THE TITS CONJECTURE FOR LOCALLY REDUCIBLE ARTIN GROUPS

RUTH CHARNEY[(*)]

ABSTRACT. Given an Artin system (A, S), a conjecture of Tits states that the subgroup $A^{(2)}$ of A generated by the squares of the generators in S is subject only to the obvious commutator relations between generators. In particular, $A^{(2)}$ is a right-angled Artin group. We prove this conjecture for a class of infinite type Artin groups, called locally reducible Artin groups, for which the associated Deligne complex has a CAT(0) geometry. We also prove that for any special subgroup A_T of A, $A^{(2)} \cap A_T = (A_T)^{(2)}$.

INTRODUCTION

It is a basic principle of geometric group theory that when a group acts on a topological space, the geometry of the space is reflected in the algebraic structure of the group. Recently, this idea has been used with great success in the case of groups acting on metric spaces with nice curvature properties. Some examples of this involve the rich and interesting groups known as Artin groups. In this paper, we use curvature properties to prove a conjecture of Tits for Artin groups with particularly nice local structure.

Artin groups are closely related to Coxeter groups. Let S be a finite set and let $m : S \times S \to \{1, 2, 3, \ldots, \infty\}$ be a symmetric function such that $m(s, s) = 1$ and $m(s, t) \geq 2$ for all $s \neq t$. Then m is called a *Coxeter matrix* and the group

$$W = \langle S \mid (st)^{m(s,t)} = 1 \rangle$$

is a *Coxeter group.* (We omit relations with $m(s, t) = \infty$). The pair (W, S) is called a *Coxeter system.* We can also encode this information into a labeled graph. The *Coxeter graph* for (W, S) has vertex set S and an edge labelled $m(s, t)$ connecting vertices s and t whenever $m(s, t) \geq 3$. Associated to a Coxeter matrix (or Coxeter graph) is an *Artin system* (A, S) where A is the group given by the presentation

$$A = \langle S \mid prod(s, t; m) = prod(t, s; m) \rangle$$
$$\text{where} \quad prod(s, t; m) = \underbrace{stst\ldots}_{m(s,t)-\text{terms}}$$

1991 *Mathematics Subject Classification.* 20F36, 20F65.
Key words and phrases. Artin groups, CAT(0) spaces.
(*) Partially supported by NSF grant DMS-9803374

(Again, we omit relations with $m(s,t) = \infty$). There is a natural surjection $A \to W$ obtained by adding the relations $s^2 = 1$ to the presentation for A. The Artin group A is said to be of *finite type* (resp. *infinite type*) if the Coxeter group W is finite (resp. infinite). The rank of A (or W) is the cardinality of S. The subgroups A_T generated by subsets T of S are called a *special subgroups* of A. The pair (A_T, T) is, itself, an Artin system for any $T \subseteq S$ (see [D] and [L]).

Artin groups arise in a wide range of mathematical settings. The most well known Artin groups are the classical braid groups B_n whose associated Coxeter groups are the symmetric groups Σ_n. Another well known class of Artin groups are the *right-angled* Artin groups (also known as graph groups) for which all $m(s,t) \in \{2, \infty\}$. These groups play a central role in the stunning work of Bestvina and Brady [BB] on finiteness properties of groups.

Finite type Artin groups have been studied extensively beginning in the late '60's and early '70's with Tits [T], Garside [Ga], Brieskorn-Saito [BS], and Deligne [D]. Their results, including a normal form for elements of A and a contractible simplicial complex (known as the Deligne complex) on which the group acts, have served as the basis for much of our understanding of these groups. The normal form is used, for example, in [C1] and [C2] to give explicit automatic structures and growth functions for A. It is also used in a recent paper of Bestvina [Be] to construct another simplicial complex which satisfies a weak curvature property, leading to a variety of new results about the algebraic structure of finite type Artin groups.

Infinite type Artin groups are not as well understood. Some infinite type Artin groups, such as the "triangle-free" groups (those for which any three generators $\{s_1, s_2, s_3\} \subseteq S$ include a pair with $m(s_i, s_j) = \infty$) and the "extra-large" groups (those with all $m(s,t) \geq 4$) lend themselves to the techniques of small cancellation theory. This approach was used by Appel-Schupp [AS], Pride [P], and Peifer [Pe]. Other infinite type Artin groups act on spaces of nonpositive curvature and lend themselves to geometric arguments. In [BM], for example, Brady and McCammond construct non-positively curved 2-complexes with fundamental group A for a variety of infinite type Artin groups.

In [CD1], the author and M. Davis construct an analogue of the Deligne complex for infinite type Artin groups. They prove that, under certain conditions (namely, if (A, S) is "2-dimensional" or of "FC-type"), this complex is CAT(0), i.e., it is simply connected and nonpositively curved. The Artin group A acts cocompactly, but not properly, on this complex—the isotropy groups of the action are finite type special subgroups of A. Using the CAT(0) geometry, it is possible to reduce properties of the infinite type group A to properties of these finite type subgroups. This technique is used in [C3] to show that the monoid of positive words A^+ injects into A (under the same conditions as above) and in [AC], it is used to construct normal forms for Artin groups of FC-type.

In this paper, we use a similar technique to address a problem which has come to be known as the Tits Conjecture (see [T]). Suppose (A, S) is an Artin system with Coxeter matrix m. Define a new Coxeter matrix $\widehat{m}$ by

$$\widehat{m}(s,t) = \begin{cases} 1 & \text{if } s = t \\ 2 & \text{if } m(s,t) = 2 \\ \infty & \text{otherwise.} \end{cases}$$

Then the associated Artin system $(\widehat{A}, S)$ is a right-angled Artin system and there is a homomorphism $\widehat{A} \to A$ defined by $s \mapsto s^2,\ s \in S$. The Tits Conjecture states that this homomorphism is injective, or equivalently, that the subgroup $A^{(2)}$ of A generated by $\{s^2 \mid s \in S\}$ is isomorphic to $\widehat{A}$.

