'}.$$

By Lemmas 2.3 and 2.5, the right hand sides of both of these equations are in normal form. By uniqueness of normal forms, they must agree as words in $F(M)^+$. The last term of $(\beta_m \dots \beta_1)^*$ is $\hat{\beta}^*_m$ which cannot be Δ (since this would imply $\beta_m = 1$) so we must have $m = m'$. It then follows that $n = n', \beta_i = \beta'_i$, and $\alpha_i = \alpha'_i$.

The following is an immediate corollary of the theorem. (It also follows directly from Proposition 1.14 of [D]). In fact, this corollary is equivalent to the theorem.

Corollary 2.7. *Let $a, b \in A^+$. The set $\{c | a \succ c$ and $b \succ c\}$ has a unique maximal element.*

Lemma 2.8. *Suppose $\sigma, a, b \in A^+$ with $a \perp b$, and $\sigma \in M$.*

(1) *If $a \succ c$ and $\sigma b \succ c$, then $c = 1$ or $c \in M$.*
(2) *There exist $\mu \in M \cup \{1\}$ such that $a = a_1\mu, \sigma b = b_1\mu$ and $a_1 \perp b_1$.*

Proof. (1) Clearly it suffices to consider the case $\sigma = \Delta$. Suppose $a = a_1c$ and $\Delta b = b_1c$ for some $c \neq 1$. Let $\alpha_k \dots \alpha_1, \beta_j \dots \beta_1$, and $\gamma_m \dots \gamma_1$ be the normal forms for a, b, and c respectively. Then $\alpha_1 \succ \gamma_1$ and $\alpha_1 \perp \beta_1$. It follows that $\gamma_1 \perp \beta_1$, so by Lemma 2.3, $\bar{\beta}_1\gamma^*_1$ is in normal form. On the other hand, we have

$$\bar{\beta}_j \dots \bar{\beta}_1\gamma^*_1\gamma_1 = \bar{b}\Delta = \Delta b = b_1c = b_1\gamma_m \dots \gamma_1.$$

Cancelling γ_1 gives

$$\bar{\beta}_j \dots \bar{\beta}_1\gamma^*_1 = b_1\gamma_m \dots \gamma_2.$$

The lefthand side of this equation is in normal form so, in particular $\gamma^*_1 \succ \gamma_2$, which implies that $\gamma_2\gamma_1$ is minimal (Lemma 2.2). This contradicts the assumption that $\gamma_m \dots \gamma_2\gamma_1$ is in normal form, unless $m = 1$ and $c = \gamma_1 \in M$.

(2) By Corollary 2.7, $\{c | a \succ c, \sigma b \succ c\}$ has a unique maximal element c_0. By part (1) above, $c_0 \in M \cup \{1\}$.

We are now ready to define normal forms in $F(M)$ for all elements of A. For $g \in A$, write $g = ab^{-1}$ with $a, b \in A^+$ and $a \perp b$. Let $\mu_k \dots \mu_1$ and $\eta_j \dots \eta_1$ be the normal forms for a and b respectively. Then the **normal form** for g is $\mu_k \dots \mu_1\eta_1^{-1} \dots \eta_j^{-1} \in F(M)$.

A few basic observations about these normal forms are worth noting. First, normal forms are uniquely determined. This follows from Theorem 2.6 and the uniqueness of normal forms for A^+. Second, normal forms are closed under inverses. That is, if $\mu_k \dots \mu_1 \eta_1^{-1} \dots \eta_j^{-1}$ is the normal form for $g \in A$, then $\eta_j \dots \eta_1 \mu_1^{-1} \dots \mu_k^{-1}$ is the normal form for g^{-1}. Third, a word $\mu_1^{\epsilon_1} \mu_2^{\epsilon_2} \dots \mu_k^{\epsilon_k} \in F(M)$ is a normal form if and only if $\mu_i^{\epsilon_i} \mu_{i+1}^{\epsilon_{i+1}}$ is a normal form for $i = 1, \dots, k-1$. This follows from Proposition 2.1 and Lemma 2.3.

Let $L \subset F(M)$ be the set of words in normal form viewed as a language for A. By the remarks above, L is symmetric (i.e., $L = L^{-1}$) and one-to-one (i.e., the natural map $L \to A$ is bijective). In the next section we show that L is a geodesic language (i.e., normal forms are geodesic in the Cayley graph $\mathcal{G}(A, M)$) and gives rise to an automatic structure on A.

Remark 2.9. According to the definitions in Section 1, L should be regarded as a subset of the monoid $F(M \cup M^{-1})^+$. However, since all normal forms are, in fact, reduced words, the natural map $F(M \cup M^{-1})^+ \to F(M)$ is injective on L, so there is no harm in viewing L as a subset of $F(M)$.

