THE LANGUAGE OF GEODESICS FOR GARSIDE GROUPS

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Abstract. We prove that the language of all geodesics of any Garside group, with respect to the generating set of divisors of the Garside element, forms a regular language. In particular, the braid groups admit generating sets where the associated language of geodesics is regular.

1. Introduction

In the early 1980s Cannon introduced the following question: *For what groups G and what finite generating systems S is the language of geodesics regular?* Recall that a word w in the free monoid S^* is a *geodesic* if the corresponding path in the Cayley graph $\Gamma(G,S)$ is a minimal length edge path joining its end points. If

$$\texttt{geo}_S(g) = \{w \in S^* \mid w \text{ describes a geodesic from 1 to } g\}$$

then the language of geodesics (with respect to the generators S) is

$$\texttt{geo}_S(G) = \bigcup_{g \in G} \texttt{geo}_S(g) \ .$$

Thus Cannon's question asks when $\texttt{geo}_S(G)$ forms a language accepted by a finite state automaton. Cannon's original work used the idea of cone types. The *cone type* of $g \in G$ with respect to S is the language

$$\texttt{cone}_S(g) = \{w \in S^* \mid \alpha w \in \texttt{geo}_S(G) \text{ for all } \alpha \in \texttt{geo}_S(g)\}$$

and a group G has *finitely many cone types* with respect to S if only finitely many languages occur as $\texttt{cone}_S(g)$ for $g \in G$. The property of having finitely many cone types is equivalent to the language of geodesics being regular. If the language of geodesics for G with respect to the generating set S is regular, then (G,S) is called a *Cannon pair* [16].

Cannon essentially proved that the language of geodesics for a word hyperbolic group is regular for all finite generating sets [5]. He also provided an example of a virtually $\mathbb{Z} \times \mathbb{Z}$ group whose language of geodesics with respect to a specific generating set is not regular (see the remark in §4 of [17]). Here we show

Main Theorem. *The language of geodesics of a Garside group G, with respect to the generating set of simple divisors of Δ, is regular.*

Garside groups are a class of groups admitting a finite set of generators with a lattice structure similar to that found in braid groups. In fact, the braid groups, as well as all Artin groups of finite type, are Garside groups. (A full definition and some background is given in the next section.) In a number of papers, Dehornoy

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and others have demonstrated that many known properties of braid groups extend to all Garside groups (see [7], [9], [10], [11], [12] and the references cited there). Our Main Theorem presents a situation where the application to braid groups was previously unknown.

Braid groups have (at least) two natural lattice structures, one based on divisors of the half twist Δ, and the other based on divisors of the braid δ which crosses the first string over all the other strings.

Corollary. *The language of geodesics for the braid group B_n, with respect to the generating set of simple divisors of either Δ or δ, is regular.*

If the language of geodesics is regular it follows that the geodesic growth series is rational, so in particular the geodesic growth series for the braid groups are rational. We explore this in §4.

Question 1.1. Is the language of geodesics for the braid group B_n with respect to other standard generating sets also regular? Sabalka [20] has shown that this is the case for B_3 with respect to the common presentation $B_3 = \langle a, b \mid aba = bab \rangle$, and he explicitly computes the complete geodesic growth series for this generating set.

2. Garside groups

Let M be a monoid and let the *indivisible* elements (often called 'atoms') be those $m \in M$ such that $m \neq 1$ and if $m = ab$ then either $a = 1$ or $b = 1$. If the indivisible elements generate M one can define $||m||$ to be the supremum of the lengths of all expressions of m in terms of indivisible elements. The monoid M is *atomic* if it is generated by its indivisible elements, and the norm $||m||$ of any element is finite. In an atomic monoid one can define a *partial order* via left (or right) divisibility: $a < b$ if $ac = b$ for some $c \in M - \{1\}$; $a \leq b$ if $ac = b$ for some $c \in M$. Conversely, if M is a finitely generated monoid where the partial order given by divisibility contains no infinite descending chains, then M is atomic.

Definition 2.1 (Garside monoid)**.** An atomic monoid M is *Garside* if

(1) left and right cancellation laws hold in M,
(2) any two elements of M admit a least common multiple and a greatest common divisor on both the left and the right,
(3) there exists an element Δ such that the left and right divisors of Δ are the same, there are finitely many of them, and they form a set of generators for M.

The element Δ is called a *Garside element.* We denote the set of divisors of Δ by $\mathcal{D}$, and call this generating set the *simple divisors.*

The fact that the set of left and right divisors of Δ are the same leads to the following fact.

Proposition 2.2. (2.2 and 2.3 in [10]) *Let G^+ be a Garside monoid with Garside element Δ. Then there is a permutation σ of the set of simple divisors $\mathcal{D}$ such that*

$$\Delta\mu = \sigma(\mu)\Delta$$

for all $\mu \in \mathcal{D}$. In particular, there is an r such that Δ^r is central.

Definition 2.3 (Complements)**.** If $\mu \in \mathcal{D}$ then by definition there is an element $\mu^* \in G^+$ such that $\mu\mu^* = \Delta$. We call μ^* the *right complement* of μ, and note that because μ^* is a right divisor of Δ, $\mu^* \in \mathcal{D}$, and thus $\mathcal{D}$ is closed under right complements.

Similarly there is an element ${}^*\mu \in G^+$ such that ${}^*\mu\mu = \Delta$. We call ${}^*\mu$ the *left complement* of μ and note that $\mathcal{D}$ is closed under left complements. Since $\Delta\mu^* = {}^*\mu\mu\mu^* = {}^*\mu\Delta$, left and right complements are related by the equation ${}^*\mu = \sigma(\mu^*)$.

