ORDINARY PSEUDOREPRESENTATIONS AND MODULAR FORMS

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Abstract. In this short note, we observe that the techniques of [WWE15] can be used to provide a new proof of some of the residually reducible modularity lifting results of Skinner and Wiles [SW99]. In these cases, we have found that a deformation ring of ordinary pseudorepresentations is equal to the Eisenstein local component of a Hida Hecke algebra. We also show that Vandiver’s conjecture implies Sharifi’s conjecture.

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1. Introduction

The key technical innovation behind our previous work [WWE15] was our definition of an ordinary 2-dimensional pseudorepresentation of $G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Using this notion, we were able to study ordinary Galois deformations in the case where the residual representation is reducible. In particular, we constructed a universal ordinary pseudodeformation ring $R_{\text{ord}}^D$ with residual pseudorepresentation $\overline{D}$. We also showed that the Galois action on the Eisenstein part of the cohomology of modular curves gives rise to an ordinary pseudorepresentation valued in the ordinary Hecke algebra $\mathfrak{H}$. Studying the tangent space of $R_{\text{ord}}^D$, we showed that, if Greenberg’s conjecture holds, then certain characteristic 0 localizations $\mathfrak{H}_p$ of $\mathfrak{H}$ are Gorenstein. Under the same assumption, we also proved $R_{\text{ord}}^D \cong \mathfrak{H}_p$.

In this note, we show that the methods of [WWE15] can be extended to the whole Eisenstein component $\mathfrak{H}$, provided that we make stronger assumptions on class groups. Namely, we have to assume that the plus-part $X^+$ of the Iwasawa class group of cyclotomic fields vanishes. When the tame level $N$ is 1, then this is known as Vandiver’s conjecture. When $N > 1$, there are examples where $X^+ \neq 0$, but is still often the case that $X^+ = 0$. Assuming $X^+ = 0$, we get an isomorphism $R_{\text{ord}}^D \cong \mathfrak{H}$ (Theorem 4.2.8). As a consequence, we have a new technique to establish the residually reducible ordinary modularity theorem of Skinner and Wiles [SW99] over $\mathbb{Q}$, in some cases (Corollary 4.3.4). We also derive new results on Gorensteinness of Hecke algebras (Corollary 4.3.1) and prove new results toward Sharifi’s conjecture.
(Corollary [4.3.2]. In particular, we prove that $\mathfrak{H}$ is Gorenstein when $X^+ = 0$, an implication that was known previously only after assuming Sharifi’s conjecture.

As well as proving these new results, we review the most novel parts of [WWE15]. In this way, this note may be serve as an introduction to [WWE15].

1.1. **Ordinary pseudorepresentations.** A 2-dimensional pseudorepresentation of $G_{\mathbb{Q}}$ with values in a ring $A$ is the data of two functions $\{\text{tr}, \text{det}\}$ that satisfy conditions as if they were the trace and determinant of a representation $G_{\mathbb{Q}} \to \text{GL}_2(A)$. The (fine) moduli of pseudorepresentations may be thought of as the coarse moduli of Galois representations produced by geometric invariant theory [WE15, Theorem A]. In this respect, our results suggest that coarse moduli rings of Galois representations are most naturally comparable with Hecke algebras. Indeed, most previous $R = T$ theorems have been established where $R$ is a deformation ring for a residually irreducible Galois representation, in which case the fine and coarse moduli of Galois representations are identical.

The ordinary condition is somewhat subtle when applied to pseudorepresentations. For example, if one thinks about the case when $A$ is a field, a representation $\rho : G_{\mathbb{Q}} \to \text{GL}_2(A)$ is defined to be ordinary when $\rho|_{G_{\mathbb{Q}}^p}$ is reducible with a twist-unramified quotient. While $\{\text{tr} \rho|_{G_{\mathbb{Q}}^p}, \text{det} \rho|_{G_{\mathbb{Q}}^p}\}$ knows nothing about nothing about which of the two Jordan-Hölder factors is the quotient, $D_{\rho} = \{\text{tr} \rho, \text{det} \rho\}$ can often distinguish them. This allows for the definition of ordinary pseudorepresentation of $G_{\mathbb{Q}}$, which we extend to non-field coefficients. We overview this and other background from [WWE15] in §§2-3.

1.2. **Outline of the proof.** The étale cohomology of compactified modular curves defines a $G_{\mathbb{Q}}$-module $H$ over the cuspidal quotient $\mathfrak{h}$ of $\mathfrak{H}$. However, $H$ is a representation (i.e. locally free $\mathfrak{h}$-module) if and only if $\mathfrak{h}$ is Gorenstein, which is not always true. Nonetheless, $H$ always induces an ordinary $\mathfrak{h}$-valued pseudorepresentation deforming the residual pseudorepresentation $\bar{D}$. This pseudorepresentation extends to $\mathfrak{H}$, resulting in a surjection $R_{\bar{D}}^{\text{ord}} \to \mathfrak{H}$.

