Injectivity of the Positive Monoid for Some Infinite Type Artin Groups

*Ruth Charney**

Abstract. We use CAT(0) geometry to show that for certain classes of infinite type Artin groups A, the associated positive monoid A^+ injects into A.

1991 Mathematics Subject Classification: 20F32, 20F36

Introduction

A Coxeter system (W, S) consists of a finite set $S = \{s_1, \ldots, s_n\}$ and a group W with presentation of the form

$$W = \langle S \mid s_i^2 = 1,\ (s_i s_j)^{m_{i,j}} = 1 \rangle$$

where $m_{i,j} = m_{j,i} \in \{2, 3, \ldots, \infty\}$. The second relator in this presentaion can be rewritten in the form $\Delta_{i,j} = \Delta_{j,i}$ where

$$\Delta_{i,j} = \underbrace{s_i s_j s_i \ldots}_{m_{i,j} \text{ terms}}$$

Associated to the Coxeter system (W, S) is an Artin system (A, S) where A is the group with presentaion

$$A = \langle S \mid \Delta_{i,j} = \Delta_{j,i} \rangle \tag{0.1}$$

In particular, if W is the symmetric group on n letters, then A is the braid group on n strands. We say that A is a *finite type* (resp. *infinite type*) Artin group if W is a finite (resp. infinite) Coxeter group. Note that A itself is always an infinite group (in fact, each generator s_i has infinite order in A). Deligne in [D] and Brieskorn and Saito in [BS], study the finite type Artin groups. They show that, in the finite type case, there are nice normal forms for elements of A which give rise to algorthims solving the word and conjugacy problems for these groups. In [C1], the language consisting of these normal forms is shown to be biautomatic.

The positive monoid A^+ of an Artin system (A, S) is the monoid associated to the presentation (0.1). That is, A^+ is the quotient of the free monoid on S, $F^+(S)$,

* Partially supported by NSF grant DMS-9505003

by the equivalence relation generated by $w\Delta_{i,j}u = w\Delta_{j,i}u$ where u, v are words in $F^+(S)$. In the finite type case, the natural map $\theta : A^+ \to A$ is injective ([D], [BS]). This fact is a key ingredient in defining normal forms for these groups.

Except for a few special cases, little is known about Artin groups of infinite type. In this paper we prove that $\theta : A^+ \to A$ is injective for certain classes of infinite type Artin systems. The main result involves Artin systems for which any three elements of S generate an infinite subgroup of W (or equivalently, an infinite type subgroup of A). For reasons explained in Section 2, we call these groups two-dimensional Artin groups. These groups can also be described as follows. Form a labelled graph Γ with S as vertex set and with an edge labelled $m_{i,j}$ connecting s_i to s_j whenever $m_{i,j} < \infty$. Then A is two-dimensional if whenever s_i, s_j, s_k are joined by a triangle in Γ, the edge labels satisfy $\frac{1}{m_{i,j}} + \frac{1}{m_{j,k}} + \frac{1}{m_{k,i}} \leq 1$. The injectivity of θ was previously proved by Cho and Pride [CP] under the more restrictive assumption that either $m_{i,j} \geq 3$ for all i, j, or Γ contains no triangles. Our methods also work for another class of Artin groups known as FC groups. The Artin group A is an FC group if the vertices of any complete subgraph of Γ generate a finite subgroup of W. Injectivity of θ for these groups was previously proved by Altobelli in [A]. Altobelli uses amalgamated product decompositions for these groups to obtain a solution to the word problem and to prove the injectivity property. In this paper we take a more geometric approach.

Our proof of injectivity uses the "Deligne complex", $\mathcal{D}_A$, for A. This is a simplicial complex, similar to one used in [D] for finite type A, which is analogous to the Coxeter complex for W. Namely, $\mathcal{D}_A$ is a building-like chamber complex which has one chamber for each element of A. The group A acts on $\mathcal{D}_A$ and the stabilizers of a simplex are finite type sub-Artin groups. We also construct an analogous complex, $\mathcal{D}_A^+$, for A^+. To prove injectivity for θ, it suffices to prove that $\mathcal{D}_A^+$ embeds into $\mathcal{D}_A$. The key idea in the proof is to reduce this global embedding problem to a local problem. To do this, we use the theory of CAT(0) spaces introduced by Gromov in [G]. The CAT(0) condition on a metric space X is a combinatorial analogue of nonpositive curvature, defined by requiring triangles in X to be "thin" relative to triangles in $\mathbb{R}^2$ (see Section 1 for definitions). CAT(0) spaces have nice local to global properties. In particular, any local geodesic in a CAT(0) space is a global geodesic. In [CD1], it is shown that for the two-dimensional and FC Artin groups, there are natural metrics on $\mathcal{D}_A$ which are CAT(0). Thus, to prove that $\mathcal{D}_A^+$ embeds in $\mathcal{D}_A$, one need only show that local geodesics are mapped to local geodesics. Since the local properties of $\mathcal{D}_A^+$ and $\mathcal{D}_A$ depend only on finite type subgroups of A, we are reduced to a more familiar and tractable setting.

In Section 1, we review some facts about CAT(0) spaces. In Sections 2 and 3, we describe the Deligne complex $\mathcal{D}_A$ and introduce the positive Deligne complex $\mathcal{D}_A^+$. Sections 4 and 5 contain the proofs of the main theorems.

1. CAT(0) Spaces

In this section we give a brief review of CAT(0) spaces. For more details, see [Ba] and [Br].

A *piecewise Euclidean complex* is a connected cell complex X made up of convex polyhedral Euclidean cells glued together by isometries along faces. We do not require that the cell complex be locally finite, but we do assume that X has *finite shapes*, that is, that there are only finitely many isometry types of cells in X. A *piecewise geodesic* in X is a path $\gamma : [a, b] \to X$ for which $[a, b]$ can be divided into subintervals $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to $[t_i, t_{i+1}]$ is a geodesic path lying entirely in some cell of X. Let $l(\gamma)$ denote the length of γ. We define the *intrinsic metric* on X as follows.

$$d(x, y) = \inf\{l(\gamma) \mid \gamma \text{ is a piecewise geodesic from } x \text{ to } y\}$$

Under the finite shapes assumption, the intrinsic metric is a complete, geodesic metric; that is, there is a length minimizing path between any two points in X (see [Br]). Such a path is called a *geodesic.*

A *piecewise spherical complex* and its intrinsic metric are defined similarly, using convex polyhedral cells in a sphere $\mathbb{S}^n$ instead of Euclidean cells. If x is a point in a piecewise Euclidean complex X, then the set of unit tangent vectors to X at x is naturally a piecewise spherical complex called the *link* of x in X, or $link(x, X)$. A piecewise geodesic γ in X is called a *local geodesic* if for each point x on γ, the incoming and outgoing unit tangent vectors to γ at x are at distance at least π in $\text{link}(x, X)$.