Garside monoids satisfy Ore's criterion, hence they embed in their group of fractions, and thus we may define a *Garside group* to be the group of fractions of a Garside monoid. We denote a Garside group by G and its positive monoid by G^+.

Remark 2.4. Garside introduced the positive braid monoid and the element Δ (the half twist) to the study of the braid group [14] and the term "Garside group" honors this work. When first introduced in [12], these groups were called "small Gaussian," while "Garside" was used for a slightly more restrictive condition. The name "Garside" has now been generally adopted for this class of monoids and their groups of fractions [10].

The classic examples of Garside groups are the Artin groups of finite type, where the Garside element corresponds to the longest element of the associated Coxeter group. It has been shown by Bessis [1], and independently by Brady and Watt [4], that finite type Artin groups have another Garside structure in which the atoms correspond to reflections in the Coxeter group and the Garside element corresponds to a Coxeter element δ. (This is directly related to the Birman-Ko-Lee monoid for braid groups [2], [3].) A diverse collection of other Garside groups and a discussion of their properties may be found in [10], [12] and [19].

For two elements $a, b \in G^+$, let $a \wedge b$ denote the left gcd of a and b. For $g \in G^+$, the *left front* of g is defined to be $\Delta \wedge g$. It is the maximum element of $\mathcal{D}$ satisfying $\mu < g$. One can use left fronts to define a normal form in G^+, and this normal form is commonly referred to as the *left greedy normal form*.

Proposition 2.5. (3.5 in [10]) *Let G^+ be a Garside monoid. Then each $g \in G^+$ may be uniquely represented as a product of simple divisors, $g = \mu_1\mu_2\cdots\mu_n$, where $\mu_i = \Delta \wedge (\mu_i \cdots \mu_n)$.*

Proposition 2.6. (3.10 in [10]) *For all $g, h \in G^+$, $\Delta \wedge (gh) = \Delta \wedge (g(\Delta \wedge h))$.*

There is an analogously defined *right greedy* normal form in which one begins by taking the (oddly named) *right front* to be the right gcd of the monoid element g and the Garside element Δ. Then as in Proposition 2.5 each $g \in G^+$ can be uniquely represented as the product of simple divisors $g = \mu_1 \cdots \mu_n$ where μ_i is the right front of $\mu_1 \cdots \mu_i$.

Since G is the group of fractions of G^+, every element of G can be written in the form $g = a^{-1}b$ with $a, b \in G^+$.

Proposition 2.7. (Lemma 3.7 in [10]) *For $g \in G$ there is a unique decomposition $g = a^{-1}b$ with $a, b \in G^+$ satisfying $a \wedge b = 1$.*

This proposition gives rise to a normal form for elements of G. Namely, if $g = a^{-1}b$ is the decomposition described in the proposition, and $a = \eta_1 \cdots \eta_j$, $b = \mu_1 \cdots \mu_k$ are the left greedy normal forms for a and b, then the normal form for g is $\eta_j^{-1} \cdots \eta_1^{-1}\mu_1 \cdots \mu_k$.

3. Geodesics

By the definition of a Garside group, the set $\mathcal{D}$ of simple divisors is a generating set for G. Let w be a word in $F(\mathcal{D})$, the free group on $\mathcal{D}$, and let g be its image in G. In this section we give precise criteria for when a word w is geodesic and prove that the language of all geodesic words forms a regular language.

Lemma 3.1. *The normal form for an element $g \in G$ is geodesic.*

Proof. In Lemma 3.11 of [10], for any $h \in G$ and $\mu \in \mathcal{D}$, Dehornoy gives an explicit description of the normal form for $h\mu^{-1}$ in terms of the normal form for h, as well as the normal form for $h\Delta$. It follows from these descriptions that the normal forms for the $h\mu^{-1}$'s and the normal form for $h\Delta$ all have length at most one more than that of h. One can find the normal form of $h\mu$ by expressing μ as $\Delta(\mu^*)^{-1}$ and applying the work cited above. It follows by a case-by-case analysis — the cases being $h \in G^+$, $h^{-1} \in G^+$, or neither h nor h^{-1} are in G^+ — that the normal form for $h\mu$ has length at most one more than that of h. Hence by induction, if w is a word of length k representing g, then the normal form for g has length at most k. □

Because the normal form for an element g of G^+ is a positive word, the lemma above implies that the shortest *positive* word representing g is geodesic.

Lemma 3.2. *If $a \in G^+$ has length k and $n > 0$, then $a\Delta^{-n}$ has length $\leq max\{k, n\}$ with equality holding if and only if $\Delta \not< a$. The same holds for $\Delta^{-n}a$.*

Proof. We appeal again to Dehornoy's calculus of normal forms. If $g \in G$ has normal form $\eta_j^{-1} \cdots \eta_1^{-1}\mu_1 \cdots \mu_k$, then Lemma 3.11 of [10] shows that the normal form for $g\Delta^{-1}$ is

$$\eta_j^{-1} \cdots \eta_1^{-1}({}^*\mu_1)^{-1}\sigma(\mu_2) \cdots \sigma(\mu_k)$$

with the convention that if $k = 0$ (i.e., g is strictly negative), then $\mu_1 = 1$ so ${}^*\mu_1 = \Delta$. If $j = 0$ (i.e., g is strictly positive) and $\mu_1 = \Delta$, then ${}^*\mu_1 = 1$ so the normal form for $g\Delta^{-1}$ is just $\sigma(\mu_2) \cdots \sigma(\mu_k)$. Thus we have

$$\text{length}(g\Delta^{-1}) = \begin{cases} \text{length}(g) + 1 & \text{if } k = 0 \\ \text{length}(g) - 1 & \text{if } j = 0 \text{ and } \Delta < g \\ \text{length}(g) & \text{otherwise} \end{cases} .$$