This map is naturally a morphism of augmented $\Lambda$-algebras, where $\Lambda$ is the Iwasawa algebra. The augmentation ideals

$$I := \ker(\mathfrak{H} \to \Lambda), \quad J := \ker(R_{\bar{D}}^{\text{ord}} \to \Lambda)$$

correspond to the Eisenstein family of $\Lambda$-adic modular forms and the reducible locus of Galois representations, respectively. We can show that certain Iwasawa class groups surject onto $J/J^2$, which is the cotangent module relative to the reducible family. Vandiver’s conjecture implies that the Iwasawa class groups are small. Using a version of Wiles’s numerical criterion [W95, Appendix], with the class groups playing the role of Wiles’s $\eta$, we can show that this forces $R_{\bar{D}}^{\text{ord}} \to \mathfrak{H}$ to be an isomorphism.

One novel aspect of this proof is that we are able to control a tangent space of a pseudodeformation ring in terms of Galois cohomology. Moreover, we use Galois cohomology groups with coefficients in $\Lambda$, while the usual approach works over a field. Such control is critical to proving $R = T$ theorems in the residually irreducible case, and $R = T$ theorems for pseudorepresentations were lacking because of this control was not as available. In our situation, the relevant Galois cohomology is determined by class groups.
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2. Background: Iwasawa theory and Hecke algebras

This section is a brief synopsis of Sections 2, 3 and 6 of [WWE15]. We overview background information from Iwasawa theory and ordinary Λ-adic Hecke algebras and modular forms.

2.1. Iwasawa algebra and Iwasawa modules. We review Section 2 of [WWE15].

Let \( p \geq 5 \) be a prime number, and let \( N \) be an integer such that \( p \nmid N\phi(N) \). Let \( \theta : (\mathbb{Z}/Np\mathbb{Z})^\times \rightarrow \mathbb{Q}_p^\times \) be an even character. Let \( \chi = \omega^{-1}\theta \), where \( \omega : (\mathbb{Z}/Np\mathbb{Z})^\times \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mathbb{Z}_p^\times \) is the Teichmüller character. Our assumption on \( N \) implies that each of these characters is a Teichmüller lift of a character valued in a field extension \( \mathbb{F} \) of \( \mathbb{F}_p \). By abuse of notation, we also use \( \theta, \chi, \omega \) to refer to these characters.

We assume that \( \theta \) satisfies the following conditions:

(a) \( \theta \) is primitive,
(b) if \( \chi \mid (\mathbb{Z}/p\mathbb{Z})^\times = 1 \), then \( \chi \mid (\mathbb{Z}/N\mathbb{Z})^\times (p) \neq 1 \), and
(c) if \( N = 1 \), then \( \theta \neq \omega^2 \).

A subscript \( \theta \) or \( \chi \) on a module refers to the eigenspace for an action of \( (\mathbb{Z}/Np\mathbb{Z})^\times \).

A superscript \( \pm \) will denote the \( \pm 1 \)-eigenspace for complex conjugation. Let \( S \) denote the set of primes dividing \( Np \), and let \( G_{\mathbb{Q},S} \) be the unramified outside \( S \) Galois group. We fix a decomposition group \( G_p \subset G_{\mathbb{Q},S} \) and let \( I_p \subset G_p \) denote the inertia subgroup.

Fix a system \( (\zeta_{Np^r}) \) of primitive \( Np^r \)-th roots of unity such that \( \zeta_p^{Np^r+1} = \zeta_{Np^r} \) for all \( r \). Let \( Q_\infty = Q(\zeta_{Np^\infty}) \) and let \( \Gamma = \text{Gal}(Q_\infty/Q(\zeta_{Np})) \).

Let \( \text{Cl}(Q(\zeta_{Np^r})) \) be the class group, and let

\[
X = \lim_{\leftarrow} \text{Cl}(Q(\zeta_{Np^r})) \{p\}.
\]

There is action of \( \Gamma \) on \( X \). By class field theory, \( X = \text{Gal}(L/Q_\infty) \) where \( L \) is the maximal pro-\( p \), abelian, unramified extension. A closely related object is \( X = \text{Gal}(M/Q_\infty) \) where \( M \) is the maximal pro-\( p \) abelian extension unramified outside \( Np \).