§3. The automatic structure.

To prove Theorem 0.1, it remains to show that L is geodesic, regular, and satisfies a fellow traveller property.

To prove that L is a regular language, we construct an FSA, $\mathcal{F} = (\Gamma, Y, e)$, which recognizes L. As alphabet for $\mathcal{F}$ we take $M \cup M^{-1}$. The graph Γ has vertex set $M \cup M^{-1} \cup \{e, f\}$ where e is the start state and f is a "failure state" (no edges emanate from f to any other state). To avoid confusion, we denote the vertex corresponding to a minimal μ by $v(\mu)$. All states other than f are accept states. The edges of Γ are as follows

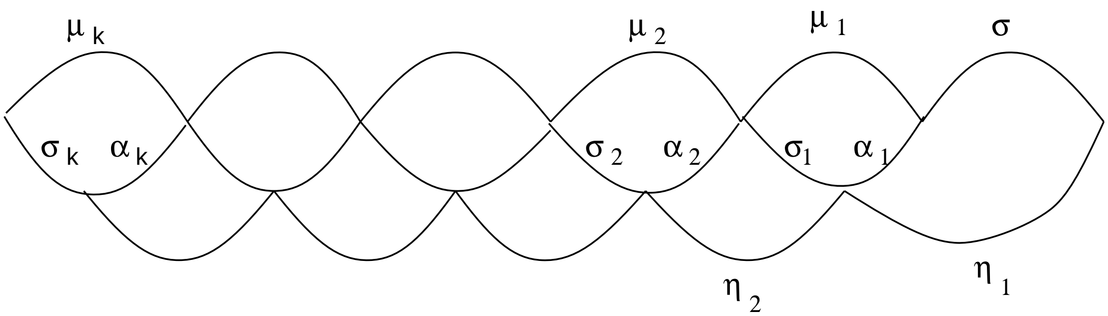
1) For every $\mu^\epsilon \in M \cup M^{-1}$, there is an edge from e to $v(\mu^\epsilon)$ labelled μ^ϵ.
2) For $\mu_1^{\epsilon_1}, \mu_2^{\epsilon_2} \in M \cup M^{-1}$ with $\mu_1^{\epsilon_1} \mu_2^{\epsilon_2}$ in normal form, there is an edge from $v(\mu_1^{\epsilon_1})$ to $v(\mu_2^{\epsilon_2})$ labelled $\mu_2^{\epsilon_2}$.
3) For $\mu_1^{\epsilon_1}, \mu_2^{\epsilon_2} \in M \cup M^{-1}$ with $\mu_1^{\epsilon_1} \mu_2^{\epsilon_1}$ not in normal form, there is an edge from $v(\mu_1^{\epsilon_1})$ to f labelled $\mu_2^{\epsilon_2}$.

It follows immediately from the remarks at the end of the last section, that $\mathcal{F}$ recognizes L.

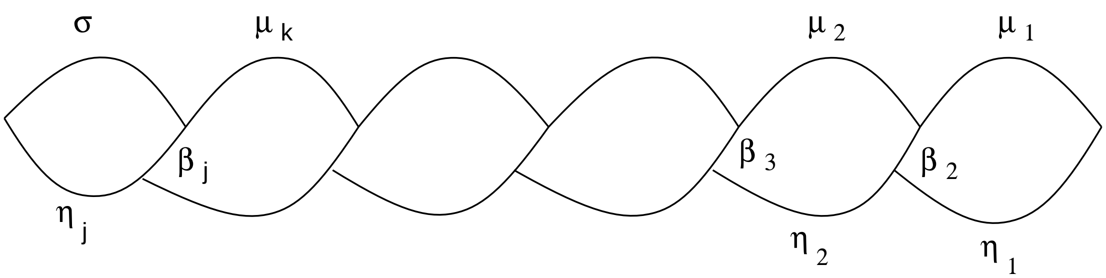
The fellow traveller property for L is an easy consequence of Propositions 3.1 and 3.2 of [C] which we summarize in the lemma below.

Lemma 3.1. *Let $g \in A^+$ and $\sigma \in M$. Let $\mu_k \dots \mu_1$ be the normal form for g.*

(i) We can decompose μ_i as a product $\mu_i = \sigma_i \alpha_i$ such that the normal form for $g\sigma$ is $\eta_j \dots \eta_1$ with $j = k$ or $k+1$ and $\eta_i = \alpha_i \sigma_{i-1}$ (where $\alpha_{k+1} = 1$ and $\sigma_0 = \sigma$).