The conjecture has been verified in a number of special cases but, surprisingly, it is not even known to hold for all finite type Artin groups.[1] It was verified for Artin groups of extra-large type by Appel and Schupp in [AS] and for triangle-free groups by Pride in [P]. The case of the classical braid groups B_n was done by Droms, Lewin and Servatius for $n \leq 5$ in [DLS1] and [DLS2], and for arbitrary n by Collins in [Co]. This was generalized by Humphries in [H] to include many other Artin groups of small type (all $m(s,t) \leq 3$).

In this paper we introduce the notion of "locally reducible" Artin groups, namely Artin groups for which all of the finite type special subgroups are direct products of rank 1 and rank 2 subgroups. This includes the right-angled groups, the triangle-free groups, the Artin groups of large type (all $m(s,t) \geq 3$), and many others. From results of [CD1], it follows that the Deligne complex for a locally reducible Artin group A is CAT(0). Using the local-to-global properties of CAT(0) spaces, we prove the Tits Conjecture for these groups. For any special subgroup A_T of A, we also prove that $A^{(2)} \cap A_T = (A_T)^{(2)}$.

1. Piecewise Euclidean Complexes

In this section we give a brief review of CAT(0) spaces. For more details, see [BH].

A *piecewise Euclidean complex* is a cell complex X made up of convex polyhedral Euclidean cells glued together by isometries along faces. We do not require that the cell complex be locally finite, but we do assume that X has *finite shapes*, that is, that there are only finitely many isometry types of cells in X. A *piecewise geodesic* in X is a path $\gamma : [a,b] \to X$ for which $[a,b]$ can be divided into subintervals $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to $[t_i, t_{i+1}]$ is an isometric embedding into some cell of X. Let $l(\gamma)$ denote the length of γ. We define the *intrinsic metric* on X as follows.

$$d(x,y) = \inf \{ l(\gamma) \mid \gamma \text{ is a piecewise geodesic from } x \text{ to } y \}$$

Under the finite shapes assumption, the intrinsic metric is a complete, geodesic metric; that is, there is a length minimizing path between any two points in the same path component of X (See [Br]). Such a path is called a *geodesic.*

[1] Between the completion and the publication of this paper, J. Crisp and L. Paris announced a proof of the Tits Conjecture for all Artin groups. Their methods are independent of this paper.

A *piecewise spherical complex* and its intrinsic metric are defined similarly, using convex polyhedral cells in a sphere $\mathbb{S}^n$ instead of Euclidean cells. If x is a point in a piecewise Euclidean complex X, then the set of unit tangent vectors to X at x is naturally a piecewise spherical complex called the *link* of x in X, or $link(x, X)$. These links determine the local structure of X: A neighborhood of x in X is isometric to the Euclidean cone on $link(x, X)$. A piecewise geodesic γ in X is called a *local geodesic* if for each point x on γ, the incoming and outgoing unit tangent vectors to γ at x are at distance at least π in $link(x, X)$. It is easy to see that a local geodesic is locally a length minimizing path (hence the terminology).

Let X be a piecewise Euclidean (resp. piecewise spherical) complex and set $M = \mathbb{R}^2$ (resp. $\mathbb{S}^2$). Let T be a geodesic triangle in X. A *comparison triangle* for T is a triangle T' in M with the same side lengths as T. We say X is a *CAT(0)* (resp. *CAT(1)*) if for any geodesic triangle T in X and any two points x, y on T, the distance from x to y in X is less than or equal to the distance in M between the corresponding points x', y' on the comparison triangle T'. (In the piecewise spherical case, we only require that this hold for triangles T of perimeter at most 2π, since no comparison triangle exists for T of perimeter greater than 2π.) We say X is *locally CAT(0)* (resp. *locally CAT(1)*) if every point in X has a neighborhood which is CAT(0) (resp. CAT(1)). We will need the following fundamental facts about CAT(0) spaces. These facts were first noted by Gromov ([G]). Proofs can be found in [BH].

Theorem 1.1. *Let X be a connected, piecewise Euclidean complex.*

(1) *X is locally CAT(0) if and only if $link(v, X)$ is CAT(1) for every vertex v in X.*
(2) *X is CAT(0) if and only if X is locally CAT(0) and simply connected.*

Theorem 1.2. *Let X be a connected, CAT(0) piecewise Euclidean complex. Then*

(1) *any two points in X are connected by a unique geodesic, and*
(2) *any local geodesic in X is a geodesic.*

We will also need a number of facts about joins of piecewise spherical complexes. These arise naturally as links in the product of two piecewise Euclidean complexes. For σ_1, σ_2 spherical simplices of dimensions k_1, k_2, we define the *orthogonal join*, $\sigma_1 * \sigma_2$, as follows. Embed $\mathbb{S}^{k_1}$ and $\mathbb{S}^{k_2}$ orthogonally in $\mathbb{S}^k$, $k = k_1 + k_2 + 1$ (so that every point in $\mathbb{S}^{k_1}$ has distance $\pi/2$ from every point in $\mathbb{S}^{k_2}$). Then $\sigma_1 * \sigma_2$ is the k-simplex in $\mathbb{S}^k$ spanned by $\sigma_1 \subset \mathbb{S}^{k_1}$ and $\sigma_2 \subset \mathbb{S}^{k_2}$. It is also convenient to define $\emptyset * \sigma = \sigma * \emptyset = \sigma$.

Suppose L_1 and L_2 are piecewise spherical complexes. The *orthogonal join*, $L_1 * L_2$, is the piecewise spherical complex whose simplices are the orthogonal joins $\sigma_1 * \sigma_2$ of (possibly empty) simplices $\sigma_1 \subset L_1$ and $\sigma_2 \subset L_2$. (Thus, combinatorially, $L_1 * L_2$ is the usual simplicial join of the two complexes.) If X_1 and X_2 are piecewise Euclidean complexes, then the link of a point (x_1, x_2) in $X_1 \times X_2$ is easily seen to be the orthogonal join of $link(x_1, X_1)$ and $link(x_2, X_2)$. Orthogonal joins are discussed in the appendix of [CD2] where the following theorem is proved.

