THE DELIGNE COMPLEX FOR THE FOUR-STRAND BRAID GROUP

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ABSTRACT. This paper concerns the homotopy type of hyperplane arrangements associated to infinite Coxeter groups acting as reflection groups on $\mathbb{C}^n$. A long-standing conjecture states that the complement of such an arrangement should be aspherical. Some partial results on this conjecture were previously obtained by the author and M. Davis. In this paper, we extend those results to another class of Coxeter groups. The key technical result is that the spherical Deligne complex for the 4-strand braid group is CAT(1).

1. INTRODUCTION

The motivation for this paper concerns the homotopy type of certain complex hyperplane arrangements, namely those associated to infinite Coxeter groups acting as reflection groups on $\mathbb{C}^n$. There is a long-standing conjecture that states that the complement of such an arrangement should be aspherical. Some partial results on this problem were obtained by the author and M. Davis in [5] and by the author and D. Peifer in [7]. Picking up where [5] left off, this paper proves the conjecture for some new classes of infinte Coxeter groups.

We begin by recalling some basic facts about Coxeter groups and Artin groups. A *Coxeter group* is a group with presentation of the form

$$W = \langle S \mid s^2 = 1, (st)^{m(s,t)} = 1, \ \forall s, t \in S \rangle$$

where S is a finite set and $m(s,t) = m(t,s) \in \{2, 3, \dots, \infty\}$. (If $m(s,t) = \infty$, we omit the relation.) The the pair (W, S) is called a *Coxeter system.*

Associated to any Coxeter system is an *Artin system* (A, S) where A is the group defined by the presentation

$$A = \langle S \mid prod(s,t; m(s,t)) = prod(t,s; m(s,t)), \ s, t \in S \rangle.$$

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where $prod(s,t;m(s,t))$ denotes the alternating word $stst...$ of length $m(s,t)$. Adding the relations $s^2 = 1$ gives back the presentation for (W, S), thus W is a quotient of A. Artin groups corresponding to finite Coxeter groups are known as *finite type* Artin groups.

If $T \subset S$, and W_T is the subgroup of W generated by T, then (W_T, T) is a Coxeter system. Likewise, if A_T denotes the subgroup of A spanned by T, then (A_T, T) is the Artin system associated to (W_T, T). The groups W_T and A_T are called *special subgroups* of W and A, respectively, and their conjugates are called *parabolic subgroups.*

Any finite Coxeter system (W, S) can be realized as a group of linear transformations of $\mathbb{R}^n$, $n = |S|$, with the elements of S acting as orthogonal reflections in the walls of a simplicial cone. For each reflection r in W (r acts as a reflection if it is conjugate to an element of S), let H_r denote the hyperplane fixed by r. Then W acts freely on the complement of these hyperplanes. Complexifying the action, we get an action of W on $\mathbb{C}^n$ which is free on the complement of the complex hyperplanes $\mathbb{C}H_r = H_r \oplus iH_r$. Let $\mathcal{H}_W = \mathbb{C}^n \backslash (\bigcup_r \mathbb{C}H_r)$ be the hyperplane complement. Deligne [11] showed that $\mathcal{H}_W/W$ is aspherical and its fundamental group is the Artin group associated to W. In other words, $\mathcal{H}_W/W$ is a $K(A,1)$-space.

More generally, any Coxeter group acts as a reflection group on an n-dimensional vector space V with the generators acting as reflections in the walls of a simplicial cone Z. In the case of an infinite Coxeter group, the reflections are orthoganal with respect to a certain bilinear form on V, but the form is not positive definite. In this case, the picture is not as nice: Z contains points with infinite isotropy group and the translates of Z do not cover V. However, Vinberg [18] proved that if we remove the points of infinite isotropy, we get a subspace Z^0 whose W-translates form an open cone Ω in V, known as the Tits cone, and the reflection hyperplanes $\mathbb{C}H_r$ form a locally finite arrangement in Ω. We define the hyperplane complement associated to W to be the space

$$\mathcal{H}_W = (\Omega \oplus i\Omega) \backslash (\bigcup_r \mathbb{C}H_r)$$

For example, if W is an irreducible Euclidean Coxeter group, then the action of W on V preserves an affine hyperplane $\mathbb{A}^{n-1}$. In this case, Z^0 is $Z - \{0\}$ and Ω is an open halfspace of V containing $\mathbb{A}$. Thus, there is a linear retraction of $\mathcal{H}_W$ onto the complement of a locally finite arrangement of hyperplanes in the complex affine space $\mathbb{A} \oplus i\mathbb{A}$.

We are interested in the following conjecture which is attributed to Arnold, Pham, and Thom.

Conjecture. For an infinte Coxeter system (W, S) with associated Artin group A, the orbit space $\mathcal{H}_W/W$ is a $K(A, 1)$-space.

It was shown by van der Lek [17] in the early 80's that $\pi_1(\mathcal{H}_W/W) = A$. Thus, to prove the conjecture, one must show that the universal covering space of the hyperplane complement is contractible. This problem was addressed by the author and M. Davis in [5]. In that paper, it is shown that the universal covering space of $\mathcal{H}_W$ is homotopy equivalent to a certain simplicial complex $\mathcal{D}_A$, known as the Deligne complex of A. To show that this simplicial complex is contractible, the authors proposed to find a metric of "nonpositive curvature" on $\mathcal{D}_A$. They considered two metrics, a cubical metric and a more natural metric based on a construction of Moussong. The cubical metric was determined to be nonpositively curved if and only if the Coxeter system (W, T) satisfies the "FC condition" (i.e., the condition that W_T is finite if and only if $m(s, t) < \infty$ for all $s, t \in T$). The Moussong metric was shown to be nonpositively curved if $\mathcal{D}_A$ is 2-dimensional (i.e., if W_T is infinite whenever $|T| > 2$) and conjectured to be nonpositively curved for all $\mathcal{D}_A$. In this paper, we prove that the Moussong metric is nonpositively curved for many of the 3-dimensional, and some higher dimensional, Deligne complexes (Corollaries 5.5 and 5.6). The $K(\pi, 1)$ conjecture and a number of algebraic properties for these groups follow immediately.

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2. Coxeter and Deligne Complexes

The notion of "non-positive curvature" on a geodesic metric space was defined by Gromov in [14] in terms of a "thin triangle" condition. A metric space satisfying this condition globally, is called a CAT(0) space. We begin with a quick review CAT(0) spaces. The reader is refered to [1] for more details.

Recall that a *piecewise Euclidean complex* is a cell complex obtained by gluing together polyhedral Euclidean cells via isometries of faces. Given such a complex, the induced metric is defined by setting $d(x, y)$ equal to the infimum of the lengths of paths from x to y. Assuming that there are only finitely many isometry types of simplicies (which will always be the case in this paper), this metric is complete and geodesic, i.e., the infimum is always realised by some path. Such a path is called a *geodesic* from x to y.

A *piecewise spherical complex* and its induced metric are defined similarly. If X is a piecewise Euclidean complex, then for any vertex

$v \in X$, $link\,(v, X)$ is defined as the unit tangent space to X at v. The piecewise Euclidean structure on X induces a piecewise spherical structure on $link\,(v, X)$ in the ovious manner.