Let T be a geodesic triangle in X. A *comparison triangle* for T is a Euclidean triangle T' with the same side lengths as T. We say X is a *CAT(0) space* if for any geodesic triangle T in X and any two points x, y on T, the distance from x to y in X is less than or equal to the Euclidean distance between the corresponding points x', y' on the comparison triangle T'. The following two fundamental facts about CAT(0) spaces will be important in the proof of our main theorem. The first follows easily from the definition of a CAT(0) space; the second is a fairly deep theorem which is due, in this setting, to Bridson [Br]. (See also Ballmann [Ba] and Alexander-Bishop [AB].)

Theorem 1.1. *Let X be a CAT(0) space. Then*
(1) *any two points in X are connected by a unique geodesic, and*
(2) *any local geodesic in X is a geodesic.*

2. The Deligne Complex

In this section we describe the Deligne complex $\mathcal{D}_A$ for an Artin group A and a natural piecewise Euclidean structure on $\mathcal{D}_A$.

Let (A, S) be an Artin system associated to the Coxeter system (W, S). For a subset $T \subseteq S$, let A_T (resp. W_T) denote the subgroup of A (resp. W) generated by T. (If $T = \emptyset$, then $A_T = W_T = \{1\}$.) These are known as *special subgroups.* For any such T, (W_T, T) is a Coxeter system and (A_T, T) is the associated Artin system (see [L] or [CD1]). Define

$$\mathcal{S}^f = \{A_T \mid W_T \text{ is finite}\}$$

$$A\mathcal{S}^f = \{aA_T \mid a \in A,\ W_T \text{ is finite}\}$$

These sets are partially ordered by inclusion. The Deligne complex, $\mathcal{D}_A$, is the flag complex associated to $A\mathcal{S}^f$. That is, the vertices of $\mathcal{D}_A$ are the elements of $A\mathcal{S}^f$ and the simplices of $\mathcal{D}_A$ are the totally ordered subsets of $A\mathcal{S}^f$.

A acts by left multiplication on $A\mathcal{S}^f$ and hence also on $\mathcal{D}_A$. A fundamental domain for this action is the flag complex associated to $\mathcal{S}^f$ which we denote by $\mathcal{K}_A$. This gives an alternate description of $\mathcal{D}_A$. Namely, let K_T denote the subcomplex of $\mathcal{K}_A$ spanned by the vertices A_R with $A_T \subseteq A_R$. Then the relative interiors of the K_T partition $\mathcal{K}_A$ (where "relative interior of K_T" means the open star of the vertex A_T in K_T, or equivalently, the points in K_T which do not lie in K_R for any $R \supseteq T$). Then

$$\mathcal{D}_A = A \times \mathcal{K}_A / \sim$$

where $(a_1, x_1) \sim (a_2, x_2)$ if $x_1 = x_2$ and $a_1 A_T = a_2 A_T$ where T is such that x_1 lies in the relative interior of K_T.

We define the *dimension of* A to be the dimension of the simplicial complex $\mathcal{D}_A$. That is,

$$\dim A = \dim \mathcal{D}_A = \max\{|T| \mid W_T \text{ is finite}\}.$$

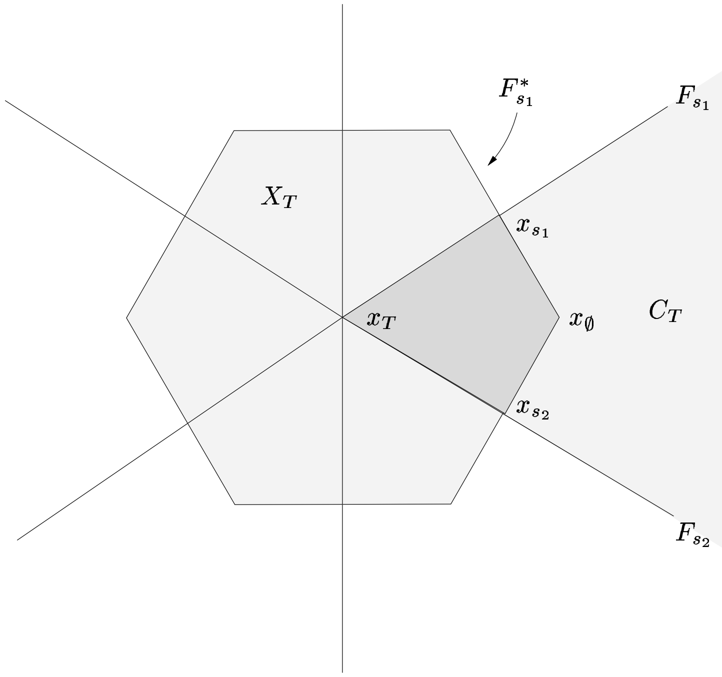
In [CD1], a natural piecewise Euclidean structure on $\mathcal{D}_A$ is defined (based on a similar structure for Coxeter complexes introduced by Moussong in [M]). This geometry is preserved by the action of A, thus it suffices to describe the piecewise Euclidean structure on $\mathcal{K}_A$. For $A_T \in \mathcal{S}^f$, the elements of $\mathcal{S}^f$ less than or equal to A_T (i.e., the special subgroups of (A_T, T)) span a combinatorial cube in $\mathcal{K}_A$ of dimension $|T|$ which we denote by *cube*(T). $\mathcal{K}_A$ is the union of these cubes. We assign a Euclidean metric to cube(T) as follows. There is a standard realization of W_T as an orthogonal reflection group acting on $\mathbb{R}^{|T|}$ such that the generators, T, act as reflections in the codimension 1 faces (or "walls") of a simplicial cone C_T (see eg., [Brn]). Let F_s denote the wall of C_T fixed by a reflection s. Then the faces of C_T are $\{F_R\}_{R \subseteq T}$ where $F_R = \cap_{s \in R} F_s$ and $F_\emptyset = C_T$.