Theorem 1.3. *The orthogonal join $L_1 * L_2$ of two piecewise spherical complexes is CAT(1) if and only if L_1 and L_2 are CAT(1).*

Definition. Let $f : L \to L'$ be a map of piecewise spherical complexes. We say f is *π-distance preserving* if $d_L(x, y) \geq \pi$ implies $d_{L'}(f(x), f(y)) \geq \pi$.

Remark. If L is not connected, it is useful to set $d_L(x, y) = \infty$ for points x, y in different path components. In particular, if L' is a 0-dimensional cell complex, then $f : L \to L'$ is π-distance preserving if and only if it is injective.

The significance of this condition is explained by the following lemma.

Lemma 1.4. *Suppose $f : X \to X'$ is a map of piecewise Euclidean complexes which takes piecewise geodesics to piecewise geodesics. Then the following are equivalent.*

(1) *The induced maps $link(x, X) \to link(f(x), X')$ are π-distance preserving for all $x \in X$.*
(2) *f is locally an isometric embedding.*

If, in addition, X' is CAT(0), then these conditions are also equivalent to

(3) *f is globally an isometric embedding.*

Proof. The first condition is equivalent, by definition, to the statement that f maps local geodesics in X to local geodesics in X'. Since local geodesics are actually geodesic (length minimizing) in a neighborhood of any point, this is equivalent to condition (2). If X' is CAT(0), then by Theorem 1.2, local geodesics are also global geodesics, hence (2) is equivalent to (3). □

Lemma 1.5. *If $f_1 : L_1 \to L_1'$ and $f_2 : L_2 \to L_2'$ are π-distance preserving, then so is $f_1 * f_2 : L_1 * L_2 \to L_1' * L_2'$.*

Proof. Lemmas A2 and A6 of the appendix of [CD2] give precise conditions for when two points in an orthogonal join have distance π. The statement above follows immediately from these conditions. □

Lemma 1.6. *Let L be a piecewise spherical simplicial complex with all edge lengths $\geq \pi/2$. If L_0 is a full subcomplex of L, then the inclusion map $L_0 \hookrightarrow L$ is π-distance preserving.*

Proof. Let x, y be two points in L_0 and let γ be a geodesic in L from x to y. If γ lies entirely in L_0, then the distance from x to y in L_0 is the same as the distance in L. If γ does not lie entirely in L_0, then it contains a segment α whose interior lies in the open star of a vertx $v \in L \backslash L_0$ and whose endpoints lie on the boundary of star(v). It follows from Lemma 9.8 of [M] (see also Lemma 8.7 of [CD1]) that α, and hence γ, has length $\geq \pi$. □

2. The Deligne Complex

In this section we describe the Deligne complex $\mathcal{D}_A$ for an Artin group A and a natural piecewise Euclidean structure on $\mathcal{D}_A$.

Let (A,S) be an Artin system associated to the Coxeter system (W,S). We introduce the following notation.

$$\mathcal{S}^f = \{T \mid T \subseteq S,\ A_T \text{ is finite type}\}$$

$$A\mathcal{S}^f = \{aA_T \mid a \in A,\ T \in \mathcal{S}^f\}$$

These sets are partially ordered by inclusion. The Deligne complex, $\mathcal{D}_A$, is the flag complex associated to $A\mathcal{S}^f$. That is, the vertices of $\mathcal{D}_A$ are the elements of $A\mathcal{S}^f$ and the simplices of $\mathcal{D}_A$ are the totally ordered subsets of $A\mathcal{S}^f$.

A acts by left multiplication on $A\mathcal{S}^f$ and hence also on $\mathcal{D}_A$. A fundamental domain for this action is the subspace spanned by the vetices $\{A_T \mid T \in \mathcal{S}^f\}$. Denote this fundamental domain by K. This gives an alternate description of $\mathcal{D}_A$. Namely, let $K_{\geq T}$ denote the subcomplex of K spanned by the vertices A_R with $A_T \subseteq A_R$. Then

$$\mathcal{D}_A = A \times K/\sim$$

where $(a_1, x_1) \sim (a_2, x_2)$ if $x_1 = x_2$, x_1 lies in the relative interior of $K_{\geq T}$, and $a_1 A_T = a_2 A_T$. (By the relative interior of $K_{\geq T}$ we mean the open star of the vertex A_T in $K_{\geq T}$, or equivalently, the points in $K_{\geq T}$ which do not lie in $K_{\geq R}$ for any $R \supsetneq T$.) K embeds in $\mathcal{D}_A$ as the image of $1 \times K$. By analogy with standard terminology for Coxeter complexes, we call the translates of K *chambers* of $\mathcal{D}_A$ and K itself the *fundamental chamber* of $\mathcal{D}_A$.

In [CD1], a natural piecewise Euclidean structure on $\mathcal{D}_A$ is defined, based on a similar structure for Coxeter complexes introduced by Moussong in [M]. This geometry is preserved by the action of A, thus it suffices to describe the piecewise Euclidean structure on K. To define this structure, it is convenient to view K as a cubical, instead of simplicial, complex. For $T \in \mathcal{S}^f$, the vertices $A_R \leq A_T$ span a combinatorial cube in K of dimension $|T|$ which we denote by $cube(T)$. K is the union of these cubes. We assign a Euclidean metric to $cube(T)$ as follows.

There is a standard realization of W_T as a group of orthogonal transformations of $\mathbb{R}^{|T|}$ with the generators acting as reflections in the codimension 1 faces (or "walls") of a simplicial cone C_T (see eg., [Brn]). There is a unique point $x_\emptyset$ in the interior of C_T whose distance from every wall of C_T is 1. The *Coxeter cell* X_T is the convex hull in $\mathbb{R}^{|T|}$ of the W_T-orbit of $x_\emptyset$. For $R \subseteq T$, let F_R^* denote the convex hull of the W_R-orbit of $x_\emptyset$. These subsets, together with their W_T-translates, $\{wF_R^*\}$, are the faces of X_T. Let F_R denote the face of C_T fixed by W_R. Then F_R and F_R^* are orthogonal and intersect at a single point x_R.