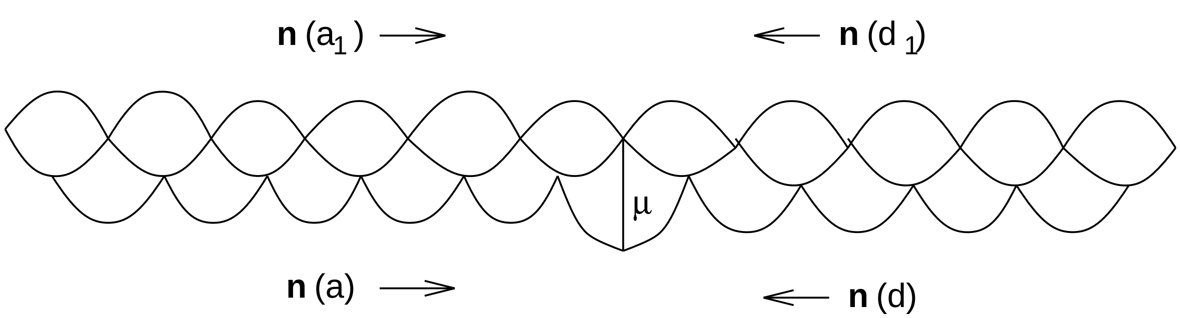


(ii) Let $\rho_j \dots \rho_1$ be the normal form for σg. Then $j = k$ or $k+1$ and there exist $\beta_i \in M \cup \{1\}$ such that $\eta_j \dots \eta_i \beta_i = \sigma \mu_k \dots \mu_i$



Theorem 3.2. *Let $g \in A, \sigma \in M$, and $\epsilon = \pm 1$. Then the normal forms for g and $g\sigma^\epsilon$ are 4-fellow travellers in the Cayley graph $\mathcal{G}(A, M)$.*

Proof. Clearly it suffices to prove the theorem for $\epsilon = -1$. (for $\epsilon = +1$, consider $h = g\sigma$ and $h\sigma^{-1} = g$). For convenience, denote by $\mathbf{n}(g)$ the normal form for g viewed either as a word in $F(M)$ or as a path in $\mathcal{G}(A, M)$. Write $g = ab^{-1}$ with $a \perp b$. Then $\mathbf{n}(g) = \mathbf{n}(a)\mathbf{n}(b)^{-1}$. Let $d = \sigma b$, so $g\sigma^{-1} = ad^{-1}$. It follows from Lemma 3.1(ii) that $\mathbf{n}(d)^{-1}$ and $\mathbf{n}(b)^{-1}$ are 2-fellow travellers, and hence likewise $\mathbf{n}(g)$ and $\mathbf{n}(a)\mathbf{n}(d)^{-1}$. By Lemma 2.8, there exists $\mu \in M \cup \{1\}$ such that $a = a_1\mu, d = d_1\mu$, and $a_1 \perp d_1$, and hence $\mathbf{n}(g\sigma^{-1}) = \mathbf{n}(a_1)\mathbf{n}(d_1)^{-1}$. It follows from Lemma 3.1(i) that $\mathbf{n}(a)\mathbf{n}(d)^{-1}$ and $\mathbf{n}(a_1)\mathbf{n}(d_1)^{-1}$ are 2-fellow travellers as indicated in the figure below.



We conclude that $\mathbf{n}(g) = \mathbf{n}(a)\mathbf{n}(b)^{-1}$ and $\mathbf{n}(g\sigma^{-1}) = \mathbf{n}(a_1)\mathbf{n}(b_1)^{-1}$ are 4-fellow travellers.

To complete the proof of Theorem 0.1, it remains only to show that L is a geodesic language for A (with respect to the generating set $M \cup M^{-1}$).

For $g \in A$, let $n(g)$ denote the length of the normal form of g. That is, if g has normal form $\mu_k \dots \mu_1 \eta_1^{-1} \dots \eta_j^{-1}$, then $n(g) = k + j$. To prove that L is a geodesic language, we must show that if g can be written as a product of m elements of $M \cup M^{-1}$, then $n(g) \le m$. We first consider the special case of $g \in A^+$.

Lemma 3.3. *Suppose $g \in A^+$ can be written as a product $g = \mu_1 \dots \mu_k, \mu_i \in M$. Then $n(g) \le k$.*

Proof. We proceed by induction on k. If $k = 1$, then $g = \mu_1$ is minimal hence it is already in normal form.

Suppose $k > 1$. Let $h = \mu_1 \dots \mu_{k-1} \in A^+$. By induction, $n(h) \le k - 1$, and by Lemma 3.1(i), the normal form for $g = h\mu_k$ has length at most one greater than that of h. That is, $n(g) \le n(h) + 1 \le k$.

