THE $K(\pi,1)$-CONJECTURE FOR THE AFFINE BRAID GROUPS

Ruth Charney and David Peifer

Abstract. The complement of the hyperplane arrangement associated to the (complexified) action of a finite, real reflection group on $\mathbb{C}^n$ is known to be a $K(\pi,1)$ space for the corresponding Artin group $\mathcal{A}$. A long-standing conjecture states that an analogous statement should hold for infinite reflection groups. In this paper we consider the case of a Euclidean reflection group of type $\tilde{A}_n$ and its associated Artin group, the affine braid group $\tilde{\mathcal{A}}$. Using the fact that $\tilde{\mathcal{A}}$ can be embedded as a subgroup of a finite type Artin group, we prove a number of conjectures about this group. In particular, we construct a finite, n-dimensional $K(\pi,1)$-space for $\tilde{\mathcal{A}}$, and use it to prove the $K(\pi,1)$ conjecture for the associated hyperlane complement. In addition, we show that the affine braid groups are biautomatic and give an explicit biautomatic structure.

Introduction

We begin by recalling some basic facts about Coxeter groups and Artin groups. A *Coxeter group* is a group with presentation

$$W = \langle s_1, \dots s_n \mid s_i^2 = (s_i s_j)^{m(i,j)} = 1 \rangle$$

where $m(i,j) \in \{2,3,\dots\} \cup \{\infty\}$. Let S be the generating set $\{s_1, \dots s_n\}$. The pair (W,S) is called a *Coxeter system.* We often encode this information in a labelled graph, called the *Coxeter diagram* whose vertices are the elements of S and two vertices s_i, s_j are joined by an edge labelled $m(i,j)$ if and only if $m(i,j) > 2$. (It is common to omit the label when $m(i,j) = 3$.) If Γ is a Coxeter graph, we denoted the associated Coxeter group by $W(\Gamma)$

The irreducible finite Coxeter systems are classified by the well-known Dynkin diagrams. The two which will primarily concern us in this paper are the Coxeter groups of type A_n and type B_n given by the diagrams in Figure 1. The Coxeter group $W(A_n)$ is the symmetry group of the n-simplex (or equivalently the symmetric group on $(n+1)$ letters) and $W(B_n)$ is the symmetry group of the n-cube.

Euclidean Coxeter groups, which act as affine reflection groups on $\mathbb{R}^n$, are also classified. Their Coxeter diagrams have the property that any proper subdiagram corresponds to a finite Coxeter group. Of primary interest in this paper is the

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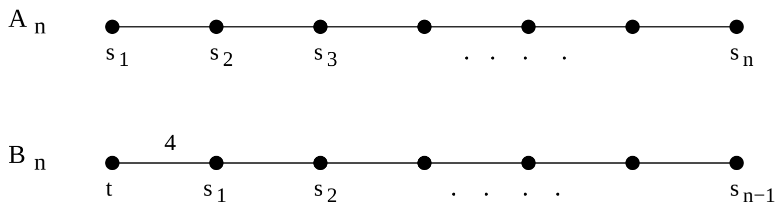

FIGURE 1. Coxeter diagrams of type A_n and B_n

Euclidean Coxeter group of type $\tilde{A}_n$ given by the diagram in Figure 2. For $n = 3$, $W(\tilde{A}_n)$ is the group of affine transformations of the plane which preserve a tiling by equilateral triangles.

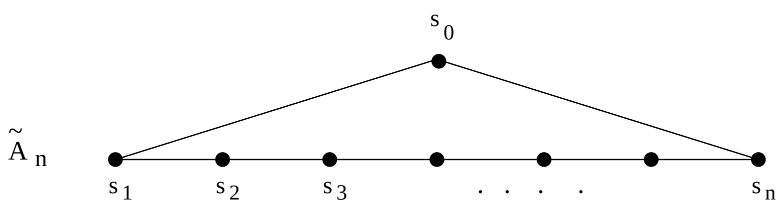

FIGURE 2. Coxeter diagram of type $\tilde{A}_n$

Associated to any Coxeter system is an *Artin system* $(\mathcal{A}, S)$ where $\mathcal{A}$ is the group defined by the presentation

$$\mathcal{A} = \langle S \mid prod(s_i, s_j; m_{i,j}) = prod(s_j, s_i; m_{i,j}),\ m_{i,j} \neq \infty \rangle.$$

where $prod(s_i, s_j; m_{ij})$ denotes the word $s_i s_j s_i ...$ of length $m_{i,j}$. Adding the relations $s_i^2 = 1$ gives back the presentation for (W, S), thus W is a quotient of $\mathcal{A}$. For a Coxeter graph Γ, we write $\mathcal{A}(\Gamma)$ for the associated Artin group. Artin groups corresponding to finite Coxeter groups are known as *spherical* or *finite type* Artin groups. The Artin group $\mathcal{A}(A_n)$ is the braid group on $n+1$ strands. The Artin group $\mathcal{A}(\tilde{A}_n)$ can also be thought of as a group of braids, as we will describe in section 1 below, and is sometimes called the *affine braid group.*

If $T \subset S$, and W_T is the subgroup of $W(\Gamma)$ generated by T, then (W_T, T) is a Coxeter system whose Coxeter diagram is the full subgraph of Γ spanned by the vertices T. Likewise, if $\mathcal{A}_T$ denotes the subgroup of $\mathcal{A}$ spanned by T, then $(\mathcal{A}_T, T)$ is the Artin system associated to (W_T, T) [L1]. The groups W_T and $\mathcal{A}_T$ are called *special subgroups* of W and $\mathcal{A}$ respectively.

The Artin groups of finite type were studied extensively in the 1970's. They are closely related to the classical braid groups, and a great deal is known about them. With a few exceptions, those corresponding to infinite Coxeter groups remain mysterious and difficult to handle. For example, the following properties are known to hold for finite type Artin groups but are only conjectured for infinite type. (For the finite type case, see [De], [BS], [CW], [Di], and [C].)

(1) $\mathcal{A}$ is torsion free.
(2) $\mathcal{A}$ is linear.

(3) $\mathcal{A}$ is biautomatic.
(4) If $\mathcal{A}$ is irreducible, the center of $\mathcal{A}$ is trivial ($\mathcal{A}$ of infinite type) or infinite cyclic ($\mathcal{A}$ of finite type).
(5) $\mathcal{A}$ has a finite $K(\pi,1)$ space of dimension n.
(6) $\mathcal{A}$ has cohomological dimension n.
(7) Let $\mathcal{H}_W$ be the hyperplane complement associated to W acting as a reflection group on $\mathbb{C}^n$. Then $\mathcal{H}_W/W$ is a $K(\mathcal{A},1)$-space.