We say that X is *CAT(0)* if geodesic triangles in X are "at least as thin" as comparison triangles in $\mathbb{R}^2$. That is, the distance between any two points on a geodesic triangle Δ in X is no greater than the distance between the corresponding points on a triangle Δ' of the same side lengths in $\mathbb{R}^2$. It is immediate from the definition that geodesics in a CAT(0) space are unique and vary continuously with their endpoints. *It follows that any CAT(0) space is contractible.*

To verify that a metric on X is CAT(0), we must study the induced piecewise spherical metric on the links of vertices. We say a peicewise spherical complex L is CAT(1) if geodesic triangles in L (of perimeter $\leq 2\pi$) are "at least as thin" as comparison triangles in $\mathbb{S}^2$. Proofs of the following properties can be found in [1].

Theorem 2.1. *(i) A piecewise Euclidean complex X is CAT(0) if and only if X is simply connected and the link of every vertex in X is CAT(1).*
(ii) A piecewise spherical complex L is CAT(1) if and only if the link of every vertex in L is CAT(1) and L has no locally geodesic loops of length less than 2π.

Now let (W, S) be a Coxeter system and let A be the associated Artin group. Two complexes will play a key role in the proof of the main theorem: the Coxeter-Davis complex $\mathcal{C}_A$ for W and the Deligne complex $\mathcal{D}_A$ for A.

We begin with some notation. Let

$$\begin{aligned} \mathcal{S}^f &= \{T \mid T \subseteq S,\ W_T \text{ is finite}\} \\ W\mathcal{S}^f &= \{wW_T \mid w \in W,\ T \in \mathcal{S}^f\} \end{aligned}$$

Here, we include $T = \emptyset$ with the convention that $W_\emptyset = \{1\}$. These sets are partially ordered by inclusion. The *Coxeter-Davis complex* for W, which we denote by $\mathcal{C}_W$, is the flag complex associated to the poset $W\mathcal{S}^f$. That is, $\mathcal{C}_W$ is the simplicial complex whose vertices are the elements of $W\mathcal{S}^f$ and whose k-simplices are totally ordered subsets (or "flags") of length k+1. This complex, introduced by M. Davis in [10], is a modification of the classical Coxeter complex defined by Tits. W acts by left multiplication on $\mathcal{C}_W$ and a fundamental domain for this action is the subcomplex K spanned by the vertices W_T. Note that this subcomplex is isomorphic to the flag complex associated to $\mathcal{S}^f$.

Now suppose W is finite. Then $\mathcal{S}^f$ is the set of all subsets of S and hence K can be combinatorially identified with a cube of dimension

$n = |S|$. (To see the cubical structure, ignore all edges except those corresponding to inclusions $W_T \subset W_{T'}$ with $|T'| = |T|+1$.) Moreover, $W\mathcal{S}^f$ has a unique maximal element $W = W_S$, so $\mathcal{C}_W$ is a cone.

For finite W, $\mathcal{C}_W$ can be given the metric of a convex polyhedral cell as follows. Realize W as a group of orthogonal transformations of $\mathbb{R}^n$ with the generators S acting as reflections in the walls of a simplicial cone Z. There is a unique point $x_\emptyset$ in the interior of Z whose distance from every wall is 1. Let $X \subset \mathbb{R}^n$ be the convex hull of the orbit $Wx_\emptyset$. Then X is equivariantly isomorphic to $\mathcal{C}_W$ with the origin corresponding to the cone point in K, $x_\emptyset$ corresponding to $W_\emptyset$, and K identified with $Z \cap X$. The complex $\mathcal{C}_W$, with this metric, is known as the *Coxeter cell* for W.

Returning to the case of a (possibly) infinite W, Moussong [15] defined a piecewise Euclidean metric on the Coxeter-Davis complex as follows. We can cover $\mathcal{C}_W$ by the closed stars of the maximal vertices wW_T (maximal with respect to the partial order on $W\mathcal{S}^f$). This star can be naturally identified with the Coxeter-Davis complex for W_T (translated by w). Since W_T is finite, we can assign this star the metric of the Coxeter cell $\mathcal{C}_{W_T}$. These cells fit together to give a peicewise Euclidean structure on $\mathcal{C}_W$. Moussong proved that the induced metric is CAT(0) for all W.

The Deligne complex for the Artin group A is defined analogously. Let

$$A\mathcal{S}^f = \{aA_T \mid a \in A,\ T \in \mathcal{S}^f\}$$

partially ordered by inclusion. Then the *Deligne complex*, $\mathcal{D}_A$, is the flag complex associated to this poset. There is a left action of A on $\mathcal{D}_A$ and the fundamental domain for this action is again K (viewed as the subcomplex spanned by the vertices A_T). The quotient map $A \to W$ induces an equivariant projection of $\mathcal{D}_A \to \mathcal{C}_W$ which is an isomorphism on K. We define a peicewise Euclidean structure on $\mathcal{D}_A$ by declaring each translate of K to be isometric to its image in $\mathcal{C}_W$. The induced metric d on $\mathcal{D}_A$ is called the *Moussong metric.*

To prove that the Moussong metric on $\mathcal{D}_A$ is CAT(0), we need to look at the links of vertices in $\mathcal{D}_A$. First consider the case in which W is finite. The link of the cone point in the Coxeter cell $\mathcal{C}_S$ is an $(n-1)$-sphere with the standard, round metric. Simplicially (with respect to the cubical structure on $\mathcal{C}_S$), this link has one top-dimensional simplex (or "chamber") for each element of W and one k-simplex for each coset wW_T with $|T| = n-1-k$. This simplicial decomposition of the sphere is called the *spherical Coxeter complex* for W. We denote it by $\mathcal{C}^0_W$.

Similarly, in the case where W is finite (so A is finite type), $\mathcal{D}_A$ is a cone with cone point $A = A_S$. The link of the cone point, with the induced piecewise spherical metric, is the *spherical Deligne complex* for A which we denote by $\mathcal{D}_A^0$. It has one k-simplex for each coset aA_T with $|T| = n - 1 - k$. Note that $\mathcal{C}_W^0$ and $\mathcal{D}_A^0$ are defined only for finite Coxeter groups W.

Now consider the link of a vertex in $\mathcal{D}_A$ for an infinite type Artin group A. Any vertex of $\mathcal{D}_A$ is the translate of some vertex $v = A_T$ in K. The star of v is spanned by two types of vertices: those greater than A_T (with respect to the partial order on $A\mathcal{S}^f$) and those less than A_T. This gives rise to a decomposition of *link* $(v, \mathcal{D}_A)$ as an orthogonal join as described in the following lemma. (For a proof of this lemma see [2], Lemma 2.2.)