Lemma 3.4. *Let $a, b \in A^+$, then*

$$\max\{n(a), n(b)\} \le n(ab) \le n(a) + n(b)$$

Proof. The right hand inequality follows from Lemma 3.3. The left hand inequality follows from Lemma 3.1: right or left multiplication by a minimal can never decrease the length of the normal form.

For the general case, $g \in A$, we need the following easy lemma.

Lemma 3.5. *Let $\mu, \eta \in M$. Then, as elements of $A, \eta^{-1}\mu = \bar{\eta}^*(\bar{\mu}^*)^{-1}$.*

Proof. $\eta^{-1}\mu\Delta = \eta^{-1}\Delta\bar{\mu} = \bar{\eta}^*\bar{\mu}$, hence $\eta^{-1}\mu = \bar{\eta}^*\bar{\mu}\Delta^{-1} = \bar{\eta}^*(\bar{\mu}^*)^{-1}$.

Theorem 3.6. *Let $g \in A$. If $g = \mu_1^{\epsilon_1}\mu_2^{\epsilon_2} \dots \mu_k^{\epsilon_k}$ is any expression for g as a word in $F(M)$, then $n(g) \le k$. In other words, L is a geodesic language.*

Proof. By Lemma 3.5, we can move all of the positive minimals to the left and all of the negative minimals to the right without increasing the total number of minimals in the expression for g. (The total could decrease since if some $\mu_i = \Delta$, then $\mu_i^* = 1$ so we omit it from the expression). This gives us a new expression for g

$$g = \alpha_j \dots \alpha_1 \beta_1^{-1} \dots \beta_\ell^{-1}$$

with $j + \ell \le k$. Let $a = \alpha_j \dots \alpha_1, b = \beta_\ell \dots \beta_1$.

By Corollary 2.7, there is a unique maximal element c_0 of $\{c | a \succ c \text{ and } b \succ c\}$. Say $a = a_1c_0$ and $b = b_1c_0$. Then $a_1 \perp b_1$, so the normal form for g is a product of the normal forms for a_1 and b_1^{-1}. By Lemmas 3.3 and 3.4, we have

$$n(g) = n(a_1) + n(b_1) \le n(a) + n(b) \le j + \ell \le k.$$

This completes the proof of Theorem 0.1

§4. Growth Series.

Let G be a group with a fixed finite generating set S. Then the **growth series** of G (with respect to S) is the power series

$$gr_{G,S}(t) = \sum_{n=0}^{\infty} a_n t^n$$

where a_n is the number of elements of G of length n (with respect to S).

Suppose G has an automatic structure over S whose language L is one-to-one and geodesic. Then the growth series for G is a rational function and it can be explicitly computed from any (deterministic) FSA $\mathcal{F} = (\Gamma, Y, v_0)$ which recognizes L. To see this, note that for such a language, a_n is equal to the number of directed paths in Γ of length n which begin at v_0 and end at an accept state $v \in Y$. We can compute these numbers as follows. Let $V = \{v_0, v_1, \dots, v_m\}$ be the non-failure states of $\mathcal{F}$. Set c_{ij} = number of directed edges in Γ from v_i to v_j, and let C be the $(m+1) \times (m+1)$-matrix $C = (c_{ij})$, indexed from 0 to m. Let $c_{ij}(n)$ denote the ijth entry of C^n. Then it is easy to check that $c_{ij}(n)$ = number of directed paths of length n from v_i to v_j. It follows that

$$a_n = \mathbf{e}C^n\mathbf{w}$$

where $\mathbf{w}$ is the column vector

$$\mathbf{w} = \begin{pmatrix} w_0 \\ \vdots \\ w_m \end{pmatrix} \qquad w_i = \begin{cases} 1 & \text{if } v_i \in Y \\ 0 & \text{if } v_i \notin Y \end{cases}$$

and $\mathbf{e}$ is the row vector $(1, 0, \dots, 0)$. Thus we have

$$\begin{aligned} gr_{G,S}(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} (\mathbf{e}C^n\mathbf{w})t^n \\ &= \mathbf{e}(\sum_{n=0}^{\infty} C^n t^n)\mathbf{w} \\ &= \mathbf{e}(I - tC)^{-1}\mathbf{w}. \end{aligned}$$

Using the adjoint formula for the inverse matrix, we can now compute $gr_G(t)$ explicitly as a rational function.