The last of these is called the $K(\pi,1)$ Conjecture and is discussed in more detail below. A number of special cases of these conjectures have been proved for infinite type Artin groups ([A],[AS], [BM], [CD1], [CD2], [Pe]), but nearly all of them fall into one of two categories: the 2-dimensional Artin groups and the Artin groups of FC type. The former are groups for which W_T is infinite for any $T \subset S$ with at least three elements, and the latter are those for which W_T is infinite if and only if some $s_i, s_j \in T$ has $m(i,j)=\infty$. In particular, little is known for the Artin groups associated to Euclidean Coxeter groups.

There is a curious, but little known fact that the affine braid group $\mathcal{A}(\tilde{A}_n)$ can be realized as a subgroup of the finite type Artin group $\mathcal{A}(B_{n+1})$. This has been observed, for example, in [KP], [tD] and [Al]. In [KP] it is shown that $\mathcal{A}(B_{n+1})$ is a semi-direct product of $\mathcal{A}(\tilde{A}_n)$ and an infinite cyclic factor generated by an element δ which acts on $\mathcal{A}(\tilde{A}_n)$ by a cyclic permutation of its Coxeter diagram $\tilde{A}_n$. In this paper, we use this fact to prove all of the above properties for the groups $\mathcal{A}(\tilde{A}_n)$.

Clearly, the first two properties follow immediately from the embedding of $\mathcal{A}(\tilde{A}_n)$ as a subgroup of a finite type Artin group. The third and fourth properties are also easy to prove. Thus, the main content of the paper is the proof of the last three properties.

Property (7) requires some explanation. Any finite Coxeter system (W,S) can be realized as a group of linear transformations of $\mathbb{R}^n$, $n=|S|$, with the elements of S acting as orthogonal reflections in the walls of a polyhedral cone. For each reflection r in W (r acts as a reflection if it is conjugate to an element of S), let H_r denote the hyperplane fixed by r. Then W acts freely on the complement of these hyperplanes. Complexifying the action, we get an action of W on $\mathbb{C}^n$ which is free on the complement of the complex hyperplanes $\mathbb{C}H_r = H_r \oplus iH_r$. Let $\mathcal{H}_W = \mathbb{C}^n \backslash (\bigcup_r \mathbb{C}H_r)$ be the hyperplane complement. Deligne [De] showed that $\mathcal{H}_W/W$ is a $K(\mathcal{A},1)$-space where $\mathcal{A}$ is the Artin group associated to W, i.e., $\mathcal{H}_W/W$ has fundamental group A and its universal covering space is contractible.

Infinite Coxeter groups also act as reflection groups on $\mathbb{R}^n$. The $K(\pi,1)$ Conjecture states that an analogous statement about the hyperplane complement should be true for infinite Coxeter groups. The "analogous statement", however, requires that we replace $\mathbb{R}^n$ with the Tits cone, an open cone in $\mathbb{R}^n$ on which W acts properly. We refer the reader to [CD1] for a complete discussion of this conjecture and discuss it here only for the case which concerns us, namely the case of a Euclidean Coxeter group. For a Euclidean Coxeter system (W,S) with $|S|=n+1$, the action on $\mathbb{R}^{n+1}$ preserves an n-dimensional affine space $\mathbb{E}^n$ and the elements of S act on $\mathbb{E}^n$ as (affine) reflections in the walls of a simplex. (In this case the Tits cone is the upper half space in $\mathbb{R}^{n+1}$ and it equivariantly retracts onto $\mathbb{E}^n$.) If we identify $\mathbb{C}^n$

with $\mathbb{E}^n \oplus i\mathbb{E}^n$ then we can define the hyperplane complement associated to (W,S) as above,

$$\mathcal{H}_W = \mathbb{C}^n \backslash (\bigcup_r \mathbb{C}H_r)$$

where $\mathbb{C}H_r$ is the fixed set of the reflection r.

Conjecture. *For a Euclidean Coxeter system (W,S) with associated Artin group $\mathcal{A}$, the orbit space $\mathcal{H}_W/W$ is a $K(\mathcal{A},1)$-space.*

It was shown by van der Lek [L2] in the early 80's that $\pi_1(\mathcal{H}_W/W) = \mathcal{A}$. Thus, to prove the conjecture, it remains to show that the universal cover of the hyperplane complement is contractible.

In section 3, we use a construction from [CMW] to find a contractible, n-dimensional complex on which $\mathcal{A}(\tilde{A}_n)$ acts freely and cocompactly, and we prove that this contractible complex is homotopy equivalent to the universal covering space of the associated hyperplane complement $\mathcal{H}_W$. This proves properties (5), (6), and (7) for these groups.

1. The group $\tilde{\mathcal{A}}$

In this section we recall the results of [KP] and prove properties (3) and (4) for the affine braid groups.

Let $\mathcal{B} = \mathcal{A}(B_{n+1})$, the Artin group of type B_{n+1} with generating set $S = \{t, s_1, \ldots, s_n\}$. We may think of elements of $\mathcal{B}$ as $(n+1)$-strand braids drawn on a cylinder. From this point of view, the generators are represented in Figure 3. (It is more common to view elements of $\mathcal{B}$ as $(n+2)$-strand braids which fix the first strand. Letting the fixed strand expand to form a cylinder, we get the representation as cylindrical braids. This representation has a certain symmetry to it which makes the relation between $\tilde{\mathcal{A}}$ and $\mathcal{B}$ more transparent.)[1]

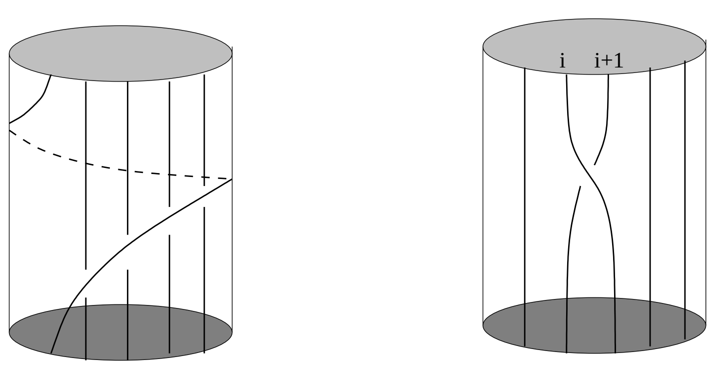

Figure 3. The generators t and s_i of $\mathcal{B}$

[1]These braids are sometimes known as annular braids since $\mathcal{B}$ can be identified with the fundamental group of the configuration space of $n+1$ points in an annulus.