Lemma 2.2. *Let $T \in \mathcal{S}^f$ and let $v = A_T$. Let $K_{\geq T}$ be the subcomplex of K spanned by $\{A_R \mid R \geq T,\ R \in \mathcal{S}^f\}$. Then link $(v, \mathcal{D}_A)$ is isometric to the orthonganal join*

$$\mathit{link}\,(v, \mathcal{D}_A) = \mathit{link}\,(v, K_{\geq T}) * \mathcal{D}^0_{A_T}.$$

Proposition 2.3. *The Moussong metric on the Deligne complex $\mathcal{D}_A$ is CAT(0) if and only if the spherical Deligne complex $\mathcal{D}^0_{A_T}$ is CAT(1) for every $T \in \mathcal{S}^f$.*

Proof. In [5] it is shown that $\mathcal{D}_A$ is simply connected, thus by Theorem 2.1 it suffices to prove that the link of every vertex is CAT(1). It was shown by Moussong [15] that link$(A_T, K_{\geq T})$ is CAT(1) for all $T \in \mathcal{S}^f$. By [5] (appendix), the join of two CAT(1) spaces is CAT(1), so the proposition follows from Lemma 2.2. □

Thus, to prove the hyperplane complement conjecture, it would suffice to show that the spherical Deligne complex for every finite type Artin group is CAT(1). It seems that a good place to start is with the Artin groups of *small type*, that is, Artin groups for which all $m(s,t) \in \{2,3\}$. Crisp [9] has shown that every finite type Artin group A injects into a finite type Artin group A^{small} of small type. These homomorphisms induce equivariant maps between the corresponding Deligne complexes.

Proposition 2.4. *Suppose that the spherical Deligne complexes for every small, finite type Artin group are CAT(1). Then the same holds for every finite type Artin group containing no irreducible factor of type H_3 or H_4. (where H_i refers to the standard Dynkin diagrams for finite irreducible Coxeter groups).*

Proof. Suppose A is an irreducible, finite type Artin group. For A of rank 2, the spherical Deligne complex was shown to be CAT(1) in [5]. For rank ≥ 3, if A is not of type H_3 or H_4, then A can be realized as the fixed set of an involution of some small, finite type Artin group. Thus, the proposition follows immediately from a theorem of Crisp, [8], Theorem 23. □

The basis of Crisp's argument is that the fixed set of a finite group action on a CAT(0) space is a convex subset, so the inclusion of $\mathcal{D}_A$ into $\mathcal{D}_{A^{small}}$ is an isometric embedding. We conjecture that for A of type H_3 and H_4, the inclusions into small type Artin groups (of type D_6 and E_8) also induce isometric embeddings on Deligne complexes, but as these groups do no arise as fixed point sets, a different method of proof will be required.

In this paper we consider the first interesting case of a spherical Deligne complex for an Artin group of small type, namely, the two-dimensional complex associated to the braid group on four strands. (This group is the only rank 3 Artin group which is irreducible, small, and finite type.) Deligne complexes are "building-like" in the sense that they can be covered by Coxeter complexes and links of vertices are lower dimensional (spherical) Deligne complexes. We hope that some inductive argument, along the lines of [6], Theorem 4.1, will allow us to extend these results to higher rank Artin groups of small type in the future. Our experience with spherical buildings suggests that the 2-dimensional case is special and any induction argument will have to start at dimension two.

3. The Coxeter Complex for the symmetric group

We now specialize to the case where $W = Sym_4$ is the symmetric group on four letters and the associated Artin group A is the four-strand braid group. In this section we describe an explicit model for the spherical Coxeter complex $\mathcal{C}^0 = \mathcal{C}^0_W$ and the spherical Deligne complex $\mathcal{D}^0 = \mathcal{D}^0_A$.

Let V be the 3-dimensional vector space

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^n \mid \Sigma x_i = 0\}$$

Then W acts on V by permuting the coordinates. A transposition (i, j) in W fixes the hyperplane $H_{i,j} \subset V$ defined by $x_i = x_j$. Consider the unit sphere $\mathbb{S}(V) = \mathbb{S}^4 \cap V$. The six hyperplanes $H_{i,j}$ divide $\mathbb{S}(V)$ into simplicies. This simplicial decomposition of the 2-sphere is a realization of the spherical Coxeter complex for W. From now on, we identify $\mathcal{C}^0$ with this realization.

The 2-dimensional simplices of $\mathcal{C}^0$ are called *chambers.* W acts freely, transitively on the set of chambers, thus all chambers are isometric to each other. Consider the chamber defined by $x_1 \geq x_2 \geq x_3 \geq x_4$. We call this the *fundamental chamber* and denote it by K^0. The sides of this chamber are formed by the hyperplanes $H_{1,2}$, $H_{2,3}$, and $H_{3,4}$.

The standard Coxeter presentation for W is

$$W = \langle r, s, t \mid r^2 = s^2 = t^2 = 1,\ (rt)^2 = (rs)^3 = (st)^3 = 1 \rangle$$

where r, s, t are the transpositions

$$r = (1,2) \quad s = (2,3) \quad t = (3,4).$$

With this notation, the sides of the triangle K^0 are formed by the intersection of $\mathbb{S}(V)$ with the hyperplanes H_r, H_s, and H_t and we refer to them as *edges of type* r, s, *or* t respectively. The vertex between the edges of type r and s, will be called a *vertex of type* $\{r,s\}$ and similarly for vertices of type $\{r,t\}$ and $\{s,t\}$. Thus, a vertex or edge of type $T \subset \{r,s,t\}$ is fixed by the subgroup W_T. For any other chamber wK^0, we define the type of the edges and vertices of wK^0 similarly, so that the vertex or edge of type T is fixed by the parabolic subgroup wW_Tw^{-1}.

Lemma 3.1. *The chambers of* $\mathcal{C}^0$ *satisfy the following*

(1) *The angle at a vertex of type* $\{r,s\}$ *or* $\{s,t\}$ *is* $\pi/3$ *and the angle at a vertex of type* $\{r,t\}$ *is* $\pi/2$.
(2) *the length of an edge of type* r *or* t *is* $\alpha = arccos(\frac{1}{\sqrt{3}})$ *and the length of an edge of type* s *is* $\beta = arccos(\frac{1}{3})$.
(3) *These lengths satisfy* $2\alpha > \beta > \alpha$ *and* $2\alpha + \beta = \pi$.

Proof. The normal vectors to H_r, H_s, H_t are

$$\mathbf{n}_r = \frac{1}{\sqrt{2}}(1,-1,0,0) \quad \mathbf{n}_s = \frac{1}{\sqrt{2}}(0,1,-1,0) \quad \mathbf{n}_t = \frac{1}{\sqrt{2}}(0,0,1,-1)$$

The lemma now follows by an exercise in trigonometry which we leave to the reader. □

New let A be the 4-strand braid group. The Artin presentation for A is given by

$$A = \langle r, s, t \mid rsr = srs,\ sts = tst,\ rt = tr \rangle$$

Let $\mathcal{D}^0 = \mathcal{D}^0_A$ be the spherical Deligne complex for A. Then a fundamental domain for the action of A on $\mathcal{D}^0$ is the 2-simplex with vertices $A_{r,s}, A_{r,t}, A_{s,t}$. Thus, we can define the "type" of a vertex or edge analogously. Taking the metric on each 2-simplex of $\mathcal{D}^0$ to be the same as

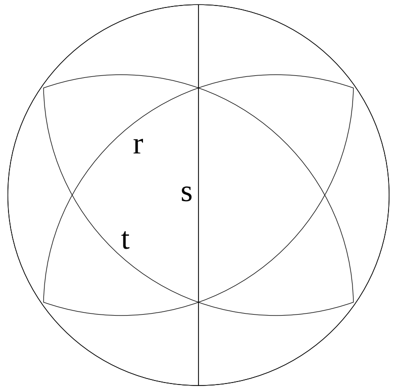

FIGURE 1. The spherical Coxeter complex for Sym_4

the metric on C_0, we see that Lemma 3.1 also applies to chambers of $\mathcal{D}^0$.