Remark 4.1. It is often the case that there are no edges that end at the start state. This means that the entire first column of C is zero, so C is of the form

$$C = \begin{bmatrix} 0 & b_1 \dots b_m \\ \vdots & \tilde{C} \\ 0 & \end{bmatrix}$$

$$b_i = \text{ number of edges from } v_0 \text{ to } v_j.$$

In this case, for $n > 0$, $\mathbf{e}\ C^n$ has first entry zero and remaining entries $\mathbf{b}\tilde{C}^n$ where $\mathbf{b} = (b_1, \dots, b_m)$. Let $\tilde{\mathbf{w}} \in \mathbb{Z}^m$ be the column vector obtained by omitting the first entry of $\mathbf{w}$. Then

$$\begin{aligned} gr_{G,S}(t) &= \sum_{n=0}^{\infty} \mathbf{e}C^n t^n \mathbf{w} \\ &= 1 + t \sum_{n=0}^{\infty} \mathbf{b}\tilde{C}^n t^n \tilde{\mathbf{w}} \\ &= 1 + t\mathbf{b}(I - t\tilde{C})^{-1}\tilde{\mathbf{w}} \\ &= 1 + \mathbf{b}(t^{-1}I - \tilde{C})^{-1}\tilde{\mathbf{w}}. \end{aligned}$$

This formulation is more convenient for our examples.

Now let us return to the case of an Artin group A of finite type. In practice, to compute $gr_A(t)$, one would like an efficient FSA, that is, one with the least possible number of states. The FSA described in Section 3 to recognize the language L of normal forms, while easy to describe, is not efficient. In this section we describe an efficient FSA recognizing L and compute some sample growth functions.

Recall that $S = \{s_1, \dots, s_n\}$ denotes the set of generators for A in the standard presentation

$$A = \langle s_1, \dots, s_n | \text{ prod}(s_i, s_j; m_{ij}) = \text{ prod}(s_j, s_i; m_{ij})\rangle.$$

For $\mu \in M$, set

$$\mathcal{S}(\mu) = \{s \in S \,|\, \mu = s\eta \text{ for some } \eta \in A^+\}$$
$$\mathcal{E}(\mu) = \{s \in S \,|\, \mu = \eta s \text{ for some } \eta \in A^+\}.$$

We call $\mathcal{S}(\mu)$ and $\mathcal{E}(\mu)$ the start and end sets of μ. The determination of $\mathcal{S}(\mu)$ and $\mathcal{E}(\mu)$ is really a problem in the Coxeter group W associated to A. Recall that the natural surjection $A \to W$ gives a one-to-one correspondence between M and the non-identity elements of W. We denote an element of M and its image in W by the same notation.

We recall some facts about Coxeter complexes. The reader is referred to [B] for details. The Coxeter group W acts on $\mathbb{R}^n$ as a group of isometries generated by reflections. The hyperplanes fixed by the reflections in W give a simplicial decomposition of the sphere $S^{n-1} \subset \mathbb{R}^n$ which is preserved by the action of W. This simplicial complex, $X(W)$, is called the **Coxeter complex** of W. The top dimensional simplices of $X(W)$ are called **chambers** and the reflection hyperplanes are called **walls**. If we fix a chamber $C_0 \subset X(W)$, there is a natural identification of S with the walls of C_0 (that is, the walls containing a codimension one face of C_0). Under this identification, we have

$$\mathcal{S}(\mu) = \{\text{walls of } C_0 \text{ separating } C_0 \text{ and } \mu C_0\}$$
$$\mathcal{E}(\mu) = \{\text{walls of } C_0 \text{ separating } C_0 \text{ and } \mu^{-1} C_0\}$$

Lemma 4.2. *For any $\eta \in M, \mathcal{S}(\eta)$ and $\mathcal{E}(\eta^*)$ are complementary subsets of S.*

Proof. By the discussion above we can identify

$$\begin{aligned}\mathcal{S}(\eta) &= \{\text{walls of } C_0 \text{ separating } C_0 \text{ and } \eta C_0\}\\ \mathcal{E}(\eta^*) &= \{\text{walls of } C_0 \text{ separating } C_0 \text{ and } (\eta^*)^{-1} C_0\}.\end{aligned}$$

Now C_0 and ΔC_0 are antipodal chambers, that is, they lie on opposite sides of every wall of $X(W)$ (See [B]). Translating by $(\eta^*)^{-1}$, we see that $(\eta^*)^{-1}C_0$ and $(\eta^*)^{-1}\Delta C_0 = \eta C_0$ are also antipodal chambers. The lemma follows.