To prove the lemma, suppose $a \in G^+$ has normal form $\mu_1 \cdots \mu_k$. Then for $n \leq k$, repeated application of the formula above gives $\text{length}(a\Delta^{-n}) \leq k$ with equality holding if and only if $\Delta \not< a$. For $n > k$, $a\Delta^{-k}$ is strictly negative, so

$$\begin{aligned} \text{length}(a\Delta^{-n}) &= \text{length}(a\Delta^{-k}\Delta^{k-n}) \\ &= \text{length}(a\Delta^{-k}) + (n-k) \\ &\leq k + (n-k) = n \end{aligned}$$

with equality holding if and only if $\Delta \not< a$. This proves the first statement of the lemma. The last statement follows since $\Delta^{-n}a = \sigma^{-n}(a)\Delta^{-n}$ and σ is length preserving. □

Recall our notation from the introduction:

$$\begin{aligned} \mathtt{geo}_{\mathcal{D}}(G) &= \{w \in F(D) \mid w \text{ is geodesic}\} \\ \mathtt{geo}_{\mathcal{D}}(G^+) &= \{w \in \mathtt{geo}_{\mathcal{D}}(G) \mid w \text{ is a positive word}\} . \end{aligned}$$

In order to establish that $\mathtt{geo}_{\mathcal{D}}(G)$ is regular, we need explicit criteria for when a word belongs to $\mathtt{geo}_{\mathcal{D}}(G)$. We begin with the case of a positive word.

Essentially by definition, ${}^*\mu = \Delta\mu^{-1}$, for any $\mu \in \mathcal{D}$. For a positive word $w = \mu_1 \cdots \mu_m$ in $F(\mathcal{D})$, let ${}^*w$ denote the positive word obtained from $\Delta^m w^{-1}$ by "sliding" the Δ's to the right so as to replace each μ_i^{-1} by ${}^*\mu_i$. That is,

$$\begin{aligned} {}^*w &= \sigma^{m-1}({}^*\mu_m) \cdot \sigma^{m-2}({}^*\mu_{m-1}) \cdots \sigma^1({}^*\mu_2) \cdot {}^*\mu_1 \\ &= \sigma^m(\mu_m^*)\sigma^{m-1}(\mu_{m-1}^*) \cdots \sigma^1(\mu_1^*). \end{aligned}$$

In particular, if a is the image of w in G^+, then the image $\Delta^m a^{-1}$ of ${}^*w$ also lies in G^+.

Proposition 3.3. *Let w be a positive word of length m and let a be its image in G^+. Let $b = \Delta^m a^{-1}$. Then $w \in \mathtt{geo}_{\mathcal{D}}(G^+)$ if and only if $\Delta \not< b$.*

Proof. Suppose w is not geodesic. Then the left greedy normal form for a is a word $u = \eta_1 \cdots \eta_k$ with $k < m$. Hence $b = \Delta^m \eta_k^{-1} \cdots \eta_1^{-1} = \Delta^{m-k} \cdot {}^*u$, so $\Delta < b$.

Conversely, suppose $\Delta < b$. Let $u = \Delta\tau_2 \cdots \tau_k$ be the left greedy normal form for b. Since b is represented by ${}^*w$, a word of length m, Lemma 3.1 implies that $k \leq m$ and we have $a = b^{-1}\Delta^m = \tau_k^{-1} \cdots \tau_2^{-1}\Delta^{m-1}$. Applying Lemma 3.2 to $a^{-1} = \Delta^{-m}b = \Delta^{1-m}\tau_2 \cdots \tau_k$, gives

$$\text{length}(a) = \text{length}(a^{-1}) \leq \max\{m-1, k-1\} = m-1.$$

□

To recognize whether a positive word w is geodesic, we thus need to determine whether the left front of the word ${}^*w$ (or more precisely the image of ${}^*w$ in G^+) is Δ. This can be done by a finite state automaton. At first glance this may seem unlikely since the powers of σ appearing in the definition of ${}^*w$ increase with the length of the word. Recall, however, that some power r of Δ is central, so σ^r is the trivial automorphism. Thus we only need to keep track of the power of σ *mod* r.

Corollary 3.4. *The language $\mathtt{geo}_{\mathcal{D}}(G^+)$ is regular.*

Proof. We construct a finite state automaton $\mathfrak{F}$ that recognizes $\mathtt{geo}_{\mathcal{D}}(G^+)$. The automaton reads a word $w = \mu_1\mu_2 \cdots \mu_k$ from left to right and ends in a failure state if and only if the left front of ${}^*w$ is Δ. The states of $\mathfrak{F}$ are labeled by pairs (η, j) with $\eta \in \mathcal{D} \cup \{e\}$ and $j \in \mathbb{Z}/r\mathbb{Z}$, where r is the order of the permutation σ (cf. Proposition 2.2) and e corresponds to the identity. The start state is $(e, 0)$ and (η, j) is an accept state if and only if $\eta \neq \Delta$.