Let \( Z_{p,N} = \lim Z/Np^rZ \). Let \( \Lambda_0 = Z_0[\mathbb{Z}_p^\times, N]_{\theta} \), where we write \( \Lambda \) for \( \Lambda_0 \) when \( \theta \) is implicit. This \( \Lambda \) is a local component of the semilocal ring \( Z_p[\mathbb{Z}_p^\times, N] \), and is abstractly isomorphic to \( \mathcal{O}[\Gamma] \simeq \mathcal{O}[T] \), where \( \mathcal{O} \) is the extension of \( Z_p \) generated by values of \( \theta \). Notice that the action on \( (\zeta_{Np^r}) \) gives an isomorphism \( \Gamma \simeq \ker(\mathbb{Z}_p^\times \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times) \).

Let \( M \mapsto M^\# \) and \( M \mapsto M(r) \) be the functors on \( Z_{p,N} \)-modules as defined in [Wak15a, Section 2.1.3]. We sometimes, especially when using duality, are forced to consider \( Z_{p,N} \)-modules with characters other than \( \theta \), but we use these functors to make the actions factor through \( \Lambda \) so we can treat all modules uniformly.
We define $\xi_\chi \in \Lambda$ to be a generator of the principal ideal $\text{Char}_\Lambda(X_\chi(1))$. By the Iwasawa Main Conjecture, this may be chosen as a Kubota-Leopoldt $p$-adic $L$-function.

Consider the $\Lambda$-valued character $(\cdot) : G_{Q,S} \to \Gamma \subset \Lambda^\times$. We will write $\Lambda^{(-)}$ for this character, and $\Lambda^\#$ for the inverse character. We will write $\Lambda^\text{triv}$ for the trivial character with values in $\Lambda$, or just $\Lambda$ when no possible confusion can arise.

2.2. Duality and consequences. We review some relevant parts of Section 6 of [WWE15]. To compare conditions on various class groups, we use the following $\Lambda$-Duality and consequences.

2.2. Let $\Lambda$ be a continuous action of $G_F$, unramified at places outside $U$, and let $M$ be a finitely generated $\Lambda$-module. Then there is a quasi-isomorphism

$$ R\Gamma(U,T \otimes_\Lambda M) \overset{\sim}{\longrightarrow} R\text{Hom}_\Lambda(R\Gamma(U,T^*(1)),M)[-3]. $$

There is a similar quasi-isomorphism when $R\Gamma$ and $R\Gamma_c$ are swapped, i.e.

$$ R\Gamma(U,T \otimes_\Lambda M) \overset{\sim}{\longrightarrow} R\text{Hom}_\Lambda(R\Gamma_c(U,T^*(1)),M)[-3]. $$

Here $T^*$ is the dual representation $\text{Hom}(T,\Lambda)$.

Proof. For the case where $M = \Lambda$, see [Nek06] Proposition 5.4.3, pg. 99. The proposition for general $M$ follows from the existence of a finite length projective resolution $P_*$ for $M$ as a $\Lambda$-module, and standard homological algebra. For some $\Lambda$-linear category $\mathcal{C}$, $T \in \mathcal{C}$, and a derived functor $RF$ from the derived category of $\mathcal{C}$ to the derived category of $\Lambda$-modules, $RF(T \otimes_\Lambda M)$ is quasi-isomorphic to $RF(T \otimes_\Lambda P_*)$. Letting $RF = R\Gamma_c(U,-)$ and using the case $M = \Lambda^\otimes n$, we get a quasi-isomorphism with $R\text{Hom}_\Lambda(R\Gamma(U,T^*(1))[-3] \otimes_\Lambda R\text{Hom}_\Lambda(P_*,\Lambda),\Lambda)$. By tensor-Hom adjunction, we have a quasi-isomorphism with $R\text{Hom}_\Lambda(R\Gamma(U,T^*(1))[-3],P_*)$. Finally, we may replace $P_*$ by $M$ in the derived category. \qed

The theorem yields spectral sequences with second page

\begin{align}
E_2^{i,j} &= \text{Ext}_\Lambda^i(H^{3-j}(U,T^*(1)),M) \implies H_\Lambda^{i+j}(U,T \otimes_\Lambda M), \\
E_2^{i,j} &= \text{Ext}_\Lambda^i(H^{3-j}(U,T^*(1)),M) \implies H_\Lambda^{i+j}(U,T \otimes_\Lambda M).
\end{align}

We record the influence of the assumption that $X_0 = 0$. In the proof, we make use of the following lemma on the structure of $\Lambda$-modules.