Lemma 4.3. *Let $\mu, \eta \in M$. Then*
(i) $\mu\eta$ is in normal form if and only if $\mathcal{E}(\mu) \subset \mathcal{S}(\eta)$
(ii) $\mu \perp \eta$ if and only if $\mathcal{E}(\mu) \cap \mathcal{E}(\eta) = \emptyset$

Proof. Part (ii) is obvious. For part (i), note that by Remark 2.4, $\mu\eta$ is in normal form if and only if $\mu \perp \eta^*$. By part (ii) and Lemma 4.2 this holds if and only if $\mathcal{E}(\mu) \subset \mathcal{E}(\eta^*)^c = \mathcal{S}(\eta)$. (Here $\mathcal{E}(\eta^*)^c$ denotes the complement of $\mathcal{E}(\eta^*)$ in S).

It follows from Lemma 4.3 that it is precisely the subsets $\mathcal{E}(\mu)$ (resp. $\mathcal{S}(\mu)$) of S that determine which minimals may follow μ (resp. μ^{-1}) in canonical form. Thus, these subsets must constitute the states of an efficient FSA. Define $\mathcal{F}(A) = (\Gamma, Y, e)$ as follows. As vertices (or states) of Γ take a start state e, a failure state f and two states T^+ and T^- for each nonempty subset $T \subseteq S$. (It is convenient to view e as the empty subset of S). All states except f are accept states. For edges of Γ we take

1) a directed edge labeled μ from T_1^+ to T_2^+ whenever $T_1 \subseteq \mathcal{S}(\mu)$ (including $T_1 = \emptyset = e$) and $T_2 = \mathcal{E}(\mu)$,
2) a directed edge labeled μ^{-1} from T_1^- to T_2^- whenever $\mathcal{E}(\mu) \subseteq T_1$ and $T_2 = \mathcal{S}(\mu)$,
3) a directed edge labeled μ^{-1} from T_1^+ to T_2^- whenever $T_1 \subseteq \mathcal{E}(\mu)^c$ (including $T_1 = \emptyset = e$) and $T_2 = \mathcal{S}(\mu)$,
4) a directed edge labeled μ^ϵ from $T^\pm$ to f whenever no other edge with that label emanates from $T^\pm$.

Note that in general this FSA is considerably smaller than the one constructed in Section 3 in which the states were $M \cup M^{-1} \cup \{e, f\}$. In particular, for the braid group B_n on n strands, our efficient FSA has 2^n states whereas the FSA in Section 3 had $2(n!)$ states.

Example 4.4. Consider the braid group

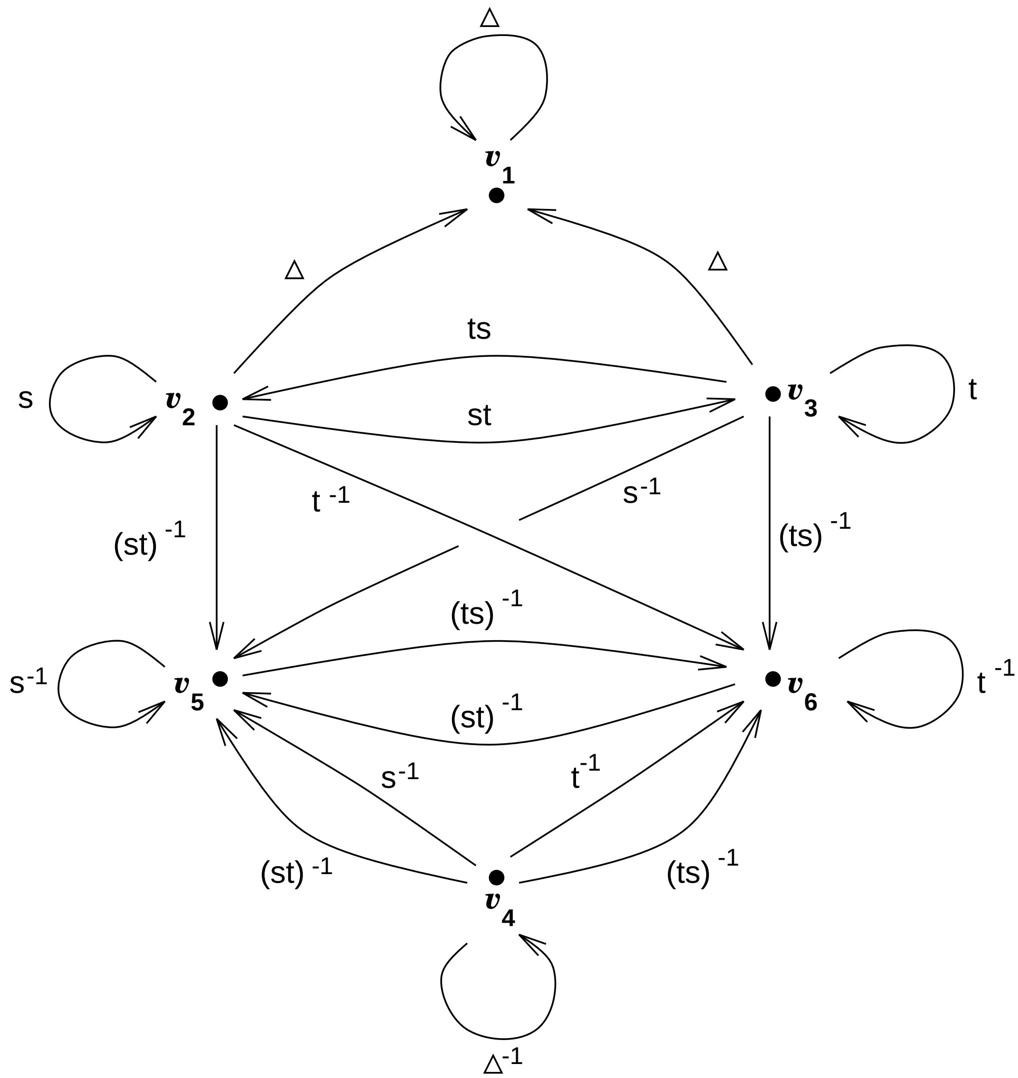
$$B_3 = \langle s, t | sts = tst \rangle.$$