Define a homomorphism

$$\phi : \mathcal{B} \to \mathbb{Z} \quad \text{by } \phi(t) = 1 \text{ and } \phi(s_i) = 0.$$

Thus, for any $b \in \mathcal{B}$, $\phi(b)$ is the exponent sum of t in some (any) word in the free group on S representing b. That this is a homomorphism follows from the fact that every relation in the presentation for $\mathcal{B}$ preserves this exponent sum. We can split ϕ by lifting the generator of $\mathbb{Z}$ to the element $\delta = ts_1s_2\cdots s_n$. It follows that $\mathcal{B}$ is isomorphic to the semidirect product, $\mathcal{B} \cong K \rtimes \mathbb{Z}$, where K is the kernel of ϕ and $\mathbb{Z}$ is the cyclic group generated by δ.

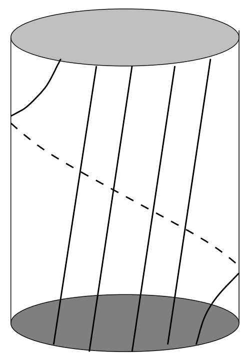

FIGURE 4. The element δ in $\mathcal{B}$

Now let $\tilde{\mathcal{A}} = \mathcal{A}(\tilde{A}_n)$, the Artin group of type $\tilde{A}_n$ with generating set $\tilde{S} = \{s_0, s_1, \ldots, s_n\}$. In [KP], Kent and Peifer show that the homomorphism $h : \tilde{\mathcal{A}} \to \mathcal{B}$ taking s_i to the generator of the same name in S for $i = 1, \ldots, n$, and taking s_0 to $\delta s_n \delta^{-1}$, maps $\tilde{\mathcal{A}}$ isomorphically onto a normal subgroup of $\mathcal{B}$. Since the image of h clearly lies in the kernel of ϕ and is a normal subgroup containing $s_1, \ldots s_n$, the image of h must be all of K. Thus we can identify $\mathcal{B}$ with $\tilde{\mathcal{A}} \rtimes \langle \delta \rangle$. Under this identification, s_0 is the braid which crosses the n^{th} string over the first string in the back of the cylinder.

It is straightforward to check that the action of δ on $\tilde{\mathcal{A}}$ (by conjugation) is given by $\delta s_i \delta^{-1} = s_{i+1} \bmod n+1$, or in other words, δ acts on $\tilde{\mathcal{A}}$ by cyclic permutation of its Coxeter diagram. In particular, δ^{n+1} acts trivially on $\tilde{\mathcal{A}}$, hence $\mathcal{B}$ contains the direct product $\tilde{\mathcal{A}} \times \langle \delta^{n+1} \rangle$ as a subgroup of finite index.

Properties (3) and (4) listed in introduction are now easy to prove for $\tilde{\mathcal{A}}$. We refer the reader to [E] for definitions and background on (bi)automatic groups.

Proposition 1.1. *The affine braid group $\tilde{\mathcal{A}}$ is biautomatic.*

Proof. The group $\mathcal{B}$ is a finite type Artin group so it is biautomatic [C]. By [E], Theorem 4.1.4, a finite index subgroup of a biautomatic group is also biautomatic. Therefore, $\tilde{\mathcal{A}} \times \langle \delta^{n+1} \rangle$ is biautomatic. In [M], L. Mosher proves that a direct factor of a biautomatic group is again biautomatic. Therefore $\tilde{\mathcal{A}}$ is biautomatic. □

An explicit biautomatic structure for $\tilde{\mathcal{A}}$ is described in Section 2 below.

To prove properties (4) and (6), we will need a bound on the rank of abelian subgroups in $\tilde{\mathcal{A}}$. For finite type Artin groups, the bound is well known.

Lemma 1.2. *Let $(\mathcal{A}, T)$ be an Artin system of finite type. Then the maximal rank of an abelian subgroup of $\mathcal{A}$ is $n = |T|$*

Proof. It suffices to consider an irreducible finite type Artin system $(\mathcal{A}, T)$. In this case, the Coxeter diagram for $(\mathcal{A}, T)$ is a tree and we can order the generators $T = \{t_1, \dots t_n\}$ so that for $1 \leq i \leq n$, $T_i = \{t_1, \dots t_i\}$ spans a connected subgraph. Then the special subgroup $\mathcal{A}_i$ generated by T_i is an irreducible finite type Artin group and hence has infinite cyclic center (generated by the Δ-element in $\mathcal{A}_i$ or the square of this element). Let z_i be a generator of this center. Then no power of z_i belongs to $\mathcal{A}_j$ for $j < i$. (This is easily seen using the normal form for $\mathcal{A}$ given in [C].) Hence $\{z_1, z_2, \dots, z_n\}$ generates a rank n abelian subgroup of $\mathcal{A}$. Since $\mathcal{A}$ has an n-dimensional $K(\pi, 1)$-space ([De], [Sa]), its cohomological dimension is at most n, so this group is of maximal rank. $\square$

Property (4), which states that the center of $\tilde{\mathcal{A}}$ is trivial, was proved by D. Johnson and M. Albar in [JA]. We give a simpler proof below.

Proposition 1.3. *The maximal rank of an abelian subgroup of the affine braid group $\tilde{\mathcal{A}}$ is n and the center of $\tilde{\mathcal{A}}$ is trivial.*

Proof. Recall that $\mathcal{A}(A_n)$ is the braid group on $(n+1)$-strands. The first statement of the proposition follows immediately from the preceding lemma since $\mathcal{A}(A_n) \subset \tilde{\mathcal{A}} \subset \mathcal{B}$, and if H is a rank k abelian subgroup of $\tilde{\mathcal{A}}$, then $H \times \langle \delta^{n+1} \rangle$ is a rank $k+1$ abelian subgroup of $\mathcal{B}$.

For the second statement, assume that $z \neq 1$ is in the center of $\tilde{\mathcal{A}}$. Let $T_i = \{s_0, \dots, \hat{s}_i, \dots, s_n\}$, the subset of $\tilde{S}$ consisting of all the generators except s_i, and let $\tilde{\mathcal{A}}_{T_i}$ be the special subgroup generated by T_i. Notice that each $\tilde{\mathcal{A}}_{T_i}$ is a copy of the braid group $\mathcal{A}(A_n)$.