There is a unique point $x_\emptyset$ in the interior of C_T whose distance from every wall of C_T is 1. The *Coxeter cell* X_T is the convex hull in $\mathbb{R}^{|T|}$ of the W_T-orbit of $x_\emptyset$. The faces of X_T are $\{F_R^*\}_{R \subseteq T}$ where

$$F_R^* = \text{the convex hull of the } W_R\text{-orbit of } x_\emptyset.$$

F_R and F_R^* are orthogonal and intersect at a single point x_R.

The intersection of X_T with C_T is combinatorially a cube with vertices $\{x_R\}_{R \subseteq T}$. The Euclidean structure on cube(T) is defined by identifying it with $X_T \cap C_T$ so that the vertex A_R of cube(T) is identified to the vertex x_R of $C_T \cap X_T$.



If $R \subset T$, it follows from the orthogonality of F_R and F^*_R that F^*_R is isometric to the Coxeter cell X_R. In other words, the face of cube(T) spanned by A_R and $A_\emptyset$ is isometric to cube(R). Thus, the metrics on the cubes fit together to give a piecewise Euclidean structure on $\mathcal{K}_A$ and hence on $\mathcal{D}_A$. We call the induced metric the *Moussong metric* and denote it by d_M. In [CD1] it is conjectured that the Moussong metric on $\mathcal{D}_A$ is CAT(0) for any Artin system (A, S). The conjecture is proved in the 2-dimensional case.

Theorem 2.1 [CD1, Proposition 4.4.5]. *If A is a 2-dimensional Artin group, then the Moussong metric on $\mathcal{D}_A$ is CAT(0).*

If A is finite type, then $A\mathcal{S}^f$ has a unique maximal element, namely A_S $(= A)$. Thus $\mathcal{K}_A = \text{cube}(S)$ and $\mathcal{D}_A$ is a cone with cone point the vertex A_S. Let $\mathcal{B}_A$ denote the link of the cone point. Then $\mathcal{B}_A$ is a simplicial complex of dimension $n-1$ (where $n = |S|$) with one top-dimensional simplex for each element of A. The piecewise spherical structure on $\mathcal{B}_A$ is given by identifying each top-dimensional simplex with the link of the origin in the cone C_S (or in other words, with a fundamental chamber of the standard Coxeter complex for (W,S)).

Returning to the case of an infinite type A, we now describe the link of a point in $\mathcal{D}_A$ as an orthogonal join. For details about orthogonal joins of piecewise spherical complexes, see the appendix of [CD2].

Lemma 2.2. *If x lies in the relative interior of K_T, then link$(x, \mathcal{D}_A)$ is isometric to the orthogonal join of link(x, K_T) and $\mathcal{B}_{A_T}$.*

Proof. For $T \subseteq R$, let cube(T, R) denote the face of cube(R) spanned by A_T and A_R. Then cube(T) and cube(T, R) intersect orthogonally at A_T. If x is a point in the relative interior of K_T, and $x \in$ cube(R), then x lies in the face cube(T, R). Moreover, a (small) neighborhood of x in cube(R) can be identified with a neighborhood of (A_T, x) in the orthogonal product cube$(T) \times$ cube(T, R). Taking the union over all such R, we see that a neighborhood of x in $\mathcal{K}_A$ can be identified with a neighborhood of (A_T, x) in the orthogonal product cube$(T) \times K_T$. Thus,

$$\text{link}(x, \mathcal{K}_A) = \text{link}(A_T, \text{cube}(T)) * \text{link}(x, K_T).$$

Taking the orbit of $\mathcal{K}_A$ under the A_T-action, we obtain a neighborhood of x in $\mathcal{D}_A$. The A_T-action fixes K_T and hence,

$$\text{link}(x, \mathcal{D}_A) = \mathcal{B}_{A_T} * \text{link}(x, K_T).$$

□

3. The Positive Deligne Complex

In this section we construct a simplicial complex $\mathcal{D}_T^+$ which plays the role of the Deligne complex for the positive monoid A^+. We begin by recalling some facts about A^+. Proofs can be found in [BS].

Lemma 3.1 [BS]. *The left and right cancellation laws hold in A^+. That is, if $ab_1 = ab_2$ (resp. $b_1a = b_2a$), then $b_1 = b_2$.*

There are two partial orderings on A^+ defined by

$$b \preceq_l a \text{ if } a = bc \text{ for some } c \in A^+.$$

$$b \preceq_r a \text{ if } a = cb \text{ for some } c \in A^+.$$

In the case of a finite type Artin group, A^+ is a lattice with respect to either of these partial orderings. That is, any two elements a and b of A^+ have a least upper bound $a \wedge_l b$ (resp. $a \wedge_r b$) and a greatest lower bound $a \vee_l b$ (resp. $a \vee_r b$). In the case of an infinite type Artin group, we have the following.

Lemma 3.2 [BS].
(1) *Any finite set of elements of A^+ has a greatest lower bound.*
(2) *A finite set of elements of A^+ either has no upper bound or it has a least upper bound.*

Since the relations in the presentation of A^+ have the property that the same generators appear on both sides of the relation, a word can only be related to another word involving the same generators. It follows that for any $T \subset S$, A_T^+ is a submonoid of A^+. Assume A_T is finite type. Define an equivalence relation $\sim_T$ on A^+ by

$$a_1 \sim_T a_2 \iff \exists \quad b_1, b_2 \in A_T^+ \quad \text{such that} \quad a_1 b_1 = a_2 b_2.$$

Lemma 3.3. *The relation $\sim_T$ is an equivalence relation.*

Proof. Suppose $a_1 b_1 = a_2 b_2$ and $a_2 b_3 = a_3 b_4$ with $b_i \in A_T^+$. Let $c = b_2 \wedge_l b_3$. Then $c = b_2 c_2 = b_3 c_3$ for some $c_2, c_3 \in A_T^+$, so $a_1 b_1 c_2 = a_2 b_2 c_2 = a_2 b_3 c_3 = a_3 b_4 c_3$ hence $a_1 \sim_T a_3$. □

Let $[a]_T$ denote the $\sim_T$ equivalence class of a. The next lemma shows that these equivalence classes may be thought of as "cosets" for A_T^+.