When $\mathfrak{F}$ reads a letter μ, it moves from state (η, j) to state $(\eta', j+1)$ where $\eta' = \Delta \wedge (\sigma^{j+1}(\mu^*)\eta)$ ($\Delta^* = e$ by definition). When reading a string $\mu_1 \cdots \mu_k$ one starts at the start state $(e, 0)$ and transitions to $(\sigma(\mu_1^*), 1)$, then to $(\Delta \wedge \sigma^2(\mu_2^*)\sigma(\mu_1^*), 2)$, and so on. An induction using Proposition 2.6 shows that the terminal state is $(\Delta \wedge {}^*w, k)$. Thus $\mathfrak{F}$ accepts w if and only if $\Delta \not< {}^*w$, which by Proposition 3.3, is equivalent to w being geodesic. □

Example 3.5. The finite state automaton for the positive three-strand braid monoid, with $\Delta = aba = bab$ and simple divisors $\mathcal{D} = \{a, b, ab, ba, \Delta\}$, is shown in Figure 1. Note that in the FSA constructed in Corollary 3.4, any transition from a state (Δ, i) ends in

$$(\Delta \wedge \sigma^{i+1}(\mu^*)\Delta, i+1) = (\Delta \wedge \Delta\sigma^i(\mu^*), i+1) = (\Delta, i+1)$$

Thus one never returns to an accept state after arriving at a state (Δ, i). These states have been removed from Figure 1 and the missing edges correspond to transitions to these fail states. The FSA in Figure 1 has also been divided into two parts in order to reduce clutter; one identifies the states with identical labels to form the actual FSA.

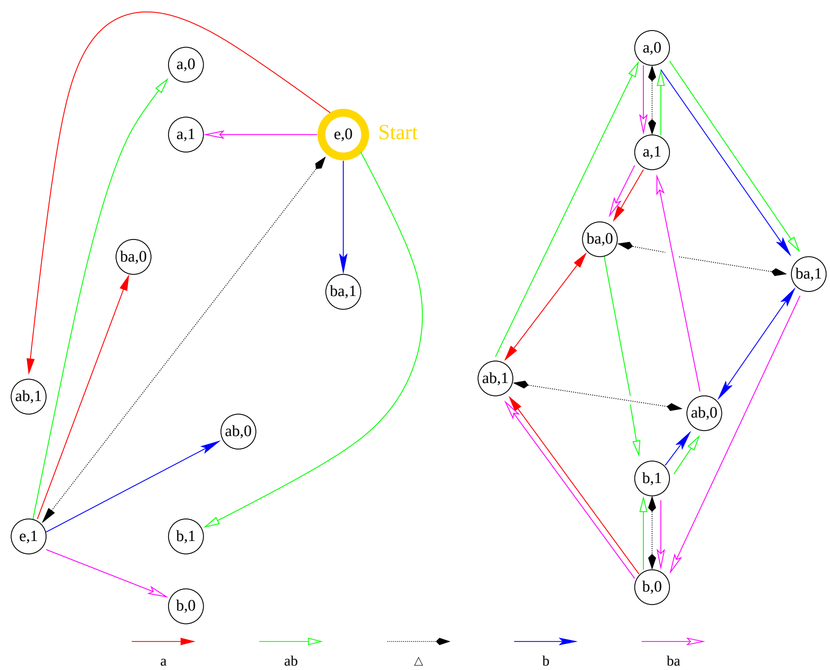

FIGURE 1. The FSA for B_3^+ with $\mathcal{D} = \{a, b, ab, ba, \Delta\}$

We now turn to the general case of a word $w = \mu_1^{\epsilon_1} \mu_2^{\epsilon_2} \cdots \mu_k^{\epsilon_k}$ where $\epsilon_i = \pm 1$. Let g be the image of w in G. Define

$$\begin{aligned} m_i &= \#\{j < i \mid \epsilon_j = -1\} \text{ and} \\ m &= m_{k+1} = \#\{j \mid \epsilon_j = -1\}. \end{aligned}$$

Note that $g\Delta^m$ lies in G^+. More explicitly, let $\overline{w} = \eta_1 \cdots \eta_k$ be the positive word defined by

$$\eta_i = \begin{cases} \sigma^{-m_i}(\mu_i) & \text{if } \epsilon_i = 1 \\ \sigma^{-m_i}(\mu_i^*) & \text{if } \epsilon_i = -1 \end{cases} .$$

That is, $\overline{w}$ is the word obtained from $w\Delta^m$ by sliding the Δ's to the left so as to replace each μ^{-1} by μ^*. Similarly, define $\underline{w} = \overline{(w^{-1})}$, that is, $\underline{w}$ is the word obtained from $w^{-1}\Delta^{k-m}$ by sliding the Δ's to the left as needed to replace each μ^{-1} by μ^*. Thus, in G we have

$$w = g \qquad \overline{w} = g\Delta^m \qquad \underline{w} = g^{-1}\Delta^{k-m} .$$

Note that if $\mu_i^{\epsilon_i} = \Delta^{-1}$, then $\eta_i = e$, the identity element in G. For this reason, we will view $\overline{w}$ and $\underline{w}$ as strings in the larger alphabet $\mathcal{A} = \mathcal{D} \cup \{e\}$.