Lemma 2.2.4. Let $M$ be a finitely generated $\Lambda$-module. Say that $M$ is type 0 if $M$ is free, type 1 if $M$ has projective dimension 1, and type 2 if $M$ is finite. Then $M$ is type 1 if and only if $\text{Ext}_\Lambda^i(M,\Lambda) = 0$ for all $j \neq i$. Moreover, $M$ is type 1 if and only if $M$ is torsion and has no non-zero finite submodule.
Proof. See [Jan89] Section 3]. \(\square\)

**Proposition 2.2.5.** \(X_\theta = 0\) if and only if \(X_{\lambda-1}^\#(1)\) is a free \(\Lambda\)-module of rank 1.

**Proof.** As in [WWE15] Section 6.1, we have \(X_\theta = H^2(\mathbb{Z}[1/Np], \Lambda^\#(1))\). Since \(X\) is the Pontryagin dual of \(H^1(\mathbb{Z}[1/Np], \mathbb{Q}_p/\mathbb{Z}_p)\), (classical) Poitou-Tate duality implies that
\[
X = \varprojlim \, H^2(\mathbb{Z}[1/Np, \zeta_{Np'}], \mathbb{Z}_p(1)).
\]
We can then deduce \(X_{\lambda-1}^\#(1) = H^2(\mathbb{Z}[1/Np], \Lambda^{(-)})\), as in [WWE15] Section 6.1.

Analyzing spectral sequence \(2.2.3\) above with \(T = \Lambda^{(-)}\) and \(M = \Lambda\), we see that \(E_2^{i,j} = 0\) for cohomological dimension reasons unless \(i, j \in \{0, 1, 2\}\). We find
\[
\text{Ext}_\Lambda^1(X_{\lambda-1}^\#(1), \Lambda) = X_\theta, \quad \text{Ext}_\Lambda^2(X_{\lambda-1}^\#(1), \Lambda) = 0.
\]
Then \(X_{\lambda-1}^\#(1)\) is a free \(\Lambda\)-module if and only if \(X_\theta = 0\) by Lemma 2.2.4.

The fact that the rank is then 1 follows from class field theory and Iwasawa’s theorem. Indeed, class field theory implies that there is an exact sequence
\[
0 \to U_{\lambda-1}^\#(1) \to X_{\lambda-1}^\#(1) \to X_{\lambda-1}^\#(1) \to 0,
\]
where \(U\) is an Iwasawa local units group, and Iwasawa’s theorem implies that \(U_{\lambda-1}^\#(1)\) is free of rank 1 over \(\Lambda\) (see [WWE15] Section 2.1 and the references given there). Since \(X_{\lambda-1}^\#(1)\) is \(\Lambda\)-torsion, this implies that \(X_{\lambda-1}^\#(1)\) has rank 1. \(\square\)

### 2.3. Hecke algebras.

We review Section 3 of [WWE15]. Let
\[
\hat{H}' = \varprojlim \, H^1(Y_1(Np'), \mathbb{Z}_p)^{\text{ord}}, \quad H' = \varprojlim \, H^1(X_1(Np'), \mathbb{Z}_p)^{\text{ord}}
\]
where the subscript \(\theta\) denotes the eigenspace for the diamond operators. Let \(\mathfrak{S}'\) and \(\mathfrak{b}'\) denote the Hida Hecke algebras acting on \(\hat{H}'\) and \(H'\), respectively. There is a unique maximal ideal of \(\mathfrak{S}'\) containing the Eisenstein ideal; let \(\mathfrak{S}\) and \(\mathfrak{b}\) be the localizations of \(\mathfrak{S}'\) and \(\mathfrak{b}'\) at the Eisenstein maximal ideal, and let \(\hat{H} = H' \otimes_{\mathfrak{b}'} \mathfrak{S}\) and \(H = H' \otimes_{\mathfrak{b}} \mathfrak{S}\). Let \(\mathcal{I} \subset \mathfrak{S}\) be the Eisenstein ideal, and let \(I \subset \mathfrak{b}\) be the image of \(\mathcal{I}\).

By Hida theory, each of \(\hat{H}, H, \mathfrak{S}\) and \(\mathfrak{b}\) is finite and flat over \(\Lambda\). There are also canonical isomorphisms of \(\mathfrak{S}\)-modules \(\mathfrak{S}/\mathcal{I} \cong \Lambda, \mathfrak{b}/I \cong \Lambda/\zeta_{\chi}\) and \(\mathcal{I} \cong \mathcal{I}\) (see [WWE15] Proposition 3.2.5)), making \(\mathfrak{S}\) an augmented \(\Lambda\)-algebra.

### 3. Ordinary Pseudorepresentations.

We define ordinary pseudorepresentations and show that they are representable by an ordinary pseudodeformation ring \(R^\text{ord}_D\), recapping results of [WWE15]. In particular, we will review background on pseudorepresentations, Cayley-Hamilton algebras, and generalized matrix algebras from §5 of loc. cit.