We claim that $\langle z \rangle \cap \tilde{\mathcal{A}}_{T_i} = \{1\}$, for some i. Suppose that this is not the case. Then for all i, there exists an integer m_i such that $z^{m_i} \in \tilde{\mathcal{A}}_{T_i}$. Let $m = m_1 m_2 \cdots m_n$. Then $z^m \in \tilde{\mathcal{A}}_{T_i}$ for all i and hence $z^m \in \tilde{\mathcal{A}}_{T_1} \cap \dots \cap \tilde{\mathcal{A}}_{T_n} = \tilde{\mathcal{A}}_{T_1 \cap \dots \cap T_n} = \{1\}$. Therefore, $z^m = 1$ and since $\tilde{\mathcal{A}}$ is torsion free, $z = 1$. This proves the claim.

Say $\langle z \rangle \cap \tilde{\mathcal{A}}_{T_j} = \{1\}$. Take a rank n abelian subgroup G of $\tilde{\mathcal{A}}_{T_j}$. The direct product, $\langle z \rangle \times G$ is a rank $n+1$ abelian subgroup of $\tilde{\mathcal{A}}$. This contradicts the first statement of the proposition and completes the proof. $\square$

2. Garside groups

Before proving our main theorems, we need to review the notion of a Garside group, introduced by Dehornoy and Paris in [DP]. For any monoid G^+, we can define a partial order on G^+ by $a < b$ if there exists $c \in G^+$ with $ac = b$. We say that G^+, together with $<$, is a *Garside monoid* if it satisfies the following conditions.

(1) There are no infinite descending chains in G^+.
(2) Left and right cancellation laws hold.
(3) $(G^+, <)$ is a lattice, that is, any two elements of G^+ have a least upper bound and a greatest lower bound.

(4) There exists an element Δ such that the left and right divisors of Δ are the same, there are finitely many of them, and they form a set of generators for G^+.

The element Δ is called a *Garside element* and the set of left divisors of Δ is denoted by M_Δ. It follows from property (3) that conjugation by Δ preserves M_Δ.

Given a Garside monoid G^+, we can form the group of fractions G whose elements are of the form $a\Delta^i$ for some $a \in G^+$, $i \in \mathbb{Z}$. If we require that $\Delta \not\leq a$, then every element of G can be uniquely written in this form. In particular, the monoid G^+ imbeds in the group G. Such a group is called a *Garside group*.[2] These have been studied extensively by Dehornoy, Picantin, and others. ([D1], [D2], [DP], [P1])

The most well known Garside monoids are the positive monoids A^+ of the finite type Artin groups. This is the monoid defined by the standard presentation for A. It consists of all elements of A which can be written as a product of positive powers of the standard generators. The element Δ projects to the longest element of the Coxeter group W corresponding to A. More recently, a different Garside structure for these groups was introduced by Bessis, Digne, and Michel in [BDM] and [Be]. It is this new Garside structure which will be relevant to our discussion.

Suppose (W, S) is a finite Coxeter system and R is the set of reflections in W. (An element of W is a reflection if it is conjugate to an element of S.) Let δ be the product of the generators, $\delta = s_1 s_2 \dots s_n$ (in any order). Then δ is called a *Coxeter element* of W. There are many ways to decompose δ as a product of n reflections, $\delta = r_1 r_2 \dots r_n$. Call a word in the free group on R *allowable* if it is an initial segment of such a decomposition. Let A_δ^+ be the monoid defined by the presentation

$$A_\delta^+ = \langle R \mid \; w_1 = w_2 \rangle$$

where the relations $w_1 = w_2$ run through all pairs w_1, w_2 of allowable words which represent the same element of W. Bessis [Be] has shown that this monoid is a Garside monoid with Garside element δ, and that the Artin group A associated to W is isomorphic to the group of fractions of A_δ^+ via the map induced by the inclusion of S into R. This gives a new Garside structure on A which we will call the δ-Garside structure. The generating set M_δ for this Garside structure can be identified with the subset of W corresponding to allowable words.

A word of caution is in order here: although elements of M_δ correspond to (allowable) elements of W, products of these elements do not satisfy all of the relations in W. It is best, therefore, to think of them as elements of $\mathcal{A}$. We describe how to do this for the Artin groups of type A_n and B_{n+1} in the example below.

Example 2.1. (i) Let W be the symmetric group on $n+1$ letters. Then the cyclic permutaion (1 2 3 … n+1) is a Coxeter element and the reflections are transpositions (i j). Now let $\mathcal{A}$ be the braid group on $(n+1)$-strands. To understand

[2] When first introduced in [DP] these groups were called "small Gaussian", and "Garside" was used for a slightly stronger condition. Since then, the definition above has become the accepted notion of a Garside group. In the case of interest here, namely the Artin groups, both definitions apply.

the δ-Garside structure on $\mathcal{A}$, we need to lift the reflections in W to elements of $\mathcal{A}$. For $1 \leq i < j \leq n+1$, let $a_{i,j}$ denote the braid which crosses the i^{th} string over the j^{th} string as shown below.

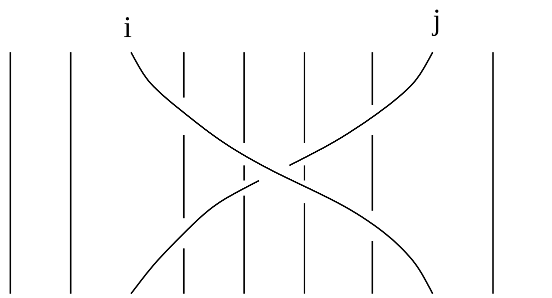

FIGURE 5. The braid $a_{i,j}$

This is a lift of the transposition (i j), and we will refer to it as a reflection. Note that $s_i = a_{i,i+1}$, $i = 1, \ldots, n$ are the standard generators for $\mathcal{A}$. Set $\delta = s_1 s_2 \ldots s_n$. It is shown in [BKL] that $\mathcal{A}_\delta^+$ can be identified with the submonoid of $\mathcal{A}$ generated by the reflections $R = \{a_{i,j}\}$. This is known as the Birman-Ko-Lee monoid.

(ii) We can do a similar construction for $\mathcal{B}$. The Coxeter group $W(B_{n+1})$ is now the symmetry group of the $(n+1)$-cube $[-1,1]^{n+1}$. The reflections in W are of three types: i) interchanging two factors of the product $[-1,1]^{n+1}$, ii) flipping a single factor, or iii)interchanging and flipping two factors. The first type we lift to the elements $a_{i,j}$ as above (now thought of as braids on a cylinder). The second type we lift to the element c_i which wraps the i^{th} string around the cylinder and the third type we lift to $a'_{i,j} = c_i a_{i,j} c_i^{-1}$ which has the effect of crossing the j^{th} string over the i^{th} around the back of the cylinder. As Coxeter element, we take $\delta = t s_1 s_2 \cdots s_n = c_1 s_1 s_2 \cdots s_n$.