The intersection of X_T with C_T is combinatorially a cube with vertices $\{x_R\}_{R \subseteq T}$. The Euclidean structure on $cube(T)$ is defined by identifying it with $X_T \cap C_T$ so that the vertex A_R of $cube(T)$ is identified to the vertex x_R of $C_T \cap X_T$. (Figure 1.) If $R \subset T$, then F_R^* is isometric to the Coxeter cell X_R, so the face of $cube(T)$ spanned by A_R and $A_\emptyset$ is isometric to $cube(R)$. Thus, the metrics on the cubes fit together to give a piecewise Euclidean structure on K and hence on $\mathcal{D}_A$. We call the induced metric the *Moussong metric* and denote it by d_M.

If $R \subset T$, then F_R^* is isometric to the Coxeter cell X_R, so the face of $cube(T)$ spanned by A_R and $A_\emptyset$ is isometric to $cube(R)$. Thus, the metrics on the cubes fit

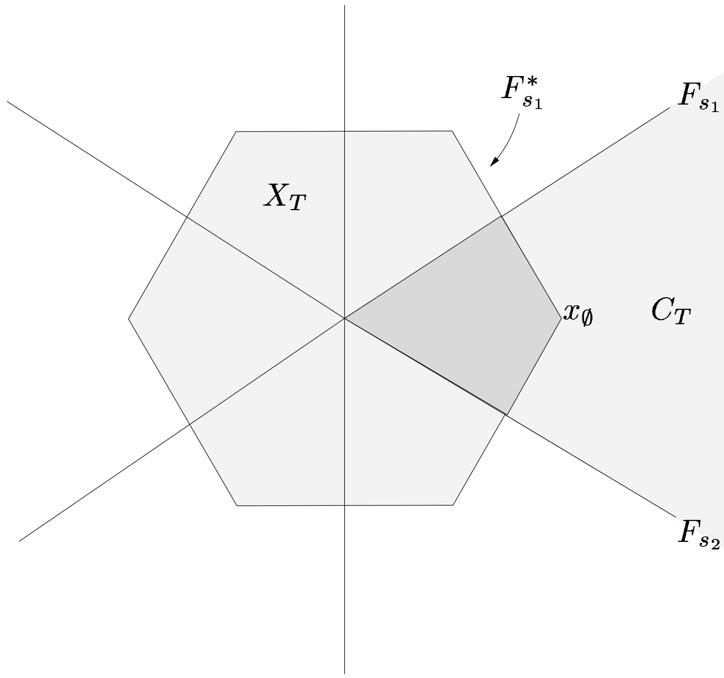

FIGURE 1

together to give a piecewise Euclidean structure on K and hence on $\mathcal{D}_A$. We call the induced metric the *Moussong metric* and denote it by d_M.

If A is finite type, then $A\mathcal{S}^f$ has a unique maximal element, namely A_S $(= A)$. Thus $K = cube(S)$ and $\mathcal{D}_A$ is a cone with cone point the vertex A_S. Let $\mathcal{B}_A$ denote the link of the cone point. Then $\mathcal{B}_A$ is a simplicial complex of dimension $n-1$ (where $n = |S|$) with one top-dimensional simplex for each element of A. The piecewise spherical structure on $\mathcal{B}_A$ is given by identifying each top-dimensional simplex with the link of the origin in the cone C_S (or in other words, with a fundamental chamber of the standard Coxeter complex for (W,S)).

Returning to the case of an infinite type A, we now describe the link of a point in $\mathcal{D}_A$. For $T \subseteq R$, we denote by $cube(T, R)$ the face of $cube(R)$ spanned by A_T and A_R. Note that $cube(T, R)$ lies in the subcomplex $K_{\geq T}$ of K.

Lemma 2.2. *If x lies in the interior of $cube(T, R)$ and $j = |R| - |T|$, then*

$$\begin{aligned} link(x, \mathcal{D}_A) &= link(x, K_{\geq T}) * \mathcal{B}_{A_T} \\ &= \mathbb{S}^{j-1} * link(A_R, K_{\geq R}) * \mathcal{B}_{A_T} \end{aligned}$$

Proof. The first equality is proved in Lemma 2.2 of [C3]. For the second, note that since x lies in the interior of $cube(T, R)$, the link of x in $cube(T, R)$ is a $(j-1)$-sphere. Hence,

$$\begin{aligned} link(x, K_{\geq T}) &= link(x, cube(T, R)) * link(cube(T, R), K_{\geq T}) \\ &= \mathbb{S}^{j-1} * link(cube(T, R), K_{\geq T}) \\ &= \mathbb{S}^{j-1} * link(A_R, K_{\geq R}) \end{aligned}$$

where the last equality follows from the fact that $cube(T, R)$ intersects $K_{\geq R}$ orthogonally at A_R. $\square$

3. Locally Reducible Artin Systems

Definition. An Artin system (A, S) is *totally reducible* if its Coxeter diagram is a disjoint union of vertices and single edges. Equivalently, (A, S) is totally reducible if it is a direct product of Artin systems $(A_1, S_1) \times \cdots \times (A_k, S_k)$ of rank ≤ 2. An Artin system (A, S) is *locally reducible* if (A_T, T) is totally reducible for every $T \in \mathcal{S}^f$.

Examples. (1) If (A, S) is a right-angled Artin system, then $T \in \mathcal{S}^f$ implies A_T is free abelian, so A is locally reducible.