Remark 3.6. Since $\mathcal{A} = \mathcal{D} \cup \{e\}$, there is a natural map $\mathcal{A}^* \twoheadrightarrow G^+$, taking e to the identity. Since $\mathcal{D}^* \subset \mathcal{A}^*$, the language $\texttt{geo}_\mathcal{D}(G^+)$ embeds in $\mathcal{A}^*$. Because any string containing e is not geodesic, we note that the image of $\texttt{geo}_\mathcal{D}(G^+)$ is $\texttt{geo}_\mathcal{A}(G^+)$. Lemma 3.3 still applies in this situation since if u is a string containing e, then ${}^*u$ is a string containing Δ.

Lemma 3.7. *Let $w = \mu_1^{\epsilon_1} \cdots \mu_k^{\epsilon_k}$ be a word in $F(\mathcal{D})$ and let m be defined as above. Then w is geodesic (i.e., $w \in \texttt{geo}_\mathcal{D}(G)$) if and only if one of the following holds.*

1. *$m = 0$ and $w \in \texttt{geo}_\mathcal{D}(G^+)$,*
2. *$m = k$ and $w^{-1} \in \texttt{geo}_\mathcal{D}(G^+)$,*
3. *$0 < m < k$ and $\overline{w}$ and $\underline{w}$ both lie in $\texttt{geo}_\mathcal{A}(G^+)$.*

Proof. The cases $m = 0$ and $m = k$ are obvious, so assume that $0 < m < k$. As was remarked above, $\overline{w}$ or $\underline{w}$ may not be words in $\mathcal{D}^*$, but they are words in $\mathcal{A}^*$, hence the change from $\texttt{geo}_\mathcal{D}(G^+)$ to $\texttt{geo}_\mathcal{A}(G^+)$ in item (3). Let $a = g\Delta^m$ be the image of $\overline{w}$ in G. Let $\tau_1 \cdots \tau_j$ be the left greedy normal form for a. Then by Lemma 3.2, $\text{length}(g) \leq \max\{j, m\} \leq k$ and w is geodesic if and only if both inequalities are actually equalities. Since $m < k$, the second inequality is an equality if and only if $j = k$ or equivalently, $\overline{w}$ is geodesic.

By Lemma 3.2, the first inequality is an equality if and only if $\Delta \not\leq a$. On the other hand, in G we have ${}^*(\underline{w}) = \Delta^k \underline{w}^{-1} = \Delta^m g = \sigma^m(a)$. So by Proposition 3.3, $\underline{w}$ is geodesic $\Leftrightarrow \Delta \not\leq \sigma^m(a) \Leftrightarrow \Delta \not\leq a$. We conclude that the first equality holds if and only if $\underline{w}$ is geodesic. ☐

Theorem 3.8. *The language $\texttt{geo}_\mathcal{D}(G)$ is regular.*

Proof. Recall that $\mathfrak{F}$ is an finite state automaton that reads a positive word and accepts it if and only if it is geodesic. We first extend $\mathfrak{F}$ to an FSA that reads an arbitrary word $w \in F(\mathcal{D})$ and accepts it if and only if $\overline{w}$ is geodesic. The states remain the same, and as before, when $\mathfrak{F}$ reads a positive generator μ, it moves from state (η, j) to state $(\eta', j+1)$ where $\eta' = \Delta \wedge (\sigma^{j+1}(\mu^*)\eta)$. When it reads a negative generator μ^{-1}, on the other hand, it moves from (η, j) to (η'', j) where $\eta'' = \Delta \wedge (\sigma^j(\mu)\eta)$.

Let us check by induction on k that, after reading the word $w = \mu_1^{\epsilon_1} \cdots \mu_k^{\epsilon_k}$, this FSA ends in the state $(\Delta \wedge {}^*\overline{w}, k - m)$. We can begin our induction with the trivial case $k = m = 0$ and $w =^* \overline{w}$ =the empty string. Here $\mathfrak{F}$ ends in the start state $s_0 = (e, 0)$.

Now suppose $k \geq 1$. Recall that $\overline{w}$ is obtained from $w\Delta^m$ by sliding the Δ's to the left while ${}^*\overline{w}$ is obtained from $\Delta^k \overline{w}^{-1}$ by sliding Δ's to the right. Thus, the net effect is that ${}^*\overline{w}$ is the word obtained from $\Delta^{k-m} w^{-1}$ by sliding Δ's to the right to get a positive word

$$ {}^*\overline{w} = \tau_k \cdots \tau_1 $$

whose first letter is

$$ \tau_k = \begin{cases} \sigma^{k-m}(\mu_k^*) & \text{if } \epsilon_k = 1 \\ \sigma^{k-m}(\mu_k) & \text{if } \epsilon_k = -1 \end{cases} . $$

Set $v = \mu_1^{\epsilon_1} \cdots \mu_{k-1}^{\epsilon_{k-1}}$. Then $w = v\mu_k^{\epsilon_k}$ and ${}^*\overline{w} = \tau_k({}^*\overline{v})$. Set $\rho = \Delta \wedge^* \overline{v}$, the left front of ${}^*\overline{v}$. By Proposition 2.6, $\Delta \wedge {}^*\overline{w} = \Delta \wedge (\tau_k \rho)$.