Lemma 3.4. *With respect to $\preceq_l$, an equivalence class $[a]_T$ contains a unique minimal element a_0 and $[a]_T = a_0 A_T^+$.*

Proof. Let a_0 be a minimal element of $[a]_T$ with respect to $\preceq_l$. (Minimal elements exist since $a_1 \preceq_l a_2$ implies $\text{length}(a_1) \leq \text{length}(a_2)$.) It suffices to show that any a' in $[a]_T = [a_0]_T$ can be expressed as $a' = a_0 c$ for some $c \in A_T^+$. Since $a' \sim_T a_0$, we have $a' b_1 = a_0 b_2$ for some $b_1, b_2 \in A_T^+$. Let $d = b_1 \wedge_r b_2$ so that $d = c_1 b_1 = c_2 b_2$ for some $c_1, c_2 \in A_T^+$. Then $a_0 b_2 \succeq_r d$, so by right cancellation, $a_0 \succeq_r c_2$. But this contradicts the minimality of a_0 unless $c_2 = 1$ in which case $a_0 c_1 b_1 = a_0 b_2 = a' b_1$, and hence $a_0 c_1 = a'$. □

We are now ready to define $\mathcal{D}_A^+$. Let

$$A^+ \mathcal{S}^f = \{[a]_T \mid a \in A^+, \ W_T \text{ is finite}\}$$

with partially ordering

$$[a]_T \leq [b]_R \iff T \subseteq R \text{ and } [a]_R = [b]_R.$$

We define $\mathcal{D}_A^+$ to be the associated flag complex. Note that $\mathcal{S}^f$ can be viewed as the subset of $A^+ \mathcal{S}^f$ consisting of the equivalence classes $[1]_T = A_T^+$, thus $\mathcal{K}_A$ is a subcomplex $\mathcal{D}_A^+$. More generally, fixing $a \in A^+$, the vertices $[a]_T$ generate a subcomplex isomorphic to $\mathcal{K}_A$. Defining K_T as above to be the subcomplex of $\mathcal{K}_A$ spanned by the vertices $[1]_R$ with $T \subseteq R$, we thus have

$$\mathcal{D}_A^+ = A^+ \times \mathcal{K}_A / \sim$$

where $(a_1, x_1) \sim (a_2, x_2)$ if $x_1 = x_2$, x_i lies in the relative interior of K_T, and $[a_1]_T = [a_2]_T$.

As described in the previous section, we can decompose $\mathcal{K}_A$ into cubes and give each cube a Euclidean metric to obtain a piecewise Euclidean structure on $\mathcal{D}_A^+$.

Lemma 3.5. *If x lies in the relative interior of K_T, $a \in A^+$, and z is the image of (a,x) in $\mathcal{D}_A^+$, then link$(z, \mathcal{D}_A)$ is isometric to the orthogonal join of link(x, K_T) and $\mathcal{B}_{A_T}^+$.*

Proof. If $a = 1$, then the proof is the same as that of Lemma 2.2. Suppose $a \neq 1$ and write $[a]_T = a_0 A_T^+ = [a_0]_T$ as in Lemma 3.4. Note that the star of z (i.e., the union of cells containing z) lies in the subcomplex of $\mathcal{D}_A^+$ spanned by

$$St[a_0]_T = \{[b]_R \mid [a_0]_T \leq [b]_R \text{ or } [b]_R \leq [a_0]_T\}.$$

Left multiplication by a_0, $[b]_R \mapsto [a_0 b]_R$, defines an order-preserving map $A^+\mathcal{S}^f \to A^+\mathcal{S}^f$ which maps $St[1]_T$ isomorphically onto $St[a_0]_T$. Thus, the star of $z = (a, x)$ is isometric to the star of $(1, x)$. □

4. Injectivity for Two-dimensional Groups

The natural map $\theta : A^+ \to A$ induces a map $\Theta : \mathcal{D}_A^+ \to D_A$ which maps $a \times \mathcal{K}_A$ isometrically to $\theta(a) \times \mathcal{K}_A$. Since the vertices $[a]_\emptyset$, $a \in A^+$ are distinct in $\mathcal{D}_A^+$, to prove that θ is injective, it suffices to prove that Θ is an embedding. Since the positive monoid of a finite type Artin group injects, the natural map $\mathcal{B}_{A_T}^+ \to \mathcal{B}_{A_T}$ is an embedding for each $A_T \in \mathcal{S}^f$. In light of Lemmas 2.2 and 3.5, Θ is thus a local embedding. To prove injectivity, we will need to show that Θ is a global embedding. This will follow (in the case of a 2-dimensional Artin group) from the next two propositions.

Proposition 4.1. *Suppose that for every $A_T \in \mathcal{S}^f$, $\mathcal{B}_{A_T}$ satisfies the following property.*

(P) *If x, y are two points of distance $d(x, y) < \pi$ in $\mathcal{B}_{A_T}$ and x, y both lie in $\mathcal{B}_{A_T}^+$, then the geodesic between them also lies in $\mathcal{B}_{A_T}^+$.*

Then $\Theta : \mathcal{D}_A^+ \to \mathcal{D}_A$ takes local geodesics to local geodesics.

Proof. Recall that a piecewise geodesic γ is a local geodesic if, at the endpoint x_i of each geodesic piece, the incoming and outgoing tangent vectors to γ represent points of distance at least π in the link of x_i. Thus to show that Θ takes local geodesics to local geodesics, it suffices to show that the induced map link$(x, \mathcal{D}_A^+) \to$ link$(\Theta(x), \mathcal{D}_A)$ takes points of distance $\geq \pi$ to points of distance $\geq \pi$. By Lemma 2.2 and 3.5 and properties of orthogonal joins ([CD2], Appendix), this holds if and only if the embedding $\mathcal{B}_{A_T}^+ \hookrightarrow \mathcal{B}_{A_T}$ takes points of distance $\geq \pi$ to points of distance $\geq \pi$. This is precisely what property (P) guarantees. □

Remark. If $|T| = 1$, $\mathcal{B}_{A_T}$ is 0-dimesional. In this case, we define the distance between any two points in $\mathcal{B}_{A_T}$ to be π and condition (P) is satisfied vacuously.