(2) An Artin system is *2-dimensional* if, for all $T \in \mathcal{S}^f$, (A_T, T) is rank ≤ 2 (or equivalently, if it's Deligne complex is 2-dimensional). Clearly, every 2-dimensional Artin system is locally reducible. This class includes the "triangle-free" Artin groups mentioned in the introduction as well as the Artin groups of "large type" (i.e., those with all $m(s,t) \geq 3$). The latter follows from the well known fact that a Coxeter group on three generators r, s, t is finite if and only if

$$\frac{1}{m(r,s)} + \frac{1}{m(s,t)} + \frac{1}{m(r,t)} > 1.$$

(3) Let (A, S) be an Artin system with Coxeter graph an n-gon, $n \geq 4$, and suppose that for any three consecutive vertices r, s, t in this n-gon

$$\frac{1}{m(r,s)} + \frac{1}{m(s,t)} \leq \frac{1}{2}.$$

(where $\frac{1}{\infty}$ is defined to be 0.) Then r, s, t generate an infinite type subgroup of A (since $m(r,t) = 2$). Hence, if $T \in \mathcal{S}^f$, then (A_T, T) corresponds to a subdiagram consisting of a disjoint union of single edges and vertices. Thus, (A, S) is locally reducible.

More generally, we have the following characterization.

Lemma 3.1. *Let Γ be a Coxeter graph and let A be the associated Artin group. Then A is locally reducible if and only if Γ satisfies the following condition:*

If two consecutive edges e_1, e_2 in Γ are not contained in a common triangle, then their labels $m_1, m_2 \in \{3, 4, \ldots, \infty\}$ satisfy $\frac{1}{m_1} + \frac{1}{m_2} \leq \frac{1}{2}$.

The following is an easy generalization of Proposition 4.4.5 of [CD1].

Theorem 3.2. *If (A, S) is a locally reducible Artin system, then the Deligne complex $\mathcal{D}_A$ with the Moussong metric d_M is CAT(0).*

Proof. By Theorem 1.1, $\mathcal{D} = \mathcal{D}_A$ is CAT(0) if and only if it is simply connected and the link of every vertex is CAT(1). That $\mathcal{D}$ is simply connected follows from [CD1] Proposition 1.5.1 (where $\mathcal{D}$ is called the "modified Deligne complex" and is denoted by Φ). A vertex v of $\mathcal{D}$ is a coset aA_T, $T \in \mathcal{S}^f$. Without loss of generality, we may assume $a = 1$ so $v = A_T$. Then by Lemma 2.2, the link of v in $\mathcal{D}$ is the orthogonal join

$$link(v, \mathcal{D}) = link(A_T, K_{\geq T}) * \mathcal{B}_{A_T}.$$

It was shown by Moussong in [M] (see also [CD1], Lemma 4.4.1) that $link(A_T, K_{\geq T})$ is CAT(1). It remains to show that $\mathcal{B}_{A_T}$ is CAT(1).

Since (A, S) is locally reducible, we can write

$$(A_T, T) = (A_1, T_1) \times \cdots \times (A_k, T_k)$$

with $|T_i| \leq 2$. It follows easily that $\mathcal{D}_{A_T}$ decomposes as a product and hence

$$\mathcal{B}_{A_T} = \mathcal{B}_{A_1} * \cdots * \mathcal{B}_{A_k}.$$

If $|T_i| = 1$, $\mathcal{B}_{A_i}$ is 0-dimensional and hence (vacuously) CAT(1). If $|T_i| = 2$, $\mathcal{B}_{A_i}$ is CAT(1) by [CD1], Proposition 4.4.5. The desired result now follows from Theorem 1.3 above. □

4. The Tits Conjecture

Let m be a Coxeter matrix and (A, S) the associated Artin system. As described in the introduction, we can define a new Coxeter matrix $\widehat{m}$ whose associated Artin system $(\widehat{A}, S)$ is a right-angled Artin system. Let $\phi : \widehat{A} \to A$ be the homomorphism sending $s \in S$ to s^2. Although we now have two Artin groups, the symbols K and $\mathcal{S}^f$ will always correspond to the fundamental chamber and finite type special subgroups of the original group A. Define a new cell complex $\widehat{\mathcal{D}}_A$ by

$$\widehat{\mathcal{D}}_A = \widehat{A} \times K / \sim$$

where $(a_1, x) \sim (a_2, x)$ if $x \in K_{\geq T}$ and $a_1^{-1} a_2 \in \widehat{A}_T$. (Note that $\widehat{\mathcal{D}}_A$ is not the Deligne complex for $\widehat{A}$ since K is the fundamental chamber for A. In particular, $\widehat{A}_T$ need not be finite type for $T \in \mathcal{S}^f$) We can identify the vertices of $\widehat{\mathcal{D}}_A$ with cosets $a\widehat{A}_T$, $a \in \widehat{A}$, $T \in \mathcal{S}^f$. In particular, the vertices $a\widehat{A}_\emptyset$ are distinct for different $a \in \widehat{A}$. The Moussong metric on K induces a piecewise Euclidean metric on $\widehat{\mathcal{D}}_A$, and the homomorphism ϕ induces a map $\widehat{\phi} : \widehat{\mathcal{D}}_A \to \mathcal{D}_A$ taking $a \times K$ isometrically onto $\phi(a) \times K$. To prove that ϕ is injective, it suffices to prove that $\widehat{\phi}$ is an embedding. In fact, we will show that $\widehat{\phi}$ is an isometric embedding.

If (A, S) is finite type, then $\widehat{\mathcal{D}}_A$ has a cone point, the vertex $\widehat{A}$ $(= \widehat{A}_S)$. In this case we define

$$\widehat{\mathcal{B}}_A = link(\widehat{A}, \widehat{\mathcal{D}}_A)$$

with its natural piecewise spherical metric. Clearly, $\widehat{\phi}$ induces a map $\widehat{\phi} : \widehat{\mathcal{B}}_A \to \mathcal{B}_A$.

Let w be a word in the free group $F(S)$. Then w can be uniquely written in the form $w = s_{i_1}^{n_1} \ldots s_{i_k}^{n_k}$ with $n_i \neq 0$ and $s_{i_j} \neq s_{i_{j+1}}$. We call $s_{i_j}^{n_j}$, $j = 1, \ldots, k$, the *syllables* of w and k the *syllable length* of w. We denote the syllable length by $\|w\|$. The *word length* (or simply the *length*) of w is $|n_1| + \cdots + |n_k|$. In particular, the length of the syllable $s_{i_j}^{n_j}$ is just $|n_j|$.