Suppose $\epsilon_k = 1$. Then $m_k = m$ and by induction, after reading v, $\mathfrak{F}$ ends in the state $(\rho, k-m-1)$. On reading the last letter μ_k, it moves to

$$\begin{aligned}(\Delta \wedge (\sigma^{k-m}(\mu_k^*)\rho), k-m) &= (\Delta \wedge \tau_k \rho, k-m) \\ &= (\Delta \wedge {}^*\overline{w}, k-m).\end{aligned}$$

Similarly, if $\epsilon_k = -1$, then $m_k = m-1$ so by induction, after reading v, $\mathfrak{F}$ ends in state $(\rho, k-m)$. On reading μ_k^{-1}, it then moves to

$$\begin{aligned}(\Delta \wedge (\sigma^{k-m}(\mu_k)\rho), k-m) &= (\Delta \wedge \tau_k \rho, k-m) \\ &= (\Delta \wedge {}^*\overline{w}, k-m).\end{aligned}$$

This completes the induction argument and shows that $\mathfrak{F}$ accepts the word w if and only if $\Delta \not\leq {}^*\overline{w}$ (where as before ${}^*\overline{w}$ may be in $\mathcal{A}^*$ and not just $\mathcal{D}^*$). Equivalently, w is accepted if and only if $\overline{w}$ lies in $\mathtt{geo}_{\mathcal{A}}(G^+)$.

By the discussion above, the language of words w such that $\overline{w}$ is geodesic is a regular language. By ([13], Thm. 1.2.8 and Lemma 1.4.1), inverses, unions, and intersections of regular languages are also regular. Thus since finite state automata can check each of the three conditions of Lemma 3.7, and one can build FSAs to check if a word is strictly positive or strictly negative (or neither), there is an FSA whose regular language is $\mathtt{geo}_{\mathcal{D}}(G)$. □

Theorem 3.8 has implications outside of the class of Garside groups. As an example, let $\mathcal{A}(\widetilde{A}_n)$ denote the *affine braid group*, or Artin group of type $\widetilde{A}_n$. This is the Artin group whose associated Coxeter group consists of affine reflections in $\mathbb{R}^n$ and whose associated Coxeter graph is a cycle of $(n+1)$-vertices with all $(n+1)$ edges labelled 3. In particular, the Coxeter group of $\widetilde{A}_3$ type is the group generated by reflections in the sides of an equilateral triangle, and

$$\mathcal{A}(\widetilde{A}_3) = \langle a, b, c \mid aba = bab, bcb = cbc, cac = aca \rangle \ .$$

As is discussed in [8], $\mathcal{A}(\widetilde{A}_n)$ embeds as an infinite index subgroup in the finite type Artin group $G = \mathcal{A}(B_{n+1})$ of type B_{n+1} (*not* the braid group!), whose associated Coxeter group is the symmetry group of the $(n+1)$-cube. The Artin group G has a Garside structure whose Garside element corresponds to a "Coxeter element" δ and whose indivisible elements correspond to all reflections in the Coxeter group. In [7] it is shown that the flag complex induced by the Cayley graph of a Garside group, with generators the set of simple divisors $\mathcal{D}$, is contractible. We denote the flag complex for G associated to δ by X_δ. As described in [8], there is an action of $\mathcal{A}(\widetilde{A}_n)$ on the subcomplex X_δ^+ of X_δ spanned by elements g of G^+ such that $\delta \not\leq g$. The action of $\mathcal{A}(\widetilde{A}_n)$ on X_δ^+ is freely transitive on vertices, hence the 1-skeleton of this subcomplex is a Cayley graph for $\mathcal{A}(\widetilde{A}_n)$. The generating set corresponding to this Cayley graph can be identified with the set of simple divisors of δ that lie in the image of $\mathcal{A}(\widetilde{A}_n)$. The language of geodesics for $\mathcal{A}(\widetilde{A}_n)$ with respect to this generating set is thus the language of geodesic paths in X_δ^+. By Proposition 3.3, this is the set of positive words w such that $w \in \mathtt{geo}_{\mathcal{D}}(G^+)$ and $\underline{w} \in \mathtt{geo}_{\mathcal{A}}(G^+)$. Since the intersection of regular languages is regular, we have the following corollary.

Corollary 3.9. *The language of geodesics for the affine braid group $\mathcal{A}(\widetilde{A}_n)$ (with respect to the generating set described above) is regular.*

Remark 3.10. A well known argument establishing that hyperbolic groups are automatic proceeds by showing that the language of all geodesics is regular *and* satisfies the fellow traveller condition (see [13]). The language consisting of normal forms for elements of a Garside group G gives a geodesic automatic structure for G [10]. However, the language of all geodesics for a Garside group does not have the fellow traveller property. This is most easily seen by noting that geodesics in $\mathbb{Z}^2$ do not fellow travel.

4. Geodesic growth series

A common measure of the rate and regularity of the growth of a group G with finite generating set S is the *spherical growth series*

$$f(z) = \sum_{g \in G} z^{|g|}$$

where $|g|$ denotes the length of g with respect to S. Growth series have been explored for a number of interesting families of groups and it is known that the growth series corresponds to a rational function in a number of important cases, for example, Coxeter groups [18] and Artin groups of finite type [6].

There is a parallel notion of the geodesic growth series that is less well understood. Let

$$p(g) = |\mathsf{geo}_S(g)|$$

be the *Pascal's function* of G, counting the number of geodesic representatives of each element [21]. The *geodesic growth series* is

$$\ell(z) = \sum_{g \in G} p(g) z^{|g|} = \sum_{w \in \mathsf{geo}_S(G)} z^{|w|} = \sum_{i=0}^{\infty} \lambda_i z^i$$

where λ_i is the number of geodesic words of length $= i$.