Proposition 4.2. *If* (A, T) *is a 2-generator finite type Artin system, then* $A\,(= A_T)$ *satisfies property* (P).

The proof of this proposition will occupy the remainder of this section. For more about 2-generator Artin groups see [AS].

Let $F(T)$ denote the free group on T. For $w \in F(T)$, let $\boldsymbol{w}$ denote the image of w in A. Suppose $w = s_1^{n_1} \dots s_k^{n_k}$ with $s_i \in T$, $s_i \neq s_{i+1}$, and $n_i \neq 0$. Define

$$|w| = |n_1| + \cdots + |n_k|$$
$$\|w\| = k.$$

The former is called the *length* of w and the latter is called the *syllable length* of w. We say a word $u \in F(T)$ is an *initial substring* of w if w is the concatenation of u with some word v. (That is, we require $w = uv$ where no cancellation is possible between u and v.)

Now let $T = \{s, t\}$ and let

$$A = \langle s, t \mid \underbrace{sts\dots}_{m \text{ terms}} = \underbrace{tst\dots}_{m \text{ terms}} \rangle$$

with $m < \infty$. Since A is finite type, we may view A^+ as a subset of A. The simplicial complex $\mathcal{B}_A$ is 1-dimensional with all 1-simplicies of length π/m. The 1-simplices are indexed by the elements of A and two such, a and b, have a common vertex if and only if $a = bt^n$ or $a = bs^n$ for some $n \in \mathbb{Z}$. $\mathcal{B}_A^+$ is the subcomplex of 1-simplices corresponding to $a \in A^+$. A piecewise geodesic in $\mathcal{B}_A$ (between two vertices) is given by a sequence of edges, $a_1, \dots, a_k$ such that a_i differs from a_{i+1} by a power of a generator. The length of this piecewise geodesic is $\frac{k\pi}{m}$. To prove property (P), it suffices to consider the case where x and y are vertices in $\mathcal{B}_A^+$. (The minimal length counterexample to (P), if any, will occur when x and y are vertices.) Suppose $a_1, \dots, a_k$ is a geodesic from x to y in $\mathcal{B}_A$ of length less than π (i.e., $k < m$). To prove Proposition 4.2, we must show that a_i lies in A^+ for $i = 1, \dots, k$. Let a_0 be an edge in $\mathcal{B}_A^+$ containing x and let a_{k+1} be an edge in $\mathcal{B}_A^+$ containing y. Since a_i and a_{i-1} share a vertex for $i = 1, \dots, k+1$, we can write $a_i = a_{i-1} s_i^{n_i}$ with $s_i \in \{s, t\}$, and we may assume that $s_i \neq s_{i+1}$, $n_i \neq 0$. Let $w = s_1^{n_1} \dots s_{k+1}^{n_{k+1}}$, so $\|w\| = k + 1 \leq m$. Then $a_0 \in A^+$, $a_0\boldsymbol{w} = a_{k+1} \in A^+$, and for any i, $a_i = a_0\boldsymbol{u}$ for some initial substring u of w. Hence Proposition 4.2 will follow immediately from the next lemma.

Lemma 4.3. *Let* w *be a (reduced) word in* $F(T)$ *with* $\|w\| \leq m$ *and let* u *be an initial substring of* w. *If* $p \in A^+$ *is such that* $p\boldsymbol{w}$ *lies in* A^+, *then* $p\boldsymbol{u}$ *also lies in* A^+.

To prove Lemma 4.3, we will need some technical facts about finite type Artin groups in general and 2-generator Artin groups in particular. For two elements $a, b \in A^+$ we write $a \perp_l b$ if $a \vee_l b = 1$, that is, if the only $c \in A^+$ with $c \preceq_l a$ and $c \preceq_l b$ is $c = 1$.

Lemma 4.4. *Let A be a finite type Artin group and $x \in A$. Then (i) x can be uniquely written as $x = a^{-1}b$ with $a, b \in A^+$ and $a \perp_l b$, and (ii) if $p \in A^+$, then $px \in A^+$ if and only if $a \preceq_r p$.*

Proof. The first statement is Theorem 2.6 of [C2] and the second statement follows easily from the uniqueness part of (i). □

We will call $a^{-1}b$ the *orthogonal splitting* of x. An analogous splitting of the form $x = ab^{-1}$ with $a, b \in A^+$, $a \perp_r b$ also exists and has similar properties. In particular, if $p \in A^+$, then $xp \in A^+$ if and only if $b \preceq_l p$.

Now let us return to the case of a 2-generator Artin group as above. Let $\Delta_{s,t}$ be the m-term product $sts\dots$ and similarly for $\Delta_{t,s}$. Let Δ be the image of $\Delta_{s,t}$ (and $\Delta_{t,s}$) in A^+. Set

$$\mathcal{M} = \{a \in A^+ \mid a \preceq_l \Delta\}$$
$$\mathcal{M}_0 = \{a \in \mathcal{M} \mid a \neq 1, \Delta\}.$$

Thus, the elements of $\mathcal{M}_0$ are alternating products of length $< m$. The element Δ plays a special role in A. If m is even, then Δ commutes with s and t, hence it lies in the center of A. If m is odd, then $s\Delta = \Delta t$ and $t\Delta = \Delta s$, hence conjugation by Δ defines an involution on A^+ which preserves $\mathcal{M}_0$. In either case, we set

$$\bar{a} = \Delta a \Delta^{-1}.$$

Note, also, that for $a \in \mathcal{M}_0$, Δa^{-1} lies in $\mathcal{M}_0$. We set

$$a^* = \Delta a^{-1}.$$

The operations $a \mapsto \bar{a}$ and $a \mapsto a^*$ commute (indeed $\bar{a} = (a^*)^*$), so there is no ambiguity in the notation $\bar{a}^*$.

Remark. Analogues of Δ and $\mathcal{M}$ exist for any finite type Artin group (see [C1] or [C2]). The elements of $\mathcal{M}$ are known as "minimals".