4. Relations in the four-strand braid group

In this section we prove two key technical lemmas about relations in the four-strand braid group which will be needed in the proof of the main theorem.

Lemma 4.1. *Let $T = \{s,t\}$ and $R = \{r,s\}$. Suppose $\tau_1, \tau_2 \in A_T$ and $\rho_1, \rho_2 \in A_R$ are such that $\tau_1\rho_1 = \rho_2\tau_2$, and assume that no τ_i or ρ_i lies in A_s. Then there exists $i, j, k_1, k_2, k_3, k_4 \in \mathbb{Z}$ such that*

$$\begin{aligned} \tau_1 &= s^{k_1} t^j s^{k_2} & \rho_1 &= s^{-k_2} r^i s^{k_4} \\ \rho_2 &= s^{k_1} r^i s^{k_3} & \tau_2 &= s^{-k_3} t^j s^{k_4} \end{aligned}$$

Proof. We identify the four strand braid group A with the mapping class group of the disk with four puctures, $\Sigma = D^2 - \{x_1, x_2, x_3, x_4\}$, with r interchanging x_1, x_2, s interchanging x_2, x_3, and t interchanging x_3, x_4 (see Figure 4). Let γ_r, γ_t be the paths shown in Figure 2. Let C_r, C_s, C_t denote the closures of the three components of $\Sigma - (\gamma_t \cup \gamma_s)$ as indicated in the figure.

Set $a = \tau_1\rho_1 = \rho_2\tau_2$, and represent a by a homeomorphism $h : \Sigma \to \Sigma$ chosen as follows. First choose a representatives h_{τ_1} and h_{ρ_1} of τ_1 and ρ_1 satisfying

(1) $h_{\tau_1}|_{C_r} = id_{C_r}$,
(2) $h_{\rho_1}|_{C_t} = id_{C_t}$, and
(3) $h_{\tau_1}(\gamma_t)$ has minimal intersection with γ_t,

Now set $h = h_{\tau_1} \circ h_{\rho_1}$. Consider the action of h on γ_t. Since $\gamma_t \cap \gamma_r = \emptyset$, we have

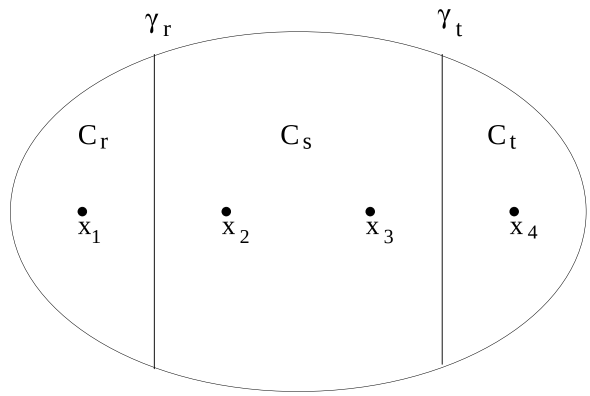

FIGURE 2. The punctured disk Σ

(1) $h(\gamma_t) \cap h(\gamma_r) = \emptyset$, and
(2) $h(\gamma_t) \cap \gamma_r = h_{\tau_1}(\gamma_t) \cap h_{\tau_1}(\gamma_r) = \emptyset$.

On the other hand, if we choose representatives h_{ρ_2} and h_{τ_2} for ρ_2 and τ_2 in a similar manner, we get a homeomorphism $h' = h_{\rho_2} \circ h_{\tau_2}$, representing a, which has the property that $h'(\gamma_r) \cap \gamma_t = \emptyset$. Since h' is isotopic to h, $h(\gamma_r)$ is homotopic to a path which is disjoint from γ_t.

We claim that we can find a path $\beta \sim h(\gamma_r)$ which is simultaneously disjoint from γ_t and $h(\gamma_t)$. To see this, set $\beta_0 = h(\gamma_r)$, so β_0 is disjoint from $h(\gamma_t)$ but may intersect γ_t in, say, n points. We may assume (by a local deformation of h_{τ_1}) that these intersections are transverse. Consider the complement of $\gamma_t \cup \beta_0$ in Σ. Since β_0 can be homotoped off of γ_t, some component D of this complement must be a disk. This disk cannot contain any segment of $h(\gamma_t)$ since this would contradict the assumption that $h_{\tau_1}(\gamma_t)$ $(=h(\gamma_t))$ has minimal intersection with γ_t. It follows that β_0 can be homotoped across D to a path β_1 which is still disjoint from $h(\gamma_t)$, but intersects γ_t in $n-2$ points. Repeat this process until no intersections remain. This proves the claim.

We now have a pair of disjoint, simple closed curves, β and $h(\gamma_t)$ such that $\beta \subset \Sigma - C_t$ and $h(\gamma_t) \subset \Sigma - C_r$. Consider the segments $\{\alpha_i\}$ of β which lie in the twice punctured disk C_s. If some α_i is homotopic to a segment of γ_r, we slide it back into C_r and ignore it. The remaining segments are mutually disjoint, non-trivial curves in C_s, hence there is an element of A_s, the mapping class group of C_s, which maps every α_i to a curve parallel to one of the three curves $\sigma_1, \sigma_2, \sigma_3$ in Figure 3. Say $s^k \in A_s$ is such an element.

Note that $s^k(\beta)$ and $s^k h(\gamma_t)$ are still disjoint, simple curves lying in $\Sigma - C_t$ and $\Sigma - C_r$ respectively. Suppose that $s^k(\beta)$ contains some segment of type σ_3, or contains segments of both types σ_1 and σ_2. In

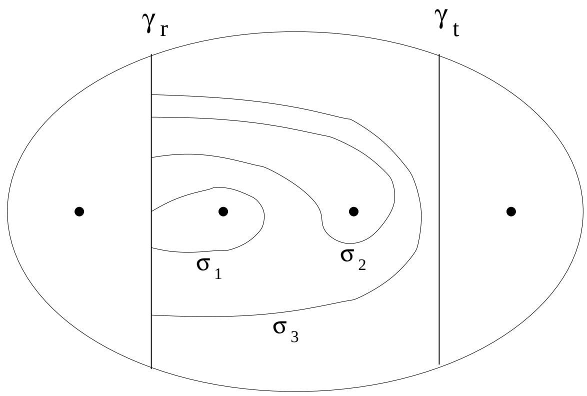

FIGURE 3. The curves $\sigma_1, \sigma_2, \sigma_3$

this case, $s^k h(\gamma_t)$ lies in a disk with only one puncture, namely x_4. Since $s^k h(\gamma_t)$ cannot be homotopic to the boundary of Σ, the only possibility is that $s^k h(\gamma_t) \sim \gamma_t$. It follows that, up to homotopy, $s^k h_{\tau_1}$ fixes both γ_r and γ_t. Thus $s^k \tau_1$ lies in A_s, the mapping class group of C_s. But this contradicts our hypothesis that $\tau_i \notin A_s$.