We highlight the following important points:
- The definition is “not local,” in the sense that it does not have the form “\(D : G_{Q,S} \to A\) is ordinary if \(D|_{G_{Q,S}}\) is ordinary.”
- When \(A\) is a field, we can say that \(D\) is ordinary if there exists an ordinary \(G_{Q,S}\)-representation \(\rho\) such that \(D\) is induced by \(\rho\).
While not every pseudorepresentation comes from a representation, we fix this problem by broadening the category of representations to include generalized matrix algebra-valued representations (GMA representations). We first define ordinary GMA representations, and then say a pseudorepresentation is ordinary when there exists an ordinary GMA representation inducing it.

We fix some notation. We use the letter $\psi$ to denote the functor that associates to a representation its induced pseudorepresentation. Let $\bar{D} = \psi(\omega^{-1} \oplus \theta^{-1})$, which is the $\mathbb{F}$-valued residual pseudorepresentation induced by the Galois action on $H$. Write $R_D$ for the pseudodeformation ring for $\bar{D}$ [WWE15 §5.1]. In this section, $A$ will denote a Noetherian local $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$. If $a \in A$, then $\bar{a} \in \mathbb{F}$ denotes the image of $a$.

### 3.1. Representations valued in generalized matrix algebras.

As in [BC09 §1.3], we say that a generalized matrix algebra over $A$ is an associative $A$-algebra $E$ such that

$$E \sim \begin{pmatrix} A & B \\ C & A \end{pmatrix}.$$ 

This means that $E \cong A \oplus B \oplus C \oplus A$ as $A$-modules, for some $A$-modules $B$ and $C$, and there is an $A$-linear map $B \otimes_A C \to A$ such that the multiplication in $E$ is given by 2-by-2 matrix multiplication. In this case, $A$ is called the scalar subring of $E$.

A GMA representation with coefficients in $A$ and residual pseudorepresentation $\bar{D}$ is a homomorphism $\rho : G_{Q,S} \to E^\times$, such that $E$ is a GMA, and such that, in matrix coordinates, $\rho$ is given by

$$(3.1.1) \quad \sigma \mapsto \begin{pmatrix} \rho_{1,1}(\sigma) & \rho_{1,2}(\sigma) \\ \rho_{2,1}(\sigma) & \rho_{2,2}(\sigma) \end{pmatrix},$$

with $\rho_{1,1}(\sigma) = \omega^{-1}(\sigma)$, $\rho_{2,2}(\sigma) = \theta^{-1}(\sigma)$, and $\rho_{1,2}(\sigma)\rho_{2,1}(\sigma) = 0$. We will fix this ordering of idempotents on all GMAs in what follows.

Given such a $\rho$, there is an induced $A$-valued pseudorepresentation, denoted $\psi_{\text{GMA}}(\rho) : G_{Q,S} \to A$, given by $\text{tr}(\rho) = \rho_{1,1} + \rho_{2,2}$ and $\text{det}(\rho) = \rho_{1,1}\rho_{2,2} - \rho_{1,2}\rho_{2,1}$.

### 3.2. Universality.

A Cayley-Hamilton representation over $A$ with residual pseudorepresentation $\bar{D}$ is the data of a pair $(\rho : G_{Q,S} \to R^\times, D : R \to A)$, where $R$ is an associative algebra such that $D \circ \rho$ is a pseudorepresentation deforming $\bar{D}$. These data must satisfy an additional Cayley-Hamilton condition that, for all $r \in R$, $r$ must satisfy the characteristic polynomial associated to $r$ by $D$. If $\rho : G_{Q,S} \to E^\times$ is a GMA representation, then $(\rho, \psi_{\text{GMA}}(\rho))$ is a Cayley-Hamilton representation.

For our purposes, the important properties of Cayley-Hamilton representations are the following (see [WWE15 Proposition 3.2.2]).

- There is a universal Cayley-Hamilton representation $(\rho^u : G_{Q,S} \to E_D^\times, D^u : E_D \to R_D)$ with residual pseudorepresentation $\bar{D}$.
- $E_D$ is a finite as an $R_D$-module, with the adic topology from $R_D$.
- $E_D$ admits a unique $R_D$-GMA structure such that $\rho^u$ is a continuous GMA representation over $R_D$.

In particular, any Cayley-Hamilton representation with residual pseudorepresentation $\bar{D}$ is a GMA representation with residual pseudorepresentation $\bar{D}$. 
We will write
\begin{equation}
E_D \cong \begin{pmatrix} R_D & B^u \\ C^u & R_D \end{pmatrix},
\end{equation}
for the decomposition of $E_D$ as in [3.1.1], and write $\rho_{i,j}^u$ for the corresponding components of $\rho^u$. Similarly, for any GMA representation $\rho$ deforming $\bar{D}$, we will write $\rho_{i,j}$ for the induced component decomposition.