If $a \in \mathcal{M}_0$, then the equivalence class of words in $F^+(T)$ representing a contains only one element, thus we may regard elements of $\mathcal{M}_0$ either as words in $F^+(T)$ or as elements of A^+. More generally we have,

Lemma 4.5. *Let $a \in A^+$. Then the following are equivalent.*
(1) $\Delta \preceq_l a$.
(2) *There exist words $w_1, w_2 \in F^+(T)$ representing a such that w_1 begins with s and w_2 begins with t.*
(3) *The equivalence class of words in $F^+(T)$ representing a contains more than one element.*
(4) *Any word $w \in F(T)$ representing a contains $\Delta_{s,t}$ or $\Delta_{t,s}$ as a subword.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) is obvious. Assuming (4), we have $a = a_1 \Delta a_2 = \Delta \bar{a}_1 a_2$ for some $a_1, a_2 \in A^+$ and (1) follows. □

Lemma 4.6. *Let* $w = \sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}\dots\sigma_n^{\epsilon_n}$, *with* $\sigma_i \in \mathcal{M}_0$, $\epsilon_i = \pm 1$, *and set* $k = \#\{i \mid \epsilon_i = -1\}$. *Then* $\Delta^k \boldsymbol{w}$ *lies in* A^+.

Proof. Using the equation $\Delta\sigma_i = \bar{\sigma}_i\Delta$ we can "slide" copies of Δ to the right until one copy of Δ precedes each σ_i (or $\bar{\sigma}_i$) with $\epsilon_i = -1$. Then the equation $\Delta\sigma_i^{-1} = \sigma_i^*$ shows that $\Delta^k\boldsymbol{w}$ is a product of elements of $\mathcal{M}_0$, hence lies in A^+. □

For a word $w \in F(T)$ let $s(w)$ denote the first letter of w and let $e(w)$ denote the last letter of w, without regard to sign. (For example, if $w = t^2 s^{-1}$, then $s(w) = t$ and $e(w) = s$.) Suppose w is a reduced word which does not contain any of the words $\Delta_{s,t}^{\pm 1}$, $\Delta_{t,s}^{\pm 1}$ as a subword. Then we can factor w into a product of subwords, $w = \sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}\dots\sigma_n^{\epsilon_n}$, $\epsilon_i = \pm 1$, $\sigma_i \in \mathcal{M}_0$ which satisfy

$$e(\sigma_i^{\epsilon_i}) = s(\sigma_{i+1}^{\epsilon_{i+1}}) \quad \text{if and only if} \quad \epsilon_i = \epsilon_{i+1}. \tag{4.7}$$

Now let $k = \#\{i \mid \epsilon_i = -1\}$ and let w_1, w_2 be the subwords of w defined by $w_1 = \sigma_1^{\epsilon_1}\dots\sigma_k^{\epsilon_k}$ and $w_2 = \sigma_{k+1}^{\epsilon_{k+1}}\dots\sigma_n^{\epsilon_n}$. Set

$$\begin{aligned} r &= \#\{i \mid \epsilon_i = -1, i \geq k+1\} \\ &= \#\{i \mid \epsilon_i = +1, i \leq k\} \end{aligned}$$

and let $a = \Delta^r\boldsymbol{w}_1^{-1}$ and $b = \Delta^r\boldsymbol{w}_2$. By Lemma 4.6, both a and b lie in A^+.

Lemma 4.8. *Suppose* $w \in F(S)$ *does not contain* $\Delta_{s,t}^{\pm 1}$ *or* $\Delta_{t,s}^{\pm 1}$ *as a subword. Let* a, b *be as above. Then* $a \perp_l b$ *and hence* $a^{-1}b$ *is the orthogonal splitting of* $\boldsymbol{w}$.

Proof. By definition, $\Delta^r w_2$ is a word in $F(S)$ representing b. As in the proof of Lemma 4.6, we can find a word in $F^+(S)$ representing b by "sliding" copies of Δ to the right and replacing $\Delta\sigma^{-1}$ by σ^*. The resulting word is of the form $\mu_{k+1}\dots\mu_n$ where

$$\mu_i = \begin{cases} \sigma_i \text{ or } \bar{\sigma}_i & \text{if } \epsilon_i = +1 \\ \sigma_i^* \text{ or } \bar{\sigma}_i^* & \text{if } \epsilon_i = -1. \end{cases}$$

The bars are determined by whether an even or odd number of Δ's slide past σ_i. We claim that

$$e(\mu_i) = s(\mu_{i+1}) \quad \text{for all } i. \tag{4.8.1}$$

To prove the claim, first note that the equations $\Delta = \sigma^*\sigma = \sigma\bar{\sigma}^*$ immediately imply that

$$e(\sigma^*) \neq s(\sigma) \quad \text{and} \quad e(\sigma) \neq s(\bar{\sigma}^*). \tag{4.8.2}$$

Now consider the word obtained by sliding the first copy of Δ through w_2. If the first negative exponent in w_2 occurs at ϵ_l, then we have

$$\begin{aligned} \Delta w_2 &= \Delta\sigma_{k+1}\dots\sigma_{l-1}\sigma_l^{-1}\sigma_{l+1}^{\epsilon_{l+1}}\dots\sigma_n^{\epsilon_n} \\ &= \bar{\sigma}_{k+1}\dots\bar{\sigma}_{l-1}\sigma_l^*\sigma_{l+1}^{\epsilon_{l+1}}\dots\sigma_n^{\epsilon_n}. \end{aligned}$$

This last product still satisfies condition (4.7). This is clear everywhere except for the pairs $\bar{\sigma}_{l-1}\sigma_l^*$ and $\sigma_l^*\sigma_{l+1}^{\epsilon_{l+1}}$. For the first of these pairs, note that $\epsilon_{l-1} \neq \epsilon_l$ and hence $e(\sigma_{l-1}) \neq s(\sigma_l^{-1}) = e(\sigma_l)$. It then follows from (4.8.2) that $e(\sigma_{l-1}) = s(\bar{\sigma}_l^*)$, or equivalently, that $e(\bar{\sigma}_{l-1}) = s(\sigma_l^*)$. For the second pair, we have $s(\sigma_l) = e(\sigma_l^{-1}) = s(\sigma_{l+1}^{\epsilon_{l+1}})$ if and only if $\epsilon_{l+1} = -1$. From (4.8.2) we conclude that $e(\sigma_l^*) = s(\sigma_{l+1}^{\epsilon_{l+1}})$ if and only if $\epsilon_{l+1} = +1$. Thus, (4.7) still holds.