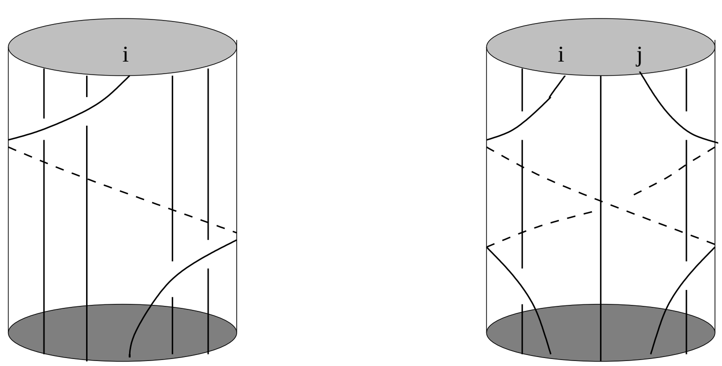

FIGURE 6. The cylindrical braids c_i and $a'_{i,j}$

It is shown in [P2] that $\mathcal{B}_\delta^+$ is the submonoid of $\mathcal{B}$ generated by the reflections $R = \{a_{i,j},\, a'_{i,j},\, c_i\}$. Note that all of these reflections lie in $\tilde{\mathcal{A}}$ except for the c_i's and that the c_i's are all δ-conjugate to t; in fact, $c_1 = t$ and $\delta c_i \delta^{-1} = c_{i+1}$. (In [Be], the positive monoid with respect to a different choice of δ is described which has the advantage that *every* reflection is δ-conjugate to one of the standard generators in S. However, for our purposes, $\delta = ts_1 \cdots s_n$ is more convenient since conjugation by this δ preserves the generating set $\tilde{S}$ of $\tilde{\mathcal{A}}$.)

In [D2], Dehornoy shows that every Garside group has a biautomatic structure based on a "left greedy" normal form. In the case of the usual Δ-structure on a finite type Artin group $\mathcal{A}$, this is the well-known normal form due to Deligne [De] which was shown to be biautomotic in [C]. In the case of the δ-structure on the classical braid group, it is the normal form given by Birman, Ko and Lee in [BKL].

We can use the δ-structure on $\mathcal{B}$ to get an explicit biautomatic structure for $\tilde{\mathcal{A}}$. The left greedy normal form for an element $b \in \mathcal{B}$ with respect to the generating set M_δ is defined as follows. First write $b = \delta^{-j} b_0$ where $b_0 \in \mathcal{B}_\delta^+$ and $\delta \not< b_0$. Then write $b_0 = \tau_1 b_1$ where τ_1 is the least upper bound of the set $\{\tau \in M_\delta \mid \tau < b_0\}$. Similarly, write $b_1 = \tau_2 b_2$ where τ_2 is the least upper bound of $\{\tau \in M_\delta \mid \tau < b_1\}$. Repeat this process with b_2, b_3, etc., until you reach the point where $b_k = 1$. This gives a decomposition of b as a word in the free group on M_δ, $b = \delta^{-j} \tau_1 \tau_2 \dots \tau_k$, satifying $\tau_i = lub\{\tau \mid \tau < \tau_i \dots \tau_k\}$. This decomposition is unique and defines a biautomatic structure on $\mathcal{B}$.

To get a biautomatic structure on $\tilde{\mathcal{A}}$, consider the left greedy normal form $a = \delta^{-j} \tau_1 \tau_2 \dots \tau_k$ for an element $a \in \tilde{\mathcal{A}} \subset \mathcal{B}$. We claim that $\phi(\tau_i) = 0$ or 1. This is because τ_i is an initial segment of a decomposition of δ into a product of $n+1$ reflections. Since each reflection $r \in \mathcal{B}$ has $\phi(r) = 0, 1$ and $\phi(\delta) = 1$, there must be exactly one reflection in this decomposition with $\phi(r) = 1$. It follows that $\phi(\tau_i) = 0, 1$. On the other hand, $\phi(a) = 0$ so the number of indicies i with $\phi(\tau_i) = 1$ must be exactly j. Since conjugation by δ preserves M_δ, we can "slide" copies of δ^{-1} through the τ_i's (replacing τ_i with its conjugate by δ as necessary) until one copy of δ^{-1} preceeds each τ_i with $\phi(\tau_i) = 1$. But if $\phi(\tau_i) = 1$, then $\delta = \tau_i^* \tau_i$ for some $\tau_i^* \in M_\delta$ (since left divisors of δ are also right divisors of δ) with $\phi(\tau_i^*) = 0$. Thus, replacing each $\delta^{-1} \tau_i$ with $(\tau_i^*)^{-1}$, we obtain a word in the free group on $M_\delta \cap \tilde{\mathcal{A}}$ representaing a. This defines a normal form on $\tilde{\mathcal{A}}$.

Theorem 2.2. *The normal form described above gives a biautomatic structure on $\tilde{\mathcal{A}}$.*

Sketch of proof. We must show that the normal form is a regular language and satisfies the "fellow traveler" properties. For $a \in \tilde{\mathcal{A}}$, let $N_{\tilde{\mathcal{A}}}(a)$ denote the normal form for a in $\tilde{\mathcal{A}}$ and $N_{\mathcal{B}}(a)$ denote the normal form for a in $\mathcal{B}$.

That $N_{\mathcal{B}}$ is a regular language follows from the fact that a word in M_δ is in normal form (with respect to $N_{\mathcal{B}}$) if and only if each adjacent pair $\tau_i \tau_{i+1}$ is in normal form [D2]. An adjacent pair of terms in $N_{\tilde{\mathcal{A}}}(a)$ determines the original pair of terms in $N_{\mathcal{B}}(a)$ up to conjugation by δ, thus the same holds for $N_{\tilde{\mathcal{A}}}$.