Thus, all segments of $s^k(\beta)$ between γ_r and γ_t are of type σ_1 or all are of type σ_2. Replacing s^k by s^{k+1} if necessary, we may assume that they are all of type σ_1. In this case, $s^k h(\gamma_t)$ lies in a disk with two punctures, x_3, x_4, whose mapping class group is A_t, so $s^k h(\gamma_t) \sim t^j(\gamma_t)$ for some j. It follows that $t^{-j} s^k h_{\tau_1}$ fixes both γ_r and γ_t up to homotopy. Thus, $t^{-j} s^k \tau_1 = s^{k_2}$ for some k_2, so $\tau_1 = s^{-k} t^j s^{k_2}$. Likewise, $s^k(\beta)$ is contained in a disk with two punctures, x_1, x_2 and, since $\beta \sim h(\gamma_r) \sim h_{\rho_2}(\gamma_r)$, a similar argument shows that $\rho_2 = s^{-k} r^i s^{k_3}$ for some i, k_3.

Set $k_1 = -k$. To complete the lemma, note that

$$\begin{aligned} \tau_1 \rho_1 &= \rho_2 \tau_2 \\ \Longrightarrow \quad s^{k_1} t^j s^{k_2} \rho_1 &= s^{k_1} r^i s^{k_3} \tau_2 \\ \Longrightarrow \quad r^{-i} s^{k_2} \rho_1 &= t^{-j} s^{k_3} \tau_2 \end{aligned}$$

The left hand side of the last equality lies in A_R while the right hand side lies in A_T, hence both must lie in $A_R \cap A_T = A_s$. We conclude that, for some $k_4 \in \mathbb{Z}$, $\rho_1 = s^{-k_2} r^i s^{k_4}$ and $\tau_2 = s^{-k_3} t^j s^{k_4}$. □

Lemma 4.2. *Let $R = \{r, s\}$ and let $\rho_1, \rho_2, \rho_3 \in A_R$. Suppose that for some $i_1, i_2, i_3 \neq 0$,*

$$t^{i_1} \rho_1 t^{i_2} \rho_2 t^{i_3} \rho_3 = 1. \tag{1}$$

Then for $j = 1, 2, 3$, there exist k_j such that $\rho_j \in A_r s^{k_j} A_r$.

Proof. We first show that it suffices to prove that *one* of the ρ's is of the form $r^i s^k r^j$. Since the relation (1) can be cyclically permuted

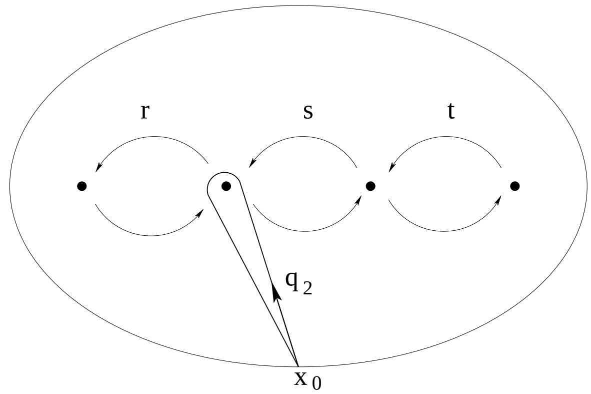

FIGURE 4. The action of r, s, and t on q_2

by conjugation, we may suppose without loss of generality, that $\rho_2 = r^i s^k r^j$. In this case we have,

$$\begin{aligned} t^{i_1}\rho_1 t^{i_2} r^i s^k r^j t^{i_3}\rho_3 &= 1 \\ (t^{i_1})(\rho_1 r^i)(t^{i_2} s^k t^{i_3})(r^j \rho_3) &= 1 \\ (t^{i_2} s^k t^{i_3})(r^j \rho_3) &= (r^{-i}\rho_1^{-1})(t^{-i_1}). \end{aligned}$$

Applying Lemma 4.1 with $\tau_2 = t^{-i_1}$, gives the desired form for ρ_1 and ρ_3.

We now prove that some ρ_j lies in $A_r s^{k_j} A_r$. Choose a basepoint x_0 which lies on the boundary of Σ. The fundamental group $F = \pi_1(\Sigma, x_0)$ is a free group on four generators, q_1, q_2, q_3, q_4, where q_i is the loop which winds once around x_i counter-clockwise. The braid group A injects into the automorphism group of F with the action of A on F given by

$$\begin{array}{lll} r(q_1) = q_2 & s(q_1) = q_1 & t(q_1) = q_1 \\ r(q_2) = q_2 q_1 q_2^{-1} & s(q_2) = q_3 & t(q_2) = q_2 \\ r(q_3) = q_3 & s(q_3) = q_3 q_2 q_3^{-1} & t(q_3) = q_4 \\ r(q_4) = q_4 & s(q_4) = q_4 & t(q_4) = q_4 q_3 q_4^{-1} \end{array}$$

Note, in particular, that for $k > 0$

$$\begin{aligned} t^k(q_3) &= (\underbrace{q_4 q_3 q_4 \ldots}_{k})(\underbrace{\ldots q_4^{-1} q_3^{-1} q_4^{-1}}_{k-1}) \\ t^{-k}(q_3) &= (\underbrace{q_3^{-1} q_4^{-1} q_3^{-1} \ldots}_{k})(\underbrace{\ldots q_3 q_4 q_3}_{k+1}) \end{aligned}$$

First assume that $|i_j| > 1$ for some $j \in \{1, 2, 3\}$. By cyclic permutation of the relation (1), we may assume that $|i_3| > 1$. Let

$$a = t^{i_1}\rho_1 t^{i_2} = (\rho_2 t^{i_3}\rho_3)^{-1}$$

and consider the action of a on q_1. We have $a(q_1) = t^{i_1}\rho_1(q_1) = t^{i_1}(w)$ where w is a word in q_1, q_2, q_3. Thus, $a(q_1) \in \langle q_1, q_2, t^{i_1}(q_3)\rangle$, the subgroup of F generated by $q_1, q_2, t^{i_1}(q_3)$. In particular, any two consecutive occurances of q_4 (or power of q_4) in the reduced form of $a(q_1)$ are separated one of the following:

(1) a power of q_3,
(2) (if $i_1 > 0$) a word in $\langle q_1, q_2\rangle$,
(3) (if $i_1 < 0$) a word in $q_3\langle q_1, q_2\rangle q_3^{-1}$.

On the other hand, $a(q_1) = \rho_3^{-1}t^{-i_3}\rho_2^{-1}(q_1) = \rho_3^{-1}(u)$ where $u \in \langle q_1, q_2, t^{-i_3}(q_3)\rangle$. Since $|i_3| > 1$, the reduced form of u contains two copies of $q_4^{\pm 1}$ separated by a power of q_3. Since ρ_3^{-1} fixes q_4 and preserves the subgroup generated by q_1, q_2, q_3, it follows that the reduced form of $a(q_1)$ contains two copies of $q_4^{\pm 1}$ separated by a power of $\rho_3^{-1}(q_3)$. Comparing this with the previous paragraph, we see that $\rho_3^{-1}(q_3)$ must be one of the three types of words listed above.