This process can now be repeated with the second copy of Δ, and so on, so that at each stage the resulting product of elements of $\mathcal{M}_0$ satisfies (4.7). In particular, after the final Δ has been used and all the factors have been made positive, we have $e(\mu_i) = s(\mu_{i+1})$ for all i. This proves the claim (4.8.1).

Equation (4.8.1) implies that the word $\mu_{k+1}\dots\mu_n$ contains no alternating product of generators of length m. Thus by Lemma 4.5, it is the *unique* positive word representing b. A completely analogous argument shows that the positive word $\eta_k\dots\eta_1$ obtained from sliding copies of Δ through w_1^{-1} in the word is the unique positive word representing a. To prove that $a \perp_l b$ it remains only to check that $s(\eta_k) \neq s(\mu_{k+1})$. If r is even, then

$$\eta_k = \begin{cases} \sigma_k & \text{if } \epsilon_k = -1 \\ \bar{\sigma}_k^* & \text{if } \epsilon_k = +1, \end{cases} \qquad \mu_{k+1} = \begin{cases} \sigma_{k+1} & \text{if } \epsilon_{k+1} = +1 \\ \bar{\sigma}_{k+1}^* & \text{if } \epsilon_{k+1} = -1, \end{cases}$$

and a case by case check using (4.7) and (4.8.2) shows that $s(\eta_k) \neq s(\mu_{k+1})$. Details are left for the reader. If r is odd, the same result is obtained by applying the bar involution. □

Proof of Lemma 4.3. Let $\boldsymbol{w} = a^{-1}b$ be the orthogonal splitting of $\boldsymbol{w}$. By Lemma 4.4(ii), it suffices to consider the case of $p = a$. If w is strictly positive or strictly negative, then the lemma is obvious. If not, then any strictly positive or strictly negative subword of w must have syllable length $< m$. Hence, w does not contain $\Delta_{s,t}^{\pm 1}$ or $\Delta_{t,s}^{\pm 1}$ as a subword and Lemma 4.8 applies to give $a = \Delta^r \boldsymbol{w}_1^{-1}$. Let u be an initial substring of w. Then either u is an initial substring of w_1, or $u = w_1 v_2$ where v_2 is an initial substring of w_2. In the first case, we have $w_1 = uv_1$ for some v_1, so $a\boldsymbol{u} = \Delta^r \boldsymbol{w}_1^{-1}\boldsymbol{u} = \Delta^r \boldsymbol{v}_1^{-1}$. Since v_1^{-1} is an initial substring of w_1^{-1}, it can be written as a product of elements of $\mathcal{M}_0$ and their inverses with at most r inverses. Thus, $\Delta^r \boldsymbol{v}_1^{-1}$ lies in A^+. In the second case, we have $a\boldsymbol{u} = \Delta^r \boldsymbol{w}_1^{-1}\boldsymbol{u} = \Delta^r \boldsymbol{v}_2$. Since v_2 is an initial substring of w_2, we similarly conclude that $\Delta^r \boldsymbol{v}_2$ lies in A^+. □

This completes the proof of Proposition 4.2.

Theorem 4.9. *If (A, S) is an infinite type two-dimensional Artin system, then the positive monoid A^+ injects into A.*

Proof. By Propositions 4.1 and 4.2, $\Theta : \mathcal{D}_A^+ \to \mathcal{D}_A$ takes local geodesics to local geodesics. by Theorems 1.1 and 2.1, any local geodesic in $\mathcal{D}_A$ is a (global) geodesic. It follows that Θ is an isometric embedding. Since the vertices $[a]_\emptyset$ for $a \in A^+$ are distinct in $\mathcal{D}_A^+$, we conclude that $\theta : A^+ \to A$ is injective. □

Remark. As mentioned above, it is conjectured in [CD] that the Moussong metric on $\mathcal{D}_A$ is CAT(0) for all Artin groups A. As explained in [CD], this conjecture is equivalent to the conjecture that the induced piecewise spherical metric on $\mathcal{B}_{A_T}$ is CAT(1) (i.e., triangles in $\mathcal{B}_{A_T}$ are thin relative to comparison triangles in $\mathbb{S}^2$) for all finite type special subgroups of A. We propose the following expanded conjecture.

Conjecture 4.10. *Let A be an Artin group of finite type. Then the natural piecewise spherical metric on $\mathcal{B}_A$ (induced by the Moussong metric on $\mathcal{D}_A$) satisfies*
(1) *$\mathcal{B}_A$ is CAT(1), and*
(2) *if x, y are two points of distance less than π in $\mathcal{B}_A$ and x, y both lie in $\mathcal{B}_A^+$, then the geodesic between them also lies in $\mathcal{B}_A^+$.*

Theorem 4.11. *Let A be an infinite type Artin group. If Conjecture 4.10 holds for all finite type special subgroups of A, then A^+ injects into A.*

5. Injectivity for FC Groups

An infinite type Artin system (A,S) is called an *FC Artin system* if it satisfies the following property.

(FC) If $T \subseteq S$ and $m_{i,j} < \infty$ for all $s_i, s_j \in T$, then A_T is finite type.

(This condition is equivalent to the condition that the link of $A_\emptyset$ in $\mathcal{K}_A$ be a "flag complex" hence the initials FC. See [CD1].) In [A], Altobelli proves the injectivity of θ for FC Artin systems. The methods of the previous section give an alternate proof which we outline below.

Recall the decomposition of $\mathcal{K}_A$ as a union of the cubes, cube(T), for $T \in \mathcal{S}^f$. If we assign each cube the metric of a regular Euclidean cube of side length 1, we obtain a different piecewise Euclidean structure on $\mathcal{D}_A$ and $\mathcal{D}_A^+$. We call the induced metric the *cubical metric*.

Theorem 5.1 [CD1, Theorem 4.3.5]. *The cubical metric on $\mathcal{D}_A$ is CAT(0) if and only if (A,S) is an FC system.*

Thus, for FC systems, it suffices to show that $\Theta : \mathcal{D}_A^+ \to \mathcal{D}_A$ takes local geodesics to local geodesics in the cubical metric. The analysis of links in $\mathcal{D}_A$ and $\mathcal{D}_A^+$ given in Lemmas 2.2 and 3.5 follows through in this situation except that under the cubical metric, $\mathcal{B}_{A_T}$ and $\mathcal{B}_{A_T}^+$ have a different geometry. Namely, they are *all-right* piecewise spherical simplicial complexes, that is, they are made up of spherical simplices with all edge lengths $= \frac{\pi}{2}$.