Lemma 4.1. *Suppose (A, S) is totally reducible and finite type. Then $\widehat{\phi} : \widehat{\mathcal{B}}_A \to \mathcal{B}_A$ is π-distance preserving.*

Proof. By hypothesis, $A = A_1 \times \cdots \times A_k$ where A_k is a special subgroup of rank 1 or 2, and hence $\widehat{A} = \widehat{A}_1 \times \cdots \times \widehat{A}_k$. It follows that we have decompositions

$$\mathcal{B}_A = \mathcal{B}_{A_1} * \cdots * \mathcal{B}_{A_k}$$
$$\widehat{\mathcal{B}}_A = \widehat{\mathcal{B}}_{A_1} * \cdots * \widehat{\mathcal{B}}_{A_k} ,$$

and $\widehat{\phi}$ decomposes as the join of maps

$$\widehat{\phi}_i : \widehat{\mathcal{B}}_{A_i} \to \mathcal{B}_{A_i} .$$

By Lemma 1.5 it suffices to show that each $\widehat{\phi}_i$ is π-distance preserving.

This is true for A_i of rank 1, since in this case, $\mathcal{B}_{A_i}$ and $\widehat{\mathcal{B}}_{A_i}$ are both 0-dimensional and the map $\widehat{\phi}_i$ can be identified with the doubling map $\mathbb{Z} \to \mathbb{Z}$. In particular, it is injective.

Suppose A_i has rank 2. Then A_i is of the form $A_i = A_T$ with $T = \{s, t\}$ and $m = m(s,t) \geq 3$. Thus $\widehat{A}_i$ is the free group on two generators. For clarity, we will denote the generators of $\widehat{A}_i$ by $\hat{s}$ and $\hat{t}$, with ϕ_i mapping $\hat{s}$ to s^2 and $\hat{t}$ to t^2. $\mathcal{B}_{A_i}$ is a 1-dimensional complex with all edges of length $\frac{\pi}{m}$. Edges of $\mathcal{B}_{A_i}$ are labelled by elements $a \in A_i$. Similarly, $\widehat{\mathcal{B}}_{A_i}$ is a 1-dimensional complex with edge lengths $\frac{\pi}{m}$ and edges labelled by $\hat{a} \in \widehat{A}_i = F(\hat{s}, \hat{t})$. The map $\widehat{\phi}_i$ takes the edge $\hat{a}$ of $\widehat{\mathcal{B}}_{A_i}$ isometrically onto the edge $\phi_i(\hat{a})$ of $\mathcal{B}_{A_i}$.

Two edges in $\widehat{\mathcal{B}}_{A_i}$ labelled $\hat{a}_1$, $\hat{a}_2$ are adjacent if and only if $\hat{a}_1^{-1}\hat{a}_2 = \hat{s}^j$ or $\hat{t}^j$ for some j. Likewise, two edges in $\mathcal{B}_{A_i}$ labelled a_1, a_2 are adjacent if and only if $a_1^{-1}a_2 = s^j$ or t^j for some j. Thus, an edge path $a_0, \ldots, a_k$ in $\mathcal{B}_{A_i}$ corresponds to a word $w \in F(s,t)$ with $\|w\| = k$ and $a_0 w = a_k$ in A_i. If this edge path is the image under $\widehat{\phi}_i$ of an edge path in $\widehat{\mathcal{B}}_{A_i}$, then all the syllables in w are of even length.

Suppose $\widehat{x}$, $\widehat{y} \in \widehat{\mathcal{B}}_{A_i}$ are points of distance $\geq \pi$. Let $\widehat{\gamma}$ be a geodesic in $\widehat{\mathcal{B}}_{A_i}$ from $\widehat{x}$ and $\widehat{y}$. Let γ, x, y denote their images in $\mathcal{B}_{A_i}$. Let α be a geodesic from x to y in $\mathcal{B}_{A_i}$, and suppose length $(\alpha) < \pi$. Then α contains at most m vertices. Consider the loop $\alpha\gamma^{-1}$ in $\mathcal{B}_{A_i}$. This loop corresponds to a word w in $F(s,t)$ representing the identity element in A_i. The word w has at most m syllables of length 1 since the subword corresponding to γ has all syllables of even length.

Following the terminology of Appel and Schupp in [AS], we consider a Dehn diagram M with ∂M labelled by w. Every region of M is labelled (up to cyclic permutation) by the alternating word $(sts\ldots)(tst\ldots)^{-1}$ of length $2m$. It follows from [AS], Lemmas 2 and 3, that M contains either

(i) two regions with all but one edge on ∂M, or

(ii) 4 regions with all but two adjacent edges on ∂M.

From this we see that w contains at least $2 \cdot (2m-3)$ syllables of length 1 (in case (i)) or $4 \cdot (2m-4)$ syllables of length 1 (in case (ii)). In either case, we get a contradiction. $\square$

Theorem 4.2. *If (A, S) is a locally reducible Artin system, then $\widehat{\phi} : \widehat{\mathcal{D}}_A \to \mathcal{D}_A$ is an isometric embedding.*

Proof. By Lemma 1.4 and Theorem 3.2 it suffices to show that the map

$$link(x, \widehat{\mathcal{D}}_A) \to link(\widehat{\phi}(x), \mathcal{D}_A)$$

induced by $\widehat{\phi}$ is π-distance preserving for every point $x \in \widehat{\mathcal{D}}_A$. Since $\widehat{\phi}$ is equivariant, we may assume without loss of generality that x lies in the fundamental chamber K of $\widehat{\mathcal{D}}_A$ and hence also $\phi(x)$ lies in the fundamental chamber of $\mathcal{D}_A$. Say x (and hence $\phi(x)$) lies in the interior of $cube(T, R)$. Then by Lemma 2.2,

$$link(\widehat{\phi}(x), \mathcal{D}_A) = link(x, K_{\geq T}) * \mathcal{B}_{A_T}\ .$$

A completely analogous argument shows that

$$link(x, \widehat{\mathcal{D}}_A) = link(x, K_{\geq T}) * \widehat{\mathcal{B}}_{A_T}\ .$$

It now follows from Lemmas 1.5 and 4.1 that the map between these links is π-distance preserving. □

The Tits Conjecture for locally reducible Artin systems is an immediate corollary.