3.3. Reductibility. It will be important to understand the notion of a reducible pseudorepresentation and the reducibility ideal of $R_D$. A pseudodeformation $D$ of $\bar{D}$ is called reducible if $D = \psi(\chi_1 \oplus \chi_2)$ for characters $\chi_i$ such that $\bar{\chi}_1 = \omega^{-1}$ and $\chi_2 = \theta^{-1}$. Otherwise, $D$ is called irreducible. Equivalently, $D$ is reducible if $D = \psi_{\text{GMA}}(\rho)$ for some GMA representation $\rho$ such that $\rho_{1,2}(G_Q,S) \cdot \rho_{2,1}(G_Q,S)$ is zero.

Since $\rho_{1,2}^u(G_Q,S)$ and $\rho_{2,1}^u(G_Q,S)$ generate $B^u$ and $C^u$, respectively, as $R_D$-modules, $D$ is reducible exactly when the image of $B^u \otimes_{R_D} C^u$ in $R_D$ under multiplication vanishes under $R_D \to A$. Consequently, we call the image of $B^u \otimes_{R_D} C^u$ in $R_D$ the reducibility ideal of $R_D$.

3.4. Ordinary GMA representations. Recall that a representation of $\text{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$ on a 2-dimensional $p$-adic vector space $V$ is ordinary if there exists a 1-dimensional quotient representation $V \to W$ such that $W(1)$ is unramified. A representation $\rho$ of $G_{Q,S}$ is ordinary if $\rho|_{G_p}$ is ordinary.

Relative to the ordering of factors in [3.1.1], we have this definition of ordinary GMA representation.

**Definition 3.4.1.** Let $\rho : G_{Q,S} \to E^\times$ be a GMA representation with scalar ring $A$ and induced pseudorepresentation $\bar{D}$. We call $\rho$ ordinary provided that
\begin{enumerate}
\item $\rho_{1,2}(G_p) = 0$, and
\item $\rho_{1,1}\mid_{I_p} \cong \kappa^{-1}_{\text{cyc}} \otimes \mathbb{Z}_p$ $A$.
\end{enumerate}

**Remark 3.4.2.** The condition that $\omega$ and $\theta$ are locally $p$-distinguished, i.e. $\omega|_{G_p} \neq \theta|_{G_p}$, is critically necessary to making this definition sensible. This is equivalent to the assumption (b) of [2.1].

**Example 3.4.3.** The motivating example of an ordinary GMA representation is the $\mathfrak{h}[G_{Q,S}]$-module $H$ given by the cohomology of modular curves. Ohta proved that there is an isomorphism of $\mathfrak{h}$-modules $H = H^+ \otimes H^- \cong \mathfrak{h} \oplus \mathfrak{h}^\vee$, where $\mathfrak{h}^\vee$ is the dualizing module of $\mathfrak{h}$ (see [WWE15, Theorem 3.1.1]). Because $\text{End}_\mathfrak{h}(\mathfrak{h}^\vee) \cong \mathfrak{h}$, we get a GMA representation $\rho_H : G_{Q,S} \to \text{Aut}_\mathfrak{h}(H)$; moreover, it is an ordinary GMA representation with residual pseudorepresentation $\bar{D}$ when the components are ordered as follows.

\[ \text{End}_\mathfrak{h}(H) \cong \begin{pmatrix} \text{End}_\mathfrak{h}(H^-) & \text{Hom}_\mathfrak{h}(H^+, H^-) \\ \text{Hom}_\mathfrak{h}(H^-, H^+) & \text{End}_\mathfrak{h}(H^+) \end{pmatrix} \cong \begin{pmatrix} \mathfrak{h} & \mathfrak{h}^\vee \\ \text{Hom}_\mathfrak{h}(\mathfrak{h}^\vee, \mathfrak{h}) & \mathfrak{h} \end{pmatrix}. \]

Here is a summary of the results of [WWE15] on ordinary GMA representations.

**Proposition 3.4.4.** (1) There is a universal ordinary GMA $E_D^{\text{ord}}$, a quotient of $E_D$, such that a GMA representation $G_{Q,S} \to E^\times$ with residual pseudorepresentation $\bar{D}$ is ordinary if and only if its map $E_D \to E$ factors through $E_D^{\text{ord}}$. 

(2) There is a universal reducible ordinary GMA $E_{D}^{\text{red}}$, a quotient of $E_{D}$, such that a GMA representation $G_{Q,S} \rightarrow E^{\times}$ with residual pseudorepresentation $D$ is reducible ordinary if and only if its map $E_{D} \rightarrow E$ factors through $E_{D}^{\text{red}}$.