The fellow traveler properties for $N_{\tilde{\mathcal{A}}}$ follow from the fellow traveler properties for $N_{\mathcal{B}}$, together with the fact that left or right multiplication by an element of M_δ can add at most one δ^{-1} to $N_{\mathcal{B}}(a)$. Thus the number of δ^{-1}'s that are slid through $N_{\mathcal{B}}(a)$ to obtain $N_{\tilde{\mathcal{A}}}$ changes by at most one. We leave the details to the reader. $\square$

Motivated by a contruction of Bestvina in [Bv], Charney, Meier, and Whittlesey show in [CMW] that for any Garside group G with Garside element Δ, one can construct a $K(G,1)$-space $X(G,\Delta)$ as follows. Recall that M_Δ denotes the set of left divisors of Δ and that these form a generating set for G. The 1-skeleton of $X(G,\Delta)$ is the Cayley graph of G with respect to this generating set. That is, the vertices are indexed by G and two vertices g_1, g_2 are joined by an edge if and only if $g_1 = g_2 m^{\pm 1}$ for some $m \in M_\Delta$. The higher dimensional simplices of $X(G,\Delta)$ are defined by requiring $X(G,\Delta)$ to be a flag complex. That is, a set of vertices spans a simplex if and only if they are pairwise joined by edges. In [CMW], it is shown that $X(G,\Delta)$ is contractible, and G acts freely on $X(G,\Delta)$, hence the orbit space $X(G,\Delta)/G$ is a $K(G,1)$-space. The dimension of this complex is the length of a maximal chain of elements in M_Δ.

In the case of an Artin group with its δ-Garside structure, the construction of $X(\mathcal{A},\delta)$ is functorial in the following sense. Let (W,S) be the Coxeter system associated to $\mathcal{A}$ with $S = \{s_1,\dots,s_n\}$ and $\delta = s_1 s_2 \cdots s_n$. For any $T = \{s_{i_1}, s_{i_2}, \dots, s_{i_k}\} \subseteq S$, let $\delta_T = s_{i_1} s_{i_2} \cdots s_{i_k}$ where $i_1 < i_2 < \cdots < i_k$. Then δ_T is a Coxeter element for W_T, and if $T \subset T'$, then $\delta_T < \delta'_T$ in A^+_δ. In particular, $\delta_T \in M_\delta$ for all T. We claim that the subspace of $X(\mathcal{A},\delta)$ spanned by the vertices in $\mathcal{A}_T$ is naturally isomorphic to $X(\mathcal{A}_T,\delta_T)$. To see this, we need the following technical lemma.

Lemma 2.3. *Let R be the set of reflections in W and let $R_T = R \cap W_T$. Suppose $|T| = k$. If $r_1,\dots,r_k \in R$ are reflections such that $\delta_T = r_1 \cdots r_k$. Then $r_i \in R_T$ for $i = 1,\dots,k$.*

Proof. We can realize W as a reflection group on $\mathbb{R}^n$ with the generators $s_0,\dots,s_n$ acting as reflections in the walls of a simplicial cone, and we may assume the action is essential (ie., fixes only the origin). For any reflection r, let H_r denote the hyperplane fixed by r. Let F be the codimension k subspace $F = H_{s_{i_1}} \cap \cdots \cap H_{s_{i_k}}$. Then $\mathbb{R}^n$ decomposes as a sum $F \oplus F^\perp$ with W_T acting trivially on F and essentially on $F^\perp$. Since δ_T is a Coxeter element for W_T, it fixes only the origin in $F^\perp$ ([H], Lemma 3.16), thus the fixed point set of δ_T is precisley F.

Now let $G = H_{r_i} \cap \cdots \cap H_{r_k}$. Then G is fixed by $r_1 \cdots r_k = \delta_T$ so $G \subseteq F$. But $\operatorname{codim} G \leq k = \operatorname{codim} F$, so $F = G$. It follows that each H_{r_i} contains F, or in other words, r_i fixes F. The isotropy group of F is W_T, so r_i lies in W_T. □

Proposition 2.4. *The subspace of $X(\mathcal{A},\delta)$ spanned by the vertices in $\mathcal{A}_T$ is naturally isomorphic to $X(\mathcal{A}_T,\delta_T)$.*

Proof. Since $X(\mathcal{A}_T,\delta_T)$ is a flag complex, it suffices to show that two vertices $a,b \in \mathcal{A}_T$ are connected by an edge in $X(\mathcal{A},\delta)$ if and only if they are connected by an edge in $X(\mathcal{A}_T,\delta_T)$. By the previous lemma and the fact that $\delta_T \in M_\delta$, a word in the free group on R_T is allowable (with respect to δ_T) if and only if it is allowable as a word in the free group on R (with respect to δ). It follows that $M_{\delta_T} = M_\delta \cap W_T$ as required. □

3. Main Theorems

Although the construction of $X(\mathcal{A},\delta)$ described in the previous section is defined only for finite type Artin groups, we can modify $X(\mathcal{B},\delta)$ to obtain a $K(\pi,1)$-space for the affine braid group $\tilde{\mathcal{A}}$. Let $X = X(\mathcal{B},\delta)$ and let $X^+ = X^+(\mathcal{B},\delta)$ denote the full subcomplex of X spanned by the vertices $a \in \mathcal{B}^+_\delta$ with $\delta \not\leq a$. In [CMW] it is shown that X decomposes as a product $X = X^+ \times \mathbb{R}$. In particular, X^+ is contractible. We can also think of the vertices of X^+ as cosets $\mathcal{B}/\langle\delta\rangle$ since any element of $\mathcal{B}$ has a unique expression of the form $a\delta^i$ with a as above. Viewing the vertices as cosets, we get an action of $\mathcal{B}$ on X^+ such that the projection of X onto X^+ is $\mathcal{B}$-equivariant.

Theorem 3.1. *$\tilde{\mathcal{A}}$ acts freely on X^+ and acts transitively on the vertices of X^+, hence $X^+/\tilde{\mathcal{A}}$ is a finite $K(\tilde{\mathcal{A}},1)$-space of dimension n.*

Proof. Identify $\tilde{\mathcal{A}}$ with the kernel of the homomorphism $\phi : \mathcal{B} \to \mathbb{Z}$. Since $\phi(\delta) = 1$, any coset $b\langle\delta\rangle$ contains a unique representative a with $\phi(a) = 0$. Thus $\tilde{\mathcal{A}}$ acts freely and transitively on the set of vertices of X^+. If $a \in \tilde{\mathcal{A}}$ fixes a point in the relative interior of a simplex σ of X^+, then it must permute the vertices of σ, hence some power of a fixes the vertices of σ. Since $\tilde{\mathcal{A}}$ is torsion-free, we conclude that a itself must be the identity element. Thus $\mathcal{A}$ acts freely on X^+.