Corollary 4.3. *If (A, S) is locally reducible, then $\phi : \widehat{A} \to A$ is injective. Thus, the subgroup of A generated by $\{s^2 \mid s \in S\}$ is a right-angled Artin group.*

There is another class of Artin groups for which the Deligne complex can be given a CAT(0) geometry, namely the Artin groups of FC type (see [CD1]). (In this case the geometry is made up of standard Euclidean cubes.) It is natural to ask whether similar methods can be used to prove the Tits conjecture for this class of Artin groups as well. Here we encounter two difficulties. First, for FC groups, the links can involve any finite type Artin group, not all of which are known to satisfy the Tits conjecture. Thus, we do not even know if the map on Deligne complexes is *locally* injective. Second, since the links are generally higher dimensional, the analogue of Lemma 4.1 is more difficult (it requires showing that $\widehat{\mathcal{B}}_A$ maps isomorphically onto a full subcomplex of $\mathcal{B}_A$). Nevertheless, it seems likely that this can be done, at least if one restricts to the case where the finite type subgroups groups are known to satisfy the Tits conjecture.

5. Special Subgroups

By Corollary 4.3, we can identify $\widehat{A}$ with its image in A and $\widehat{A}_T$ with its image in A_T. Clearly, $\widehat{A}_T \subset A_T \cap \widehat{A}$, but the reverse inclusion is not at all obvious. Using methods similar to those above, we now show that for locally reducible (A, S), the equality $\widehat{A}_T = A_T \cap \widehat{A}$ holds for every subset $T \subseteq S$. We will need the following lemma.

Lemma 5.1. *Let (A, S) be an arbitrary Artin system and $T \subset S$ any subset. The natural inclusion $\mathcal{D}_{A_T} \hookrightarrow \mathcal{D}_A$ is locally an isometric embedding. If $\mathcal{D}_A$ is $CAT(0)$, then the inclusion is globally an isometric embedding.*

Proof. Let K denote the fundamental chamber of $\mathcal{D}_A$ and let K^T denote the fundamental chamber of $\mathcal{D}_{A_T}$. For $R' \subset R \subset T \subset S$, the metric on $cube(R', R)$ depends only on R and R', so the natural inclusion $K^T \hookrightarrow K$ (and hence $\mathcal{D}_{A_T} \hookrightarrow \mathcal{D}_A$) maps each cube of K^T isometrically onto a cube of K. Thus by Lemma 1.4, we must show that the induced inclusions $link(x, \mathcal{D}_{A_T}) \hookrightarrow link(x, \mathcal{D}_A)$ are π-distance preserving for all $x \in \mathcal{D}_{A_T}$.

Without loss of generality, we may assume that x lies in the interior of $cube(R', R)$ in the fundamental chamber K^T for some $R' \subset R \subset T$. Let $j = |R'| - |R|$. Then by Lemma 2.2, we have

$$link(x, \mathcal{D}_{A_T}) = \mathbb{S}^{j-1} * link(A_R, K^T_{\geq R}) * \mathcal{B}_{A_{R'}}$$
$$link(x, \mathcal{D}_A) = \mathbb{S}^{j-1} * link(A_R, K_{\geq R}) * \mathcal{B}_{A_{R'}} .$$

In light of Lemma 1.5, it suffices to show that the inclusion of $link(A_R, K^T_{\geq R})$ into $link(A_R, K_{\geq R})$ is π-distance preserving. For this, we will verify that the hypotheses of Lemma 1.6 apply.

First note that, since $K_{\geq R}$ intersects $cube(R)$ orthogonally at A_R,

$$link(A_R, K_{\geq R}) = link(cube(R), K) .$$

An edge in this link is of the form $link(cube(R), cube(P))$ where $P = R \cup \{s, t\} \in \mathcal{S}^f$ with $s, t \notin R$. The length of this edge is the dihedral angle between the faces $cube(R \cup \{s\})$ and $cube(R \cup \{t\})$ in $cube(P)$. By construction of the Moussong metric, this dihedral angle is $\pi - \frac{\pi}{m(s,t)} \geq \frac{\pi}{2}$. Thus all edge lengths in $link(A_R, K_{\geq R})$ are $\geq \frac{\pi}{2}$.

Finally, we note that the set of simplices of $link(A_R, K_{\geq R})$ is isomorphic, as a poset, to

$$\{P \in \mathcal{S}^f \mid R \subset P \subseteq S\}$$

while the set of simplices of $link(A_R, K^T_{\geq R})$ is isomorphic to

$$\{P \in \mathcal{S}^f \mid R \subset P \subseteq T\} .$$

In either case, two simplices P, P' span a larger simplex if and only if $P \cup P' \in \mathcal{S}^f$. Thus, $link(A_R, K^T_{\geq R})$ is a full subcomplex of $link(A_R, K_{\geq R})$ and Lemma 1.6 applies. $\square$

Theorem 5.2. *Suppose (A, S) is locally reducible. Let $\widehat{A}$ be the subgroup of A generated by $\{s_1^2, \dots, s_n^2\}$. Then for any $T \subset S$, $\widehat{A} \cap A_T = \widehat{A}_T$.*