There are five minimal elements in B_3 which are listed below along with their start and end sets

μ	$\mathcal{S}(\mu)$	$\mathcal{E}(\mu)$
s	s	s
t	t	t
st	s	t
ts	t	s
Δ	s, t	s, t

The associated FSA, $\mathcal{F}(B_3)$, is displayed below. To avoid cluttering the diagram, we have omitted the start and failure states, e and f, and labelled the remaining states as follows.

$$v_1 = \{s,t\}^+ \qquad v_4 = \{s,t\}^-$$
$$v_2 = \{s\}^+ \qquad v_5 = \{s\}^-$$
$$v_3 = \{t\}^+ \qquad v_6 = \{t\}^-$$



The matrix $\tilde{C}$ and vector $\mathbf{b}$ used in Remark 4.1 to compute the growth function may be read directly off the diagram above (or directly off the chart of start and end sets).

$$\mathbf{b} = [1 \quad 2 \quad 2 \quad 1 \quad 2 \quad 2]$$

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The states $v_1, \ldots, v_6$ are all accept states so $\tilde{\mathbf{w}}$ has all entries 1. A direct computation now gives

$$\begin{aligned} gr_{B_3,M}(t) &= 1 + \mathbf{b}(t^{-1}I - \tilde{C})^{-1}\tilde{\mathbf{w}} \\ &= 1 + \frac{2t(-5+8t)}{(-1+t)(-1+2t)^2} \\ &= 1 + 10t + 34t^2 + 90t^3 + 218t^4 + 506t^5 + 1146t^6 + 2554t^7 \\ &\quad + 5626t^8 + 12282t^9 + 26618t^{10} + \ldots \end{aligned}$$

Example 4.5. The growth function for any two generator Artin group

$$A_m = \langle s, t \mid \text{prod}(s,t;m) = \text{prod}(t,s;m) \rangle$$

can be similarly determined. The FSA for A_m has the same states as that of $B_3 (= A_3)$. The minimal elements of A_m are precisely the elements of the form $\text{prod}(s,t;k)$ or $\text{prod}(t,s;k)$ for $k \leq m$. The start and end sets of these elements are summarized in the chart below.

μ	$\mathcal{S}(\mu)$	$\mathcal{E}(\mu)$
$prod(s,t;k)$ $k < m, k$ odd	s	s
$prod(s,t;k)$ $k < m, k$ even	s	t
$prod(t,s;k)$ $k < m, k$ odd	t	t
$prod(t,s;k)$ $k < m, k$ even	t	s
$prod(s,t;m)$ $(= prod(t,s;m))$	s,t	s,t

Let $a = \left[\frac{m}{2}\right]$ and $b = \left[\frac{m-1}{2}\right]$ (where $[\quad]$ denotes greatest integer less than or equal to) and set $r = a + b = m - 1$. Then

$$a = \text{ number of odd integers } k \text{ with } 1 \leq k < m,$$

$$b = \text{ number of even integers } k \text{ with } 1 \leq k < m.$$

Ordering the states as in the previous example, we get

$$\mathbf{b} = [1 \quad r \quad r \quad 1 \quad r \quad r]$$

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & a & b & 0 & b & a \\ 1 & b & a & 0 & a & b \\ 0 & 0 & 0 & 1 & r & r \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & b & a \end{bmatrix}$$