Moreover, $\rho_3^{-1}(q_3)$ can be represented by a simple closed curve in Σ, oriented counter-clockwise, which encloses a single puncture. The only words in $\langle q_1, q_2\rangle$ with this property are words of the form $r^j(q_2)$ (since the mapping class group of $D^2 - \{x_1, x_2\}$ is just A_r). It follows that the three cases above give, respectively,

(1) $\rho_3^{-1}(q_3) = q_3$,
(2) $\rho_3^{-1}(q_3) = r^j(q_2) = r^j s^{-1}(q_3)$,
(3) $\rho_3^{-1}(q_3) = q_3 r^j(q_2)q_3^{-1} = r^j s(q_3)$

In the first case, it follows that ρ_3 fixes both q_3 and q_4 and hence is supported on a disk $\Sigma' \subset \Sigma$ with only two punctures, x_1, x_2. Thus ρ_3 lies in the mapping class group A_r of Σ'. In the other two cases, the same reasoning applies to $\rho_3 r^j s^{\pm}$ to give $\rho_3 r^j s^{\pm} = r^k$ for some k.

It remains to consider the case where $|i_1| = |i_2| = |i_3| = 1$. Say all three of the i_j have the same sign. Inverting the relation (1) if necessary, we may assume that all i_j are positive. Let a be as above and consider $a(q_4)$. On the one hand,

$$a(q_4) = t\rho_1 t(q_4) = t\rho_1(q_4 q_3 q_4^{-1}) = t(q_4 u q_4^{-1}) = q_4 t(u) q_4^{-1}$$

where $u = \rho_1(q_3)$ is a word in q_1, q_2, q_3. Thus $t(u)$, and also $a(q_4)$, are words in q_1, q_2, q_4. On the other hand,

$$a(q_4) = \rho_3^{-1}t^{-1}\rho_2^{-1}(q_4) = \rho_3^{-1}t^{-1}(q_4) = \rho_3^{-1}(q_3)$$

which is a word in q_1, q_2, q_3. It follows that $a(q_4) = \rho_3^{-1}(q_3)$ must lie in $\langle q_1, q_2\rangle$, and we are in the same situation as case (2) above.

Finally, suppose two of the i_j have opposite signs. Permuting and inverting the relation as necessary, we may assume $i_1 = 1$ and $i_2 =$

$i_3 = -1$. Set $\delta = rst$. A simple computation shows that conjugation by δ takes A_R to A_T. Let $\tau_1 = \delta\rho_1\delta^{-1} \in A_T$. Conjugating the two expressions for a by (rs) gives

$$\begin{aligned}(rs)t\rho_1 t^{-1}(rs)^{-1} &= (rs)\rho_3^{-1}t\rho_2^{-1}(rs)^{-1}\\ \tau_1 &= (rs)\rho_3^{-1}t\rho_2^{-1}(rs)^{-1}\\ \tau_1(rs\rho_2) &= (rs\rho_3^{-1})t\end{aligned}$$

Applying Lemma 4.1 with $\tau_2 = t$ gives the desired result. This completes the proof of the lemma. □

5. The Main Theorem

We are now ready to prove that the spherical Deligne complex for the four strand braid group is CAT(1). By Theorem 2.1, we must verify that links of vertices in $\mathcal{D}^0$ are CAT(1) and that $\mathcal{D}^0$ has no closed geodesics of length less than 2π. The link of a vertex in $\mathcal{D}^0$ is the spherical Deligne complex of a 2-generator Artin group (namely the parabolic subgroup stabilizing that vertex) which is CAT(1) by [5], Prop. 4.4.5. Thus, it remains to show that $\mathcal{D}^0$ has no closed geodesics of length $< 2\pi$.

In [12], Elder and McCammond describe an algorithmic method for determining possible configurations for closed geodesics of length less than 2π in a given, piecewise spherical 2-complex. Their method involves "developing" geodesic segments onto a 2-sphere. We use a similar method here, but in our case, the process can be greatly simplified by developing geodesics onto $\mathcal{C}^0$, rather than onto an abstract 2-sphere, and using the simplicial structure of $\mathcal{C}^0$. This will allow us to quickly reduce to the case of a geodesic edge path.

If a local geodesic γ in $\mathcal{D}^0$ does not pass through any vertices, we can map the sequence of chambers containing γ to a sequence of chambers in $\mathcal{C}^0$ such that the image of γ is again a local geodesic. This map is unique, once an initial chamber has been chosen. This is process is called "developing" γ onto $\mathcal{C}^0$. (Note that it is not the same as the image of γ under the projection map $\mathcal{D}^0 \to \mathcal{C}^0$.)

In general, we cannot develop a geodesic across a vertex. However, in the case of a vertex of type $\{r,t\}$, the developing map still works as we now describe. Let K^0 be the fundamental chamber of $\mathcal{C}^0$. Then the union of the chambers K^0, rK^0, tK^0, rtK^0 forms a quadrilateral Q in $\mathcal{C}^0$ with all sides of type s (see Figure 5). We call any W-translate of Q an *rt-cube* of $\mathcal{C}^0$. Similarly, if $\hat{K}^0$ is the fundamental chamber of $\mathcal{D}^0$, then for any $i, j \neq 0$, the union of the chambers $\hat{K}^0, r^i\hat{K}^0, t^j\hat{K}^0, r^it^j\hat{K}^0$

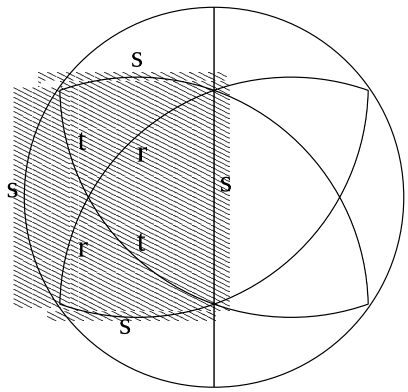

FIGURE 5. An rt-cube in $\mathcal{C}^0$

forms a quadrilateral $\hat{Q}$ in $\mathcal{D}^0$ with all sides of type s. We call any A-translate of such a quadrilateral an *rt-cube* in $\mathcal{D}^0$.

Given a (type preserving) projection of a chamber $a\hat{K}^0$ onto a chamber wK^0 of $\mathcal{C}^0$, this projection extends to a unique isometry of an rt-cube $a\hat{Q}$ onto an rt-cube of $\mathcal{C}^0$. It follows that any geodesic segment in $a\hat{Q}$ can be developed to a geodesic segment in $\mathcal{C}^0$, even if it passes through the center vertex of the cube. Thus, for the purpose of developing local geodesics in $\mathcal{D}^0$ to local geodesics in $\mathcal{C}^0$, vertices of type $\{r,t\}$ can be treated as nonsingular points. We will sometimes refer to vertices of type $\{r,t\}$ as "nonsingular" vertices and those of type $\{r,s\}$ or $\{s,t\}$ as "singular" vertices. A pair of edges of type t (respectively r) meeting at a non-singular vertex forms the diagonal of some rt-cube and will be refered to as a *t-diagonal* (respectively *r-diagonal*).

Lemma 5.1. *Let v_1 and v_2 be distinct vertices in $\mathcal{D}^0$ of type $\{r,t\}$. Then $Star(v_1) \cap Star(v_2)$ is either empty, a single vertex, or a single edge. In particular, two distinct t-diagonals (or r-diagaonals) meet in at most one vertex.*

Proof. It suffices to prove the case where $v_1 = A_{\{r,t\}}$. We first show that $Star(v_1) \cap Star(v_2)$ contains at most one vertex of type $T = \{s,t\}$ and at most one vertex of type $R = \{r,s\}$. Let $v_2 = aA_{\{r,t\}}$ and suppose $v_3 = bA_T$ lies in $Star(v_1) \cap Star(v_2)$. Then $bA_T \cap A_{\{r,t\}} \neq \emptyset$ so the coset representative b can be chosen to be r^i for some i. Since $bA_t \cap aA_{\{r,t\}}$ is also non-empty, $bA_T = r^iA_T$ contains an element of the form ar^j. Thus, $ar^j = r^i\tau_1$ for some $\tau_1 \in A_T$.