Proof. Clearly $\widehat{A}_T \subset \widehat{A} \cap A_T$. To prove the reverse inclusion, suppose $a \in \widehat{A} \cap A_T$. Lift a to an element $\widehat{a} \in \widehat{A}$. Let γ_1 be a geodesic in $\widehat{\mathcal{D}}_A$ from the vertex $A_\emptyset$ to $\widehat{a}A_\emptyset$ and let γ_2 be a geodesic in $\mathcal{D}_{A_T}$ from the vertex $A_\emptyset$ to $aA_\emptyset$. Then by Theorem 4.2

and Lemma 5.1, γ_1 and γ_2 map to geodesics $\overline{\gamma}_1$, $\overline{\gamma}_2$ from $A_\emptyset$ to $aA_\emptyset$ in $\mathcal{D}_A$. Since $\mathcal{D}_A$ is CAT(0), geodesics are unique, so $\overline{\gamma}_1 = \overline{\gamma}_2$. Given any $\varepsilon > 0$, we can perturb γ_1 inside an ε-neighborhood to a path γ_1' from $\widehat{A}_\emptyset$ to $\widehat{a}\widehat{A}_\emptyset$ which does not intersect the codimension 2 skeleton of $\widehat{\mathcal{D}}_A$ and crosses the codimension 1 faces transversely. The sequence of chambers (K-translates) through which this perturbation passes determines a word $w \in F(S)$ representing $\widehat{a}$ in $\widehat{A}$. Squaring each letter of w, we obtain a word $u \in F(s_1^2, \dots, s_n^2)$, corresponding to the image of γ_1' in $\mathcal{D}_A$, which represents $a \in A$.

The image, $\overline{\gamma}_1'$ of γ_1' in $\mathcal{D}_A$ lies in an ε-neighborhood of $\gamma_1 = \gamma_2$. Thus it lies in an ε-neighborhood of $\mathcal{D}_{A_T} \subset \mathcal{D}_A$. But if $K_{\geq \{s\}}$ is a codimension 1 face of K not contained in K^T (i.e., $s \notin T$), then $K_{\geq \{s\}}$ and K^T are disjoint, closed subspaces of K, hence there is a ε-neighborhood of K^T not intersecting $K_{\geq \{s\}}$. Taking the intersection of these neighborhoods over all $s \notin T$, we see that a sufficiently small ε-neighborhood of K^T in K intersects only those codimension 1 faces $K_{\geq \{t\}}$ with $t \in T$. The same ε works for all translates of K^T. It follows that the word u involves only generators in T, so a lies in $\widehat{A}_T$. □

References

[AC] J. Altobelli and R. Charney, *A geometric rational form for Artin groups of FC type*, Geometriae Dedicata **79** (2000), 277-289.

[AS] K. I. Appel and P. E. Schupp, *Artin groups and infinite Coxeter groups*, Invent. Math. **72** (1983), 201-220.

[BB] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **109** (1997), 445-470.

[Be] M. Bestvina, *Non-positively curved aspects of Artin groups of finite type*, preprint.

[BH] M. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer-Verlag, 1999.

[BM] T. Brady and J. McCammond, *Three-generator Artin groups of large type are biautomatic*, preprint.

[Br] M. R. Bridson, *Geodesics and curvature in metric simplicial complexes*, Group Theory from a Geometrical Viewpoint, edited by E. Ghys, A. Haefliger and A. Verjovsky, World Scientific, Singapore, 1991, pp. 373-463.

[Brn] K. Brown, *"Buildings"*, Springer-Verlag, New York 1989.

[BS] E. Brieskorn and K. Saito, *Artin-gruppen und Coxeter-gruppen*, Invent. Math. **17** (1972), 245-271.

[C1] R. Charney, *Artin groups of finite type are biautomatic*, Math. Annalen **292** (1992), 671-683.

[C2] ———, *Geodesic automation and growth functions for Artin groups of finite type*, Math. Annalen **301** (1995), 307-324.

[C3] ———, *Injectivity of the positive monoid for some infinite type Artin groups*, Geometric Group Theory Down Under, ed. by J. Cossey, C. Miller, W. Neumann, and M. Shapiro, Walter de Gruyter, Berlin, 1999.

[CD1] R. Charney and M. W. Davis, *The $K(\pi,1)$-problem for hyperplane complements associated to infinite reflection groups*, J. Amer. Math. Soc. **8** (1995), 597-627.

[CD2] ———, *Singular metrics of nonpositive curvature on branched covers of Riemannian manifolds*, Amer. J. Math **115** (1993), 929-1009.

[C] D. Collins, *Relations among the squares of the generators of the braid group*, Invent. Math. **117** (1994), 525-529.

[D] P. Deligne, *Les immeubles des groups de tresses généralisés*, Invent. Math. **17** (1972), 273-302.

[DLS1] C. Droms, J. Lewin and H. Servatius, *Tree groups and the 4-string pure braid group*, J. of Pure Appl. Alg. **70** (1991), 251-261.

[DLS2] ______, *The Tits conjecture and the five string braid group*, Topology and Combinatorial Group Theory, ed. by P. Latiolois, LNM 1440, Springer, 1990, pp. 48-51.

[G] M. Gromov, *Hyperbolic groups*, Essays in Group Theory, ed. by S. M. Gersten, MSRI Publ. 8, Springer-Verlag, New York, 1987, pp. 75-264.

[Ga] F. A. Garside, *The braid groups and other groups*, Quart. J. Math. Oxford **20** (1969), 235-254.

[H] S. P. Humphries, *On representations of the Artin groups and the Tits conjecture*, J. of Algebra **169** (1994), 847-862.

[L] H. van der Lek, *The homotopy type of complex hyperplane complements*, Ph.D. thesis, University of Nijmegen, 1983.

[M] G. Moussong, *Hyperbolic Coxeter groups*, Ph.D. thesis, The Ohio State University, 1988.

[T] J. Tits, *Normalisateurs de tores. I. Groups de Coxeter étendus*, J. of Algebra **4** (1966), 96-116.

[Pe] D. Peifer, *Artin groups of extra-large type are biautomatic*, J. of Pure Appl. Algebra **110** (1996), 15-56.

[Pr] S. J. Pride, *On Tits' conjecture and other questions concerning Artin and generalized Artin groups*, Invent. Math. **86** (1986), 347-356.