The quotient is finite since every simplex in X^+ is the translate of a simplex containing the identity vertex. The dimension of X is the length of a maximal chain in M_δ, or equivalently, the length of δ as a product of reflections. Thus, $\dim(X^+) = \dim(X) - 1 = n$. $\square$

Corollary 3.2. *The cohomological dimension of $\tilde{\mathcal{A}}$ is n.*

Proof. The dimension of a $K(\pi,1)$-space gives an upper bound for the cohomolgical dimension. A lower bound is given by the rank of a maximal abelian subgroup. Thus the corollary follows from Proposition 1.3 and Theorem 3.1. $\square$

We are now ready to prove the $K(\pi,1)$ Conjecture for the affine braid group $\tilde{\mathcal{A}}$. Let $\tilde{W} = W(\tilde{A}_n)$ be the Coxeter group corresponding to $\tilde{\mathcal{A}}$ and let $\mathcal{H}_{\tilde{W}}$ be the associated hyperplane complement. In [L2] and [CD1], it is shown that $\mathcal{H}_{\tilde{W}}/\tilde{W}$ has fundamental group $\tilde{\mathcal{A}}$ and its universal cover is homotopy equivalent to a certain simplicial complex known as the Deligne complex for $\tilde{\mathcal{A}}$. Thus, to prove the $K(\pi,1)$ Conjecture, it suffices to prove that the Deligne complex is contractible.

The Deligne complex for $\tilde{\mathcal{A}}$ is defined as follows. Let $\tilde{S} = \{s_0, s_1, \ldots s_n\}$ be the standard generating set for $\tilde{\mathcal{A}}$ and let $\mathcal{P} = \mathcal{P}(\tilde{S})$ be the set of proper subsets of $\tilde{S}$ (including the empty set). Define

$$\tilde{\mathcal{A}}\mathcal{P} = \{a\tilde{\mathcal{A}}_T \mid a \in \tilde{\mathcal{A}},\ T \in \mathcal{P}\}$$

(where $\tilde{\mathcal{A}}_\emptyset = \{1\}$) to be the poset of special cosets partially ordered by inclusion. Then the *Deligne complex*, $\mathcal{D}(\tilde{\mathcal{A}})$, is the geometric realization of this poset. (Deligne introduced this complex for finite type Artin groups in [De].)

Theorem 3.3. *Let $T_0 = \{s_1, \dots, s_n\} \subset \tilde{S}$, and let X_0^+ be the subspace of X^+ spanned by the vertices in $\tilde{\mathcal{A}}_{T_0}$. Let $\mathfrak{U}$ denote the collection of $\mathcal{B}$-translates of X_0^+, and $N(\mathfrak{U})$ the nerve of $\mathfrak{U}$. Then*

(1) *$\mathfrak{U}$ covers X^+,*
(2) *the barycentric subdivision of $N(\mathfrak{U})$ is isomorphic to the Deligne complex $\mathcal{D}(\tilde{\mathcal{A}})$, and*
(3) *every nonempty intersection of sets in $\mathfrak{U}$ is contractible.*

Thus, $\mathcal{D}(\tilde{\mathcal{A}})$ is homotopy equivalent to the contractible space X^+.

Proof. (1) It suffices to show that every maximal simplex σ in X^+ is a $\mathcal{B}$-translate of a simplex in X_0^+. There are three ways we can label vertices of X^+, as elements of $\tilde{\mathcal{A}}$, as elements of the positive monoid $\mathcal{B}_\delta^+$ with $\delta \not< a$, or as cosets of $\langle\delta\rangle$ in $\mathcal{B}$. The proof exploits the interplay among these. Note, for example, that if we label the vertices by elements of $\tilde{\mathcal{A}}$, then the left action of δ on X^+ has the effect of moving the vertex a to the vertex $\delta a \delta^{-1}$.

Let σ be a maximal simplex. Up to translation, we may assume that σ contains the identity vertex $*$. First consider the vertices of σ as elements of $\mathcal{B}_\delta^+$. Since every vertex is connected to the identity vertex by an edge, it must correspond to an admissible element $\tau < \delta$. Since any two vertices in σ are connected by an edge, the set of vertices is totally ordered under $<$, hence σ corresponds to a maximal chain of admissible elements $* < \tau_1 < \tau_2 < \cdots < \tau_n < \delta$. Any such chain is given by a decomposition of δ into a product of reflections $\delta = r_1 r_2 \cdots r_{n+1}$ where $\tau_i = r_1 r_2 \cdots r_i$.

Now let us relabel these vertices as elements of $\tilde{\mathcal{A}}$. To do so, we must find the unique element of $\tilde{\mathcal{A}}$ which lies in the coset $\tau_i\langle\delta\rangle$. Recall that $\tilde{\mathcal{A}}$ is the kernel of the homomorphism $\phi : \mathcal{B} \to \mathbb{Z}$ which counts the exponent sum of the generator t. Clearly $\phi(\delta) = 1$ and since every reflection r is a conjugate of an element of S, $\phi(r) = 0, 1$. It follows that exactly one of the r_i's in the decompostion of δ is conjugate to t, say r_j. As noted in Example 2.1, r_j is, in fact, δ-conjugate to t. Since the action of δ on X^+ has the effect of conjugating vertices by δ, we may assume that $r_j = t$. The remaining r_i's lie in $\tilde{\mathcal{A}}$. Thus, τ_i lies in $\tilde{\mathcal{A}}$ for $i < j$ and $\tau_i\delta^{-1}$ lies in $\tilde{\mathcal{A}}$ for $i \geq j$.

We claim that $\delta^{-1}\sigma$ lies in X_0^+ and hence σ lies in δX_0^+. To see this, note that

$$\delta^{-1} r_1 r_2 \cdots r_{j-1} t r_{j+1} \cdots r_{n+1} = 1.$$

Letting $\bar{r}_i = \delta^{-1} r_i \delta$, we obtain

$$\bar{r}_1 \bar{r}_2 \cdots \bar{r}_{j-1} \delta^{-1} t r_{j+1} \cdots r_{n+1} = 1,$$

and thus

$$r_{j+1} \cdots r_{n+1} \bar{r}_1 \cdots \bar{r}_{j-1} = t^{-1}\delta = s_0 s_1 \cdots s_n.$$

It now follows from Lemma 2.3 that $\bar{r}_i \in \mathcal{B}_{T_0}$ for $i < j$ and $r_i \in \mathcal{B}_{T_0}$ for $i > j$. Viewed as elements of $\tilde{\mathcal{A}}$, the vertices of $\delta^{-1}\sigma$ are $\delta^{-1}\tau_i\delta = \bar{r}_1 \dots \bar{r}_i$ for $i < j$ and

$\delta^{-1}\tau = \bar{r}_1 \dots \bar{r}_{j-1}(\delta^{-1}t)r_{j+1}\dots r_i$ for $i \geq j$. By the discussion above, these all live in $\mathcal{B}_{T_0} = \tilde{\mathcal{A}}_{T_0}$ This proves the claim and completes the proof of statement (1).