Lemma 5.2. *Suppose L is an all-right piecewise spherical simplicial complex and L_0 is a full subcomplex of L. If γ is a geodesic in L of length $< \pi$ with endpoints in L_0, then γ lies entirely in L_0.*

Proof. The proof follows standard arguments as in [G] or [M]. Let γ be a geodesic in L whose endpoints lie in L_0. If γ does not lie entirely in L_0, then it contains a segment α whose interior lies in the open star of a vertex $v \in L \setminus L_0$ and whose endpoints lie on the boundary of the the closed star of v, which we denote by $star(v)$. We claim that the length of α is at least π. The distance from v to any point on the boundary of $star(v)$ is $\frac{\pi}{2}$, so if α passes through v, it must have length π. If it does not pass through v, consider the union of the geodesic segments from v to $\partial star(v)$ which pass through a point of α. Assuming α has length less than 2π, this union forms a subspace of $star(v)$ which can be isometrically embedded in $\mathbb{S}^2$ with v as the north pole and α as a geodesic segment with endpoints on the equator and interior lying strictly above the equator. Any such geodesic in $\mathbb{S}^2$ has length π. □

Lemma 5.3. *$\mathcal{B}^+_{A_T}$ embeds in $\mathcal{B}_{A_T}$ as a full subcomplex.*

Proof. The injectivity of $A^+_T \to A_T$ for finite type A_T impiles that $\mathcal{B}^+_{A_T}$ embeds in $\mathcal{B}_{A_T}$. We must verify that the image is a full subcomplex. A k-simplex of $\mathcal{B}_{A_T}$ corresponds to a coset aA_{T_1} in A_T such that $|T_1| = |T| - k - 1$. This simplex lies in the image of $\mathcal{B}^+_{A_T}$ if and only if $aA_{T_1} \cap A^+_T \neq \emptyset$ (in which case $aA_{T_1} \cap A^+_T = [a_1]_{T_1}$ for any $a_1 \in aA_{T_1} \cap A^+_T$). Two simplicies, $a_1 A_{T_1}$ and $a_2 A_{T_2}$, in $\mathcal{B}_{A_T}$ are contained in a common simplex if and only if $a_1 A_{T_1} \cap a_2 A_{T_2} \neq \emptyset$ (in which case the intersection is a coset of $A_{T_1 \cap T_2}$). Thus, it suffices to show that if $a_1, a_2 \in A^+_T$ are such that $a_1 A_{T_1} \cap a_2 A_{T_2} \neq \emptyset$, then $[a_1]_{T_1} \cap [a_2]_{T_2} \neq \emptyset$.

Let $x \in a_1 A_{T_1} \cap a_2 A_{T_2}$, and let $x = cd^{-1}$ be the orthogonal splitting of x with $c, d \in A^+_T$ and $c \perp_r d$. Then $x \in a_1 A_{T_1}$ imples that for some $y \in A_{T_1}$, $x = a_1 y = a_1 rs^{-1}$ where rs^{-1} is the orthogonal splitting of y in A_{T_1}. Since $xs \in A^+_T$, we have $d \preceq_l s$ and hence d must lie in A_{T_1}. It follows that $c = xd \in a_1 A_{T_1} \cap A^+_T = [a_1]_{T_1}$. On the other hand, the same argument, starting with $x \in a_2 A_{T_2}$, shows that $c \in [a_2]_{T_2}$. □

Together, Lemmas 5.2 and 5.3 imply that property (P) holds with respect to the cubical metric. Thus we have proved

Theorem 5.4. *If (A,S) is an FC Artin system, then the positive monoid A^+ injects into A.*

References

[A] Altobelli, J., The word problem for Artin groups of FC type, J. of Pure Appl. Alg., to appear.

[AS] Appel, K.I., and Schupp. P.E., Artin groups and infinite Coxeter groups, Invent. Math. 72 (1983), 201-220.

[AB] Alexander, S.B., and Bishop, R.L., The Hadamard-Cartan theorem for locally convex spaces, Enseign. Math. 36 (1990), 309-320.

[Ba] Ballmann, W., Lectures on Spaces of Nonpositive Curvature, DMV Seminar 25, Birkhäuser, Basel 1995.

[Br] Bridson, M.R., Geodesics and curvature in metric simplicial complexes, in "Group Theory from a Geometrical Viewpoint" (ed. by E. Ghys, A. Haefliger and A. Verjovsky), World Scientific, Singapore 1991, 373-463.

[Brn] Brown, K.S., "Buildings", Springer-Verlag, New York 1989.

[BS] Brieskorn, E., and Saito, K., Artin-gruppen und Coxeter-gruppen, Invent. Math. 17 (1972), 245-271.

[C1] Charney, R., Artin groups of finite type are biautomatic, Math. Annalen 292 (1992), 671-683.

[C2] Charney, R., Geodesic automation and growth functions for Artin groups of finite type, Math. Annalen 301 (1995), 307-324.

[CD1] Charney, R., and Davis, M.W., The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups, J. Amer. Math. Soc. 8 (1995), 597-627.

[CD2] Charney, R., and Davis, M.W., Singular metrics of nonpositive curvature on branched covers of Riemannian manifolds, Amer. J. Math 115 (1993), 929-1009.

[CP] Cho, J.R., and Pride, S.J., Embedding semigroups into groups and the asphericity of semigroups, Int. J. Algebra Comput. 3 (1993), 1-13.

[D] Deligne, P., Les immeubles des groupes de tresses généralisés, Invent. Math. 17 (1972), 273-302.

[G] Gromov, M., Hyperbolic groups, in "Essays in Group Theory" (ed. by S. M. Gersten), M.S.R.I. Publ. 8, Springer-Verlag, New York 1987, 75-264.

[L] van der Lek, H., The homotopy type of complex hyperplane complements, Ph.D. thesis, University of Nijmegan 1983.

[M] Moussong, G., Hyperbolic Coxeter groups, Ph.D. thesis, The Ohio State University 1988.