Now if v_4 is another vertex of type T lying in $Star(v_1) \cap Star(v_2)$, then we likewise have $v_4 = r^kA_T$ and $ar^l = r^k\tau_2$. Thus, $a = r^i\tau_1 r^{-j} =$

$r^k\tau_2 r^{-l}$, and hence $\tau_1 r^{l-j} = r^{k-i}\tau_2$. By Lemma 4.1, either $l - j = k - i = 0$, or $\tau_1 = \tau_2 = t^n$ for some n.

In the latter case, $a = r^{i-j}t^n \in A_{\{r,t\}}$, so $v_1 = v_2$ which contradicts the assumption that v_1 and v_2 are distinct vertices. In the former case, we get $i = k$ so $v_3 = v_4$, and we conclude that $Star(v_1) \cap Star(v_2)$ contains at most one vertex $r^i A_T$ of type T. By a completely analogous argument, it contains at most one vertex $t^j A_R$ of type R. Since $\mathcal{D}^0$ is a flag complex, if $Star(v_1) \cap Star(v_2)$ contains the vertices $r^i A_T$ and $t^j A_R$, then it must also contain the edge $r^i t^j A_s$ connecting them. We conclude that $Star(v_1) \cap Star(v_2)$ is either empty, or consists of a single vertex or a single edge. □

We will need to analyze certain edge paths in $\mathcal{D}^0$. Suppose e_1 and e_2 are two edges meeting at a vertex v of type $S_1 \in S$. Let $a\hat{K}^0$ be a chamber containing e_1 and $b\hat{K}^0$ a chamber containing e_2. Then $b = a\rho$ for some $\rho \in A_R$. Now suppose that $e_1, e_2, \dots e_k$ is a closed edge path in $\mathcal{D}^0$ and the vertex between e_i and e_{i+1} is of type S_i. Repeating the process above at each vertex, we get a relation $\rho_1\rho_2\cdots\rho_k = 1$ where $\rho_i \in S_i$. Moreover, if e_i, e_{i+1} forms an r-diagonal (respectively t-diagonal), then we can choose our chambers so that $\rho_i \in A_t$ (respectively $\rho_i \in A_r$).

Lemma 5.2. *Suppose p is a closed edge path in $\mathcal{D}^0$ consisting of 4 edges of type s. Then p is the boundary of a unique rt-cube in $\mathcal{D}^0$. In particular, p is not locally geodesic at any of its four vertices.*

Proof. Edges of type s connect vertices of type $R = \{r, s\}$ to vertices of type $T = \{s, t\}$. It suffices to consider the case where the first edge of p is the edge A_s of the fundamental chamber. By the discussion above, p corresponds to a relation $\tau_1\rho_1\tau_2\rho_2 = 1$ where $\rho_i \in R$ and $\tau_i \in T$. By Lemma 4.1, the edges of p are then

$$\begin{aligned} e_1 &= A_s = s^{k_1}A_s \\ e_2 &= \tau_1 A_s = s^{k_1}t^i A_s \\ e_3 &= \tau_1\rho_1 A_s = s^{k_1}t^i r^j A_s \\ e_4 &= \tau_1\rho_1\tau_2 A_s = s^{k_1} r^j A_s \end{aligned}$$

Thus, p is the boundary of an rt-cube $s^{k_1}\hat{Q}$. The uniqueness of this cube follows from Lemma 5.1. □

Lemma 5.3. *Suppose p is a triangle in $\mathcal{D}^0$ consisting of three r-diagonals (respectively t-diagonals). Then p is not locally geodesic at any of its three (singular) vertices.*

Proof. By the discussion above, p corresponds to a relation of the form $t^{i_1}\rho_1 t^{i_2}\rho_2 t^{i_3}\rho_3 = 1$, where two edges of p meeting at a singular vertex are of the form $e = aA_r$ and $e' = a\rho_j A_r$ for some a. By Lemma 4.2, we have $\rho_j = r^n s^k r^m$ for some n, k, m. Thus we have $e' = ar^n s^k A_r$ and $e = ar^n A_r$, so the angle between e' and e is the same as the angle between $s^k A_r$ and A_r, namely $2\pi/3$. □

Theorem 5.4. *The spherical Deligne complex $\mathcal{D}^0$ for the braid group on four strands is CAT(1).*

Remark. Since $\mathcal{D}^0$ is the link of the cone point in $\mathcal{D}$ with respect to the Moussong metric, this theorem is equivalent to the statement that the Moussong metric on the Deligne complex for the 4-strand braid group is CAT(0).

Proof. We must show that $\mathcal{D}^0$ has no closed geodesics of length $< 2\pi$. First consider the case of a geodesic p which passes through no vertices of type R or T. Then p can be developed onto a local geodesic q in $\mathcal{C}^0$ which also contains no vertices of type R or T. Note, however, that since p has length less than 2π, q will not be a *closed* geodesic in $\mathcal{C}^0$. It is easy to see (Figure 5) that any local geodesic in $\mathcal{C}^0$ which crosses an rt-cube from the interior of one edge to the interior of the opposite edge must do the same in the adjacent rt-cube and the distance across these two rt-cubes is π. Thus, q is one of two types: either it crosses every rt-cube from one side to the opposite side, or it crosses every rt-cube from one side to an adjacent side.

In the first case, it suffices to show that p (and hence q) crosses four rt-cubes (since this forces it to have length 2π). Lemma 5.1 rules out the possibility that p closes up after crossing only 2 cubes. If it closes up after crossing three, then the three cubes shown on the left in Figure 6 form a möbius strip in $\mathcal{D}^0$ (since different colors represent different types of vertices). In this case, the edge path e_1, e_2, e_3, e_4 indicated in the figure forms forms a quadrilateral of type s-edges which is geodesic at both ends of e_2. This is impossible by Lemma 5.2.

In the case where q cuts across a corner of each rt-cube, it must cross six cubes to have length 2π. (This can be seen by looking at any great circle in $\mathcal{C}^0$ which cuts across the corner of an rt-cube.) The rt-cubes in $\mathcal{D}^0$ containing p are pairwise joined along edges as shown on the right in Figure 6. In order for p to close up after crossing less that six corners, two of the white vertices v_1, v_2, v_3 would have to be identified. But this contradicts Lemma 5.1, so p must traverse all six rt-cubes which makes q a full great circle in $\mathcal{C}^0$. This completes the case in which p contains no vertices of type R or T.