This gives a growth function

$$gr_{A_m,M}(t) = 1 + \frac{2t(-1 - 2r + 2rt + r^2t)}{(-1+t)(-1+rt)^2}$$

Example 4.6. The braid group

$$B_4 = \langle r, s, t | rsr = srs, sts = tst, rt = tr \rangle$$

contains 23 minimal elements (one for each nonidentity element of Σ_4). We can determine start and end sets for each of these. Our work here is cut nearly in half by Lemma 4.1 which tells us that $\mathcal{E}(\mu^*) = \mathcal{S}(\mu)^c$ and $\mathcal{S}(\mu^*) = \mathcal{E}(\mu^{**})^c = \overline{\mathcal{E}(\mu)}^c$. Since either μ or μ^* has length (in terms of the generators r, s, t) at most half that of Δ and the length of Δ is 6, it suffices to compute $\mathcal{S}(\mu)$ and $\mathcal{E}(\mu)$ for minimal elements of length at most 3. We list below 13 minimals μ; the "duals", μ^*, of the first 10 constitute the remaining minimals. The bar involution (= conjugation by Δ) is given by $\bar{r} = t, \bar{t} = r, \bar{s} = s$.

μ	$\mathcal{S}(\mu)$	$\mathcal{E}(\mu)$	$\mathcal{S}(\mu^*)$	$\mathcal{E}(\mu^*)$
r	r	r	r, s	s, t
s	s	s	r, t	r, t
t	t	t	s, t	r, s
rs	r	s	r, t	s, t
sr	s	r	r, s	r, t
rt	r, t	r, t	s	s
st	s	t	s, t	r, t
ts	t	s	r, t	r, s
rst	r	t	s, t	s, t
tsr	t	r	r, s	r, s
$(rts)^* = rts$	r, t	s		
$(str)^* = str$	s	r, t		
Δ	r, s, t	r, s, t		

Order the states as follows

$$v_1 = \{r, s, t\}^+ \qquad v_8 = \{r, s, t\}^-$$

$$v_2 = \{r, s\}^+ \qquad v_9 = \{r, s\}^-$$

$$v_3 = \{s, t\}^+ \qquad v_{10} = \{s, t\}^-$$

$$v_4 = \{r, t\}^+ \qquad v_{11} = \{r, t\}^-$$

$$v_5 = \{r\}^+ \qquad v_{12} = \{r\}^-$$

$$v_6 = \{t\}^+ \qquad v_{13} = \{t\}^-$$

$$v_7 = \{s\}^+ \qquad v_{14} = \{s\}^-$$

Then

$$\mathbf{b} = \begin{bmatrix} 1 & 3 & 3 & 5 & 3 & 3 & 5 & 1 & 3 & 3 & 5 & 3 & 3 & 5 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\
1 & 2 & 2 & 3 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 3 \\
1 & 2 & 2 & 3 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 3 \\
1 & 2 & 2 & 3 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 5 & 3 & 3 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2
\end{bmatrix}$$

$$\begin{aligned} gr_{B_4,M}(t) &= 1 + \frac{2t(-23 + 122t - 108t^2 - 54t^3 - 33t^4 + 108t^5 - 36t^6)}{(-1+t)(-1+2t)^2(1-6t+3t^2)^2} \\ &= 1 + 46t + 538t^2 + 4302t^3 + 29978t^4 + 196262t^5 + 1241922t^6 + \\ & 7690870t^7 + 46903378t^8 + 282717726t^9 + 1688256362t^{10} + \dots \end{aligned}$$

References

[B] K.S. Brown, **Buildings**, Springer-Verlag 1989.

[C] R. Charney, Artin groups of finite type are biautomatic, Math. Annalen 292 (1992), 671-683.

[D] P. Deligne, Les immeubles des groupes de tresses généralisés, Inventiones Math. 17 (1972), 273-302.

[E] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson and W. Thurston, **Word Processing in Groups**, Jones and Bartlett, 1992.

[Ga] F.A. Garside, The braid groups and other groups, Oxford Quat. J. Math. 20 (1969), 235-254.

[Gr] M. Gromov, Hyperbolic groups, in **Essays in Group Theory**, ed. by S.M. Gersten, MSRI Publ. 8, Springer-Verlag,1987.

[GS1] S.M. Gersten and H. Short, Small cancellation theory and automatic groups, Inventiones Math. 102 (1990), 305-334.

[GS2] S.M. Gersten and H. Short, Small cancellation theory and automatic groups, Part II, Inventiones Math. 105 (1991), 641-662.

[GS3] S.M. Gersten and H. Short, Rational subgroups of biautomatic groups, Annals of Math. 134 (1991), 125-158.

Ruth Charney
Ohio State University
Department of Mathematics
Columbus, OH 43210
charney@mps.ohio-state.edu