(3) The natural homomorphism $E_{D}^{\text{red}} \rightarrow E_{D}^{\text{ord}}/JE_{D}^{\text{ord}}$ is surjective.

(4) $\text{End}_{h}(H)$ has a unique $h$-GMA structure such that $\rho_{H} : G_{Q,S} \rightarrow \text{Aut}_{h}(H)$ is an ordinary GMA representation.

Proof. Statement (1) comes from Proposition 5.5.4; (2) and (3) come from Proposition 7.5.1; and (4) comes from Theorem 7.1.1 of [WWE15]. □

We write $R_{\text{ord}}^{\bar{D}}$, the ordinary pseudodeformation ring, for the scalar ring of $E_{D}^{\text{ord}}$. We write $J \subset R_{\text{ord}}^{\bar{D}}$ for its reducibility ideal.

3.5. Ordinary pseudorepresentations. Having established a notion of ordinary GMA representation, we can now define ordinary pseudorepresentations.

Definition 3.5.1. Let $D : G_{Q,S} \rightarrow A$ be a pseudorepresentation deforming $\bar{D}$. Then we call $D$ ordinary if there exists an ordinary GMA representation $\rho : G_{Q,S} \rightarrow E^{\times}$ with scalar ring $A$ such that $D = \psi_{\text{GMA}}(\rho)$.

The ring $R_{D}^{\text{ord}}$ represents the functor of ordinary pseudodeformations of $\bar{D}$ [WWE15, Theorem 5.6.5]. We remark that the reason for introducing GMA representations is to make Definition 3.5.1: not every pseudodeformation of $\bar{D}$ comes from a representation, but every pseudodeformation comes from some GMA representation.

By definition, we see that the modular pseudorepresentation $\psi_{\text{GMA}}(\rho_{H}) : G_{Q,S} \rightarrow h$ is ordinary. It can be extended to an $\bar{h}$-valued pseudorepresentation with the following properties.

Proposition 3.5.2. There is a pseudorepresentation $D_{H} : G_{Q,S} \rightarrow \bar{h}$ that is is ordinary, deforms $\bar{D}$, and satisfies $D_{H} \otimes_{h} \bar{h} = \psi_{\text{GMA}}(\rho_{H})$. The corresponding map $\phi : R_{D}^{\text{ord}} \rightarrow \bar{R}$ is:

(1) a map of augmented $\Lambda$-algebras, where the augmentation ideals are the reducibility ideals $\mathcal{J} \subset R_{D}^{\text{ord}}$, $\mathcal{T} \subset \bar{R}$, and

(2) surjective.

Proof. The pseudorepresentation $D_{H}$ is constructed by gluing $\psi_{\text{GMA}}(\rho_{H})$ together with the Eisenstein pseudorepresentation, and it follows that $D_{H} \otimes_{h} \bar{h} = \psi_{\text{GMA}}(\rho_{H})$ and that the reducibility ideal is $\mathcal{I} \subset \bar{h}$ ([WWE15, Corollary 7.1.2]).

Then (2) follows from [WWE15, Lemma 7.1.3], and (1) follows from the fact that $R_{D}^{\text{ord}}/\mathcal{J} \cong \Lambda$ [WWE15, Proposition 7.5.1]. □

Consequently, the functor of reducible ordinary pseudorepresentations is represented by $\Lambda$, and $\Lambda$ is the scalar ring of $E_{D}^{\text{ord}}$.

4. New Results

With the overview of [WWE15] complete, we now prove the main theorems.

4.1. Reducible representations and class groups. Let us write $E_{D}^{\text{ord}}$ and $E_{D}^{\text{red}}$ in GMA form as

$$E_{D}^{\text{ord}} \cong \left( \begin{array}{cc} R_{D}^{\text{ord}} & B_{D}^{\text{ord}} \\ C_{D}^{\text{ord}} & R_{D}^{\text{ord}} \end{array} \right), \quad E_{D}^{\text{red}} \cong \left( \begin{array}{cc} \Lambda & B_{D}^{\text{red}} \\ C_{D}^{\text{red}} & \Lambda \end{array} \right).$$
Our goal is to control $\mathcal{J}/\mathcal{J}^2$ using Galois cohomology. For this, we use the surjection $B^{\text{red}} \otimes_{B^\text{red}_D} C^{\text{ord}} \to \mathcal{J}$ discussed in Proposition 3.3 along with Proposition 3.4.4(3) to produce a surjection
\[(4.1.1) \quad B^{\text{red}} \otimes_{\Lambda} C^{\text{red}} \to B^{\text{ord}}/\mathcal{J} \otimes_{\Lambda} C^{\text{ord}}/\mathcal{J} \to \mathcal{J}/\mathcal{J}^2\]
(see [WWE15, Proposition 5.3.4]). The main result is the following

**Proposition 4.1.2.** $E_D^{\text{red}}$ is determined as follows.

1. There exists a natural isomorphism
   \[X_\chi(1) \sim \sim B^{\text{red}}.\]
2. Assume that $X_0 = 0$. Then there exists a natural isomorphism
   \[\mathfrak{X}^\#(1) \sim \sim C^{\text{red}}.\]

Moreover, $C^{\text{red}}$ is free of rank 1 over $\Lambda$.