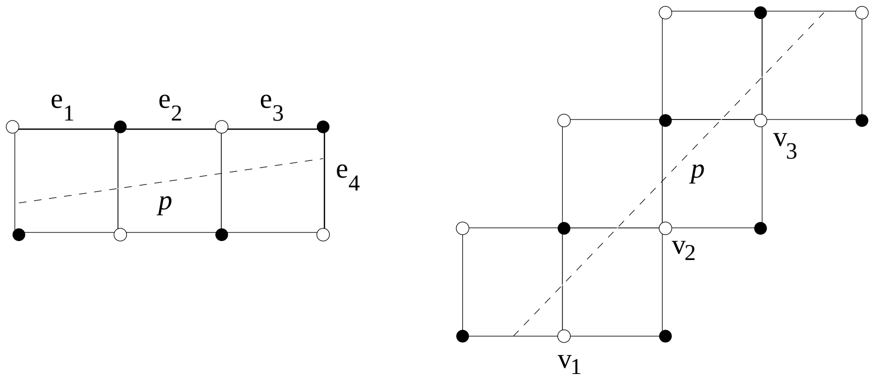

FIGURE 6. Geodesics with no singular vertices

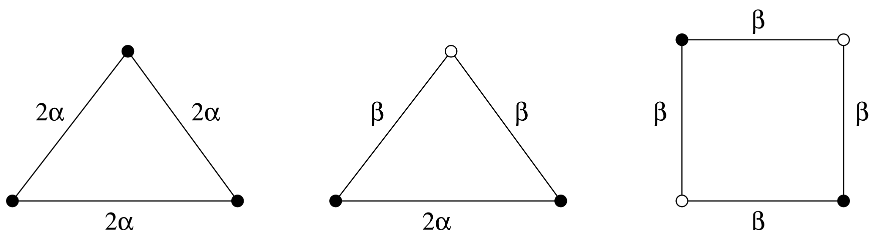

FIGURE 7. Possible closed edgepaths p

At the other extreme, suppose p is a geodesic edge path. Then every edge of type t or r in p must be followed by another of the same type. Thus p consists of t-diagonals, r-diagonals, and edges of type s. By abuse of terminology, we will refer to all three of these as "edges". By Lemma 5.1, we may assume that p consists of at least three such edges. As in Lemma 3.1, we let

$$\begin{aligned} 2\alpha &= \text{length of an } r\text{-diagonal or } t\text{-diagonal} \\ \beta &= \text{length of an edge of type } s. \end{aligned}$$

Consider the possible configurations of edges of length 2α and β which make up p. By Lemma 3.1, $2\alpha > \beta > \alpha$ and $2\alpha + \beta = \pi$. In particular, $6\beta > 4\alpha + 2\beta = 2\pi$, and $8\alpha > 2\alpha' + 2\beta = 2\pi$. Taking into account that edges of length 2α connect vertices of the same type, while edges of length β connect vertices of different types, a simple exercise shows that the only possible configurations for p are the three in Figure 7.

By Lemmas 5.3 and 5.2, the triangle with all side lengths 2α and the quadrilateral with all side lengths β fail to be locally geodesic at every vertex. That leaves the triangle Δ with side lengths $2\alpha, \beta, \beta$. Suppose that the edge e of length 2α is a t-diagonal. (The r-diagonal case is entirely analogous.) Then Δ can be translated so that e is the pair of edges $A_t, r^j A_t$ for some j. Left multiplying by t, we get a new triangle, $t\Delta$ which is also of type $2\alpha, \beta, \beta$ and still contains e as its 2α edge. Together, Δ and $t\Delta$ give rise to a quadrilateral with all sides of type s. By Lemma 5.2, this quadrilateral bounds an rt-cube which, by Lemma 5.1, must contain e as its t-diagonal. In other word, Δ bounds half of an rt-cube, so the angles at the vertices of Δ are $\pi/3$, $\pi/3$, and $2\pi/3$. We have, infact, shown that a closed edge path of length less than 2π fails to be geodesic at *every* singular vertex.

Finally, suppose p is a closed geodesic in $\mathcal{D}^0$ of length less than 2π which is not an edge path, but contains at least one singular vertex. Divide p into segments $p = p_1 p_2 \dots p_n$ such that each p_i begins and ends at a singular vertex but contains no such vertices in its interior. Developing p_i onto $\mathcal{C}^0$, we get a locally geodesic segment q_i connecting two vertices in $\mathcal{C}^0$ of type T or R. Since the point antipodal to any singular vertex in $\mathcal{C}^0$ is another singular vertex (of the opposite type), each q_i is either an edge or half of a great circle. Since length$(q) \leq 2\pi$, exactly one q_i is a half circle, say q_1. This half circle can be rotated, with endpoints fixed, until it becomes an edge path with one vertex of type T or R in its interior. This rotation lifts to $\mathcal{D}^0$, to give a geodesic edgepath p_1' with the same endpoints and same length as p_1. The resulting path $p' = p_1' p_2 \dots p_n$ is a closed edge path of length less than 2π. By the discussion above, p' fails to be locally geodesic at *every* singular vertex. But this is a contradiction since p_1' is necessarily geodesic at its interior vertex. □

Recall that Sym_4 denotes the symmetric group on 4 letters. Combining the results of [5] with the theorem above we obtain the following corollary.

Corollary 5.5. *Let (W, S) be a Coxeter system such that every finite, irreducible special subgroup W_T is either Sym_4, $\mathbb{Z}_2$, or a dihedral group, and let A be the associated Artin group. Then the Deligne complex $\mathcal{D}_A$ is CAT(0), and the hyperplane complement $\mathcal{H}_W/W$ is a $K(A,1)$-space.*

Proof. For $T \in \mathcal{S}^f$, the decomposition of (W_T, T) into irreducible factors gives a decomposition of (A_T, T) into a direct product of copies of $\mathbb{Z}$, 2-generator Artin groups, and 4-strand braid groups. It follows (cf. [3], Thm. 3.1) that the link of the vertex A_T in $\mathcal{D}_A$ decomposes as a join of spherical Deligne complexes for the factors. The spherical Deligne complex for a 2-generator Artin group is CAT(1) by [5], Prop. 4.4.5, and the spherical Deligne complex for the 4-strand braid group is CAT(1) by Theorem 5.4. □

In particular, we have

Corollary 5.6. *Let (A, S) be an Artin system whose Deligne complex $\mathcal{D}_A$ has dimension ≤ 3. If $m(s,t) \neq 4, 5$ for all $s, t \in S$, then $\mathcal{D}_A$ is CAT(0) and the associated hyperplane complement $\mathcal{H}_W/W$ is a $K(A,1)$-space.*

The primary motivation for this paper was the $K(\pi, 1)$ conjecture for hyperplane complements. In closing, however, we note that there are other, algebraic consequences of having a CAT(0) Deligne complex. For example, in [16], Salvetti proves that there is a finite cell complex, of the same dimension as $\mathcal{D}_A$ which is homotopy equivalent to the space $\mathcal{H}_W/W$ (see also [4]). Combining this with Corollary 5.5, we get

Corollary 5.7. *Let A be as in Corollary 5.5. Then A has a finite $K(\pi, 1)$-space of dimension $n = max\{|T| \mid T \in \mathcal{S}^f\}$. In particular, A is torsion-free and has cohomological dimension n.*

In his thesis [13], Godelle determines the normalizer, commensurator, and quasi-centralizer of any finite parabolic subgroup of A under the assumption that $\mathcal{D}_A$ is CAT(0). In [8], under the same hypothesis, Crisp proves that if G is a group of symmetries of an Artin system (A, S), then the subgroup of A fixed by G is itself an (explicitly described) Artin group. These results can now be applied to the Artin groups in Corollary 5.5.

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