The following result is well known in the study of formal languages:

Proposition 4.1. *If $\mathcal{L}$ is a regular language, then the formal power series $\sum_{w \in \mathcal{L}} z^{|w|}$ is rational.*

In fact, one standard method for establishing that the spherical growth series of a group G is rational is to produce a regular language of geodesic normal forms and apply 4.1. Since Garside groups admit geodesic automatic structures ([10]) it is known that they have rational spherical growth series. Similarly, combining our Main Theorem with Proposition 4.1 we get

Corollary 4.2. *The geodesic growth series for a Garside group G, with respect to the generating set of simple divisors of Δ, is rational.*

It is not practical (in general) to go from a large FSA $\mathfrak{F}$ to an explicit rational function giving the growth series of the language accepted by $\mathfrak{F}$. However, it is possible to find rational expressions for the geodesic growth series for the Artin groups of dihedral type by brute force. Recall that the Artin group of type I_m is the group presented by

$$A = \langle a, b \mid \underbrace{aba\cdots}_{m \text{ letters}} = \underbrace{bab\cdots}_{m \text{ letters}} \rangle \ .$$

These groups are often referred to as the Artin groups of *dihedral type*; they are also encountered as the fundamental groups of the complements of $(2, m)$-torus knots and links.

Example 4.3 (Dihedral type with $\Delta = \delta$)**.** The Coxeter quotient of A is the dihedral group of order $2n$, D_n. The group A is generated by two elements a, b whose images in D_n are reflections and whose product $\delta = ab$ corresponds in D_n to a rotation of order n. Let $R \subset A$ be the set of lifts of all the reflections in D_n defined by $R = \{a, b, b^{-1}ab, (ab)^{-1}b(ab), \cdots\}$ such that the product of any two consecutive elements in R is δ. Let A^+_δ be the submonoid of A generated by R. This describes a Garside structure for A with $\Delta = \delta$ and $\mathcal{D} = R \cup \{\delta\}$. In particular, the Artin group of type I_4 can be represented as the Garside group given by the presentation

$$I_4 = \langle a, b, c, d, \delta \mid \delta = ab = bc = cd = da\rangle$$

where $\{a, b, c, d\}$ correspond to the four reflections in D_4 and δ corresponds to a $90°$ rotation (see Figure 2).

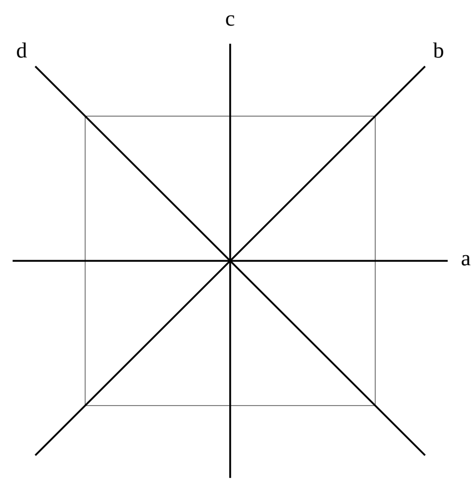

FIGURE 2. The Coxeter quotient for the Artin group of I_4 type

In terms of the nature of geodesics, the subgraph of the Cayley graph corresponding to A^+ behaves differently than the subgraph corresponding to $A \setminus A^+$. A small piece of the Cayley graph is illustrated in Figure 3, where one of the fundamental squares is fully labelled, and the larger vertex indicates the location of the identity. The entire Cayley graph for the Artin group of type I_m has roughly the structure of an m-valent tree crossed with $\mathbb{Z}$ (ignoring edge labels and the diagonal elements).

Call the subgraph of the Cayley graph induced by A^+ the *positive half* and call its complement the *mixed half*. The sphere of radius 3 is indicated (schematically) in Figure 4. Note that in the positive half, geodesics from the identity never travel along the diagonal edges. The portion labelled "level zero" corresponds to the subgraph of the Cayley graph induced by those elements of A^+ that contain no δs; this is just the m-valent tree.

The number of geodesics of length n in the level zero subgraph is $m(m-1)^{n-1}$ $(n \geq 1)$, hence the number of geodesics of length $n \geq 0$ in the positive half of the