(2) We next consider the nerve of $\mathfrak{U}$. Recall that the inclusion of $\tilde{\mathcal{A}}$ in $\mathcal{B}$ takes $\tilde{S} = \{s_0, s_1, \dots, s_n\}$ to the set $\{\delta^{-1}s_1\delta, s_1, \dots, s_n\}$ and that δ acts on this set by cyclic permutation (Example 2.1). In particular, if $T_i = \tilde{S}\backslash\{s_i\}$, then $\delta T_i \delta^{-1} = T_{i+1}$, where the indices are taken mod $(n+1)$. The T_i's are the maximal elements of $\mathcal{P}$. Let X_i^+ denote the span of the vertices $\tilde{\mathcal{A}}_{T_i}$ in X^+.

Let $b \in \mathcal{B}$ and write $b = a\delta^i$ where $a \in \tilde{\mathcal{A}}$. Then

$$bX_0^+ = a\delta^i X_0^+ = aX_i^+$$

It follows that every element of $\mathfrak{U}$ is the span of a maximal coset in $\tilde{\mathcal{A}}\mathcal{P}$, and an intersection of elements of $\mathfrak{U}$ is the span of the corresponding intersection of cosets. A non-empty intersection of special cosets in an Artin group is again a special coset and every special coset in $\mathcal{P}$ can be obtained as an intersection of maximal ones. This proves that the poset of simplices of $N(\mathfrak{U})$ is isomorphic to the poset $\mathcal{P}$, or in other words, the barycentric subdivision of $N(\mathfrak{U})$ is the Deligne complex for $\tilde{\mathcal{A}}$.

(3) Suppose Y is the intersection of a collection of sets in $\mathfrak{U}$. Up to translation by an element of $\mathcal{B}$, we may assume that one of these sets is X_0^+. Then by the discussion above, Y is the span of the vertices lying in some special coset $a\tilde{\mathcal{A}}_T$ with $T \subset T_0$. Translating by a^{-1}, we may, in fact, assume that Y is the span of $\tilde{\mathcal{A}}_T$.

We claim that Y is isomorphic to the subcomplex X_T of X spanned by $\tilde{\mathcal{A}}_T$ ($=\mathcal{B}_T$). The projection $\pi : X \to X^+$ maps the vertices of X_T isomorphically onto the vertices of Y since $\tilde{\mathcal{A}}_T \cap \langle\delta\rangle = \emptyset$. Since both X_T and Y are flag complexes, it suffices to show that every edge in Y is the image of an edge in X_T. Two vertices $a, b \in \tilde{\mathcal{A}}_T$ are connected by an edge in Y if and only if $ac = b\delta^i$ for some $c \in M_\delta$ and $i \in \mathbb{Z}$. Since $\phi(a) = \phi(b) = 0$, we must have $\phi(c) = \phi(\delta^i) = i$. But $c < \delta$ implies that $\phi(c) = 0, 1$. If $\phi(c) = 1$, then the element $c^* \in M_\delta$ with $c^*c = \delta$ must satisfy $\phi(c^*) = 0$. Thus we have two cases. Either $ac = b$ with $\phi(c) = 0$, or $a = bc^*$ with $\phi(c^*) = 0$. In either case, a and b are connected by an edge in X_T. By Proposition 2.4, X_T is contractible, so this completes the proof of statement (3). $\square$

Corollary 3.4. *The $K(\pi, 1)$ Conjecture holds for $\tilde{\mathcal{A}}$.*

Remark. Let $\delta_0 = s_1 \dots s_n$ and $\delta_i = \delta^i\delta_0\delta^{-i}$. Then δ_i is a Coxeter element for $\tilde{\mathcal{A}}_{T_i}$ and $M_{\delta_i} = M_\delta \cap \tilde{\mathcal{A}}_{T_i}$. It is clear from the proof of the theorem, that the one-skeleton of X^+ can be identified with the Cayley graph for $\tilde{\mathcal{A}}$ with respect to the generating set $M = \bigcup M_{\delta_i}$. This construction could be imitated for other infinite type Artin groups. Namely, choose a compatible set $\{\delta_T\}$ of δ-elements for the finite type special subgroups of $\mathcal{A}$ and take M to be the union of the generating sets M_{δ_T}. Let $X(\mathcal{A}, \{\delta_T\})$ be the flag complex whose one-skeleton is the Cayley graph for $\mathcal{A}$ with respect to M. It seems plausible that this space is a $K(\mathcal{A}, 1)$-space, at least for the Euclidean Artin groups, but we don't know how to prove it.

There is another finite, n-dimensional complex which is conjectured to be a $K(\tilde{\mathcal{A}}, 1)$-space, namely the Salvetti complex for $\tilde{\mathcal{A}}$. This complex, which we will denote by $\Sigma(\tilde{\mathcal{A}})$, has one cell (a "Coxeter cell") for each $T \in \mathcal{P}$. We give a brief description of $\Sigma(\tilde{\mathcal{A}})$ and refer the reader to [CD2] for details.

For any $T \in \mathcal{P}$, take the standard Garside structure on $\tilde{\mathcal{A}}_T$ with Garside element $\Delta(T)$ corresponding to the longest element in $\tilde{W}_T$. For this Garside structure, the elements of $M_{\Delta(T)}$ (i.e., the left divisors of $\Delta(T)$) are in one-to-one correspondence with elements of $\tilde{W}_T$. For $a \in \tilde{\mathcal{A}}$, $aM_{\Delta(T)}$ denotes the subset of $\tilde{\mathcal{A}}$ of elements ac, $c \in M_{\Delta(T)}$. Let

$$Sal(\tilde{\mathcal{A}}) = \{aM_{\Delta(T)} \mid a \in \tilde{\mathcal{A}},\ T \in \mathcal{P}\}$$

be the poset of such subsets, ordered by inclusion. There is a left action of $\tilde{\mathcal{A}}$ on this poset and the Savetti complex is obtained by taking the geometric realization of this poset modulo the action of $\tilde{\mathcal{A}}$.

$$\Sigma(\tilde{\mathcal{A}}) = |Sal(\tilde{\mathcal{A}})|/\tilde{\mathcal{A}}.$$

Salvetti [Sa] proved that $\Sigma(\tilde{\mathcal{A}})/\tilde{\mathcal{A}}$ is homotopy equivalent to the hyperplane complement $\mathcal{H}_{\tilde{W}}/\tilde{W}$. Thus we obtain

Corollary 3.5. *The Salvetti complex $\Sigma(\tilde{\mathcal{A}})$ is a $K(\tilde{\mathcal{A}}, 1)$-space.*

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