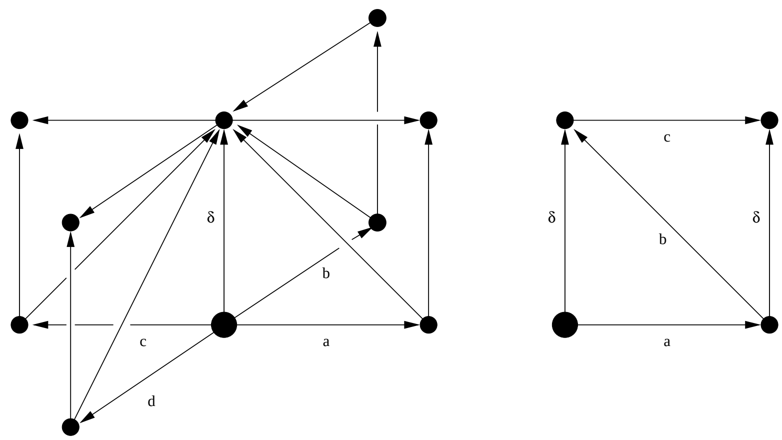

FIGURE 3. A portion of the Cayley graph

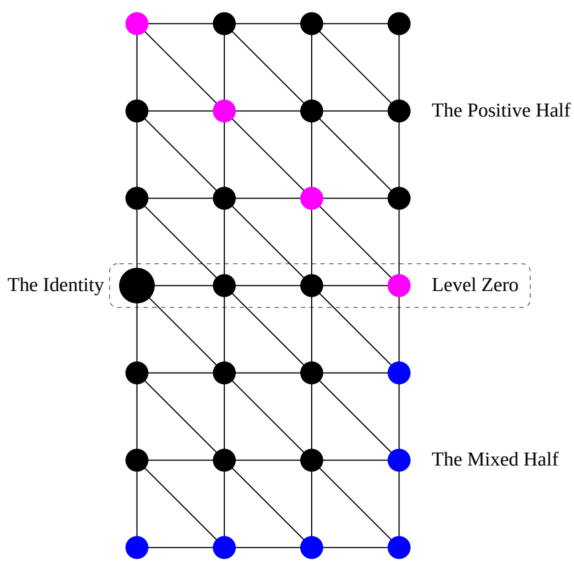

FIGURE 4. A schematic of the sphere of radius 3

Cayley graph is given by

$$\begin{aligned} \lambda_n &= 1 + \sum_{k=1}^{n} \binom{n}{k} m(m-1)^{k-1} \\ &= 1 + \frac{m}{m-1} \sum_{k=1}^{n} \binom{n}{k} (m-1)^k \\ &= 1 + \frac{m}{m-1} \left[m^n - 1 \right] . \end{aligned}$$

A routine computation then shows that the geodesic growth series corresponding to the positive half is the series for $\dfrac{1}{(1-z)(1-mz)}$.

One can divide the sphere of radius n in the mixed half into those elements whose geodesics involve some negative power of δ (those lying below the diagonal in the schematic of Figure 4) and those that use no δ (those lying on or above the diagonal in the schematic of Figure 4). The number of geodesics of length n in the mixed half that use a δ^{-1} can be counted by choosing where the non-δ generators occur:

$$1+\binom{n}{1}m+\binom{n}{2}m(m-1)+\cdots+\binom{n}{n-1}m(m-1)^{n-2}\ .$$

This simplifies to $1+\dfrac{m}{m-1}\left[m^n-(m-1)^n-1\right]=\dfrac{m}{m-1}\left[m^n-(m-1)^n-\dfrac{1}{m}\right]$.

The geodesics of length n with no δ correspond to paths in the m-valent tree of length n along with 2^n choices as to whether one multiplies by a generator or the inverse of a generator. To avoid overcounting the m-valent tree at level zero, at least one inverse must be used. This gives the total as $\dfrac{m}{m-1}\left[(2m-2)^n-(m-1)^n\right]$. In both mixed case computations we are assuming $n\geq 1$ as $n=0$ was counted in the positive half.

Finding rational expressions for the corresponding series, and combining them with the expression for the positive half, results in the rather opaque

$$\ell(z)=\frac{1-2(m-2)z+(m^2-7m+5)z^2+(m-1)(m-2)z^3+2m(m-1)z^4}{(1-z)(1-(m-1)z)(1-mz)(1-2(m-1)z)}\ .$$

In particular, if $m=3$ then A is the three strand braid group, and the corresponding geodesic growth series is

$$\ell(z)=\frac{1-2z-7z^2+2z^3+12z^4}{(1-z)(1-2z)(1-3z)(1-4z)}\ .$$

Example 4.4 (Dihedral type with $\Delta=\Delta$)**.** One can do a similar analysis for the Artin groups of dihedral type using $\Delta=\underbrace{aba\cdots}_{m\text{ letters}}=\underbrace{bab\cdots}_{m\text{ letters}}$ as Garside element. As in the previous example, one really only needs to understand the growth of the subgraph at level zero, and then apply routine techniques. In the case where $\Delta=\Delta$ the graph at level zero is just the 1-skeleton of a "tree of simplices," or more specifically, it is the 1-skeleton of a simplicial complex formed by $(m-1)$-dimensional simplices joined two at each vertex. The growth series for the graph at level zero is

$$1+2(m-1)z+2(m-1)^2z^2+\cdots=\frac{1+(m-1)z}{1-(m-1)z}\ .$$

Combining the growth series for the positive half and the mixed half one gets

$$\ell(z)=\frac{1-(3m^2-5m+3)z^2+(2m-5)(m-1)(m-2)z^3+2m(m-1)^2z^4}{(1-z)(1-(m-1)z)(1-mz)(1-2(m-1)z)}\ .$$

In the case of the three-strand braid group $(m=3)$ this simplifies to

$$\ell(z)=\frac{1-15z^2+2z^3+24z^4}{(1-z)(1-2z)(1-3z)(1-4z)}\ .$$

Remark 4.5. Grigorchuk and Nagnibeda introduced "complete" versions of these formal power series [15]. For example, the *complete geodesic growth series* for a group G with (monoid) generating set S is the element of the ring of formal power series with semigroup ring coefficients, $\mathbb{Z}S^*[[z]]$, defined by

$$L(z) = \sum_{w \in \mathtt{geo}_S(G)} w z^{|w|} = \sum_{n=0}^{\infty} \Lambda_n z^n$$

where Λ_n is the element of $\mathbb{Z}S^*$ gotten by adding the geodesic words of fixed length n. The augmentation map $\mathbb{Z}S^* \to \mathbb{Z}$ sends the complete geodesic growth series to the ordinary geodesic growth series. Our Main Theorem combined with Proposition 5 of [15] shows that the complete geodesic growth series of any Garside group, with generating set the simple divisors of a Garside element, is rational.

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