First, some lemmas. We will need the following notation, which is particular to

We will abbreviate $H^1(\mathbb{Z}/1/Np,-)$, $H^1_{(c)}(\mathbb{Z}/1/Np,-)$ and $\bigoplus_{i} H^1(\mathbb{Q}_p, -)$ to $H^1(-)$, $H^1_{(c)}(-)$ and $H^1_{NP}(-)$, respectively. Likewise, for a $\Lambda$-module $M$ (with trivial $G_{Q,S}$-action), we will write $M^\#$ for $\Lambda^\# \otimes_{\Lambda} M$ and write $M^{(-)}$ for $\Lambda^{(-)} \otimes_{\Lambda} M$.

**Lemma 4.1.3.** Functorially in finitely generated $\Lambda$-modules $M$, we have isomorphisms
\[(4.1.4) \quad \text{Hom}_{\Lambda}(B^{\text{red}}, M) \sim \sim H^1_{(c)}(M^{(-)})(-1))\]
and
\[(4.1.5) \quad \text{Hom}_{\Lambda}(C^{\text{red}}, M) \sim \sim H^1(M^{(\#)}(1)).\]

**Proof.** For $\text{Lemma 4.1.5}$, [BC09] Theorem 1.5.5] tells us that there is a natural $\Lambda$-linear injective map $i_C : \text{Hom}_{\Lambda}(C^{\text{red}}, M) \to H^1(M^{\#}(1))$ when $M$ is a cyclic module. But nothing about the proof depends upon $M$ being cyclic, so we have injectivity in general.

An element of $H^1(M^{\#}(1))$ results in a short exact sequence of $\Lambda[G_{Q,S}]$-modules $0 \to M^\# \to E \to \Lambda(-1) \to 0$. Choose an element $x \in E$ mapping to $1 \in \Lambda$ and write $f : G_{Q,S} \to M$ for the map $\gamma \mapsto \gamma \cdot x - \kappa_{\text{cyc}}^{-1}(\gamma)x$. Then we have a GMA representation
\[
\rho : G_{Q,S} \to \left(\begin{array}{cc}
\Lambda & 0 \\
M & \Lambda
\end{array}\right), \quad \rho(\gamma) = \left(\begin{array}{cc}
\kappa_{\text{cyc}}^{-1} & 0 \\
f(\gamma) & (\gamma)^{-1}
\end{array}\right).
\]

By the universal property of $E_D^{\text{red}}$, there exists a unique map $C^{\text{red}} \to M$ induced by this representation. This construction is an inverse to the construction of $i_C$ in [BC09] Theorem 1.5.5], so we have proved that $\text{Lemma 4.1.5}$ is an isomorphism.

Recall the definition of $H^1_{(c)}$ as the cohomology of a mapping cone. Note that $H^1_{NP}(M^{(-)}(1)) = 0$ for any $M$. This follows from the fact that the residual character $\chi$ is non-trivial on decomposition groups at all primes dividing $NP$. Consequently, $H^1_{(c)}(M^{(-)}(1))$ is naturally isomorphic to the subset of $H^1(M^{(-)}(1))$ whose restriction to $H^1_{NP}(M^{(-)}(1))$ is zero. We have an injection
\[
i_B : \text{Hom}_{\Lambda}(B^{\text{red}}, M) \hookrightarrow H^1_{(c)}(M^{(-)}(1)) \subset H^1(M^{(-)}(1))\]
like $i_C$ above, because any extension of $\Lambda^{(-)}$ by $M(-1)$ realized by a GMA map

$E_D^{\text{red}} \to \begin{pmatrix} \Lambda & M \\ 0 & \Lambda \end{pmatrix}$

induces a trivial extension of $G_p$-representations. The same argument as above shows that $i_B$ is surjective. □

Now, we prove Proposition 4.1.2.

Proof. We will use Yoneda’s lemma for finitely generated $\Lambda$-modules to determine $B^{\text{red}}$ and $C^{\text{red}}$. Indeed, $B^{\text{red}}$ and $C^{\text{red}}$ are finitely generated by the fact that $E_D$ is a finite $R_D$-module (see §3.2), and the construction of $E^{\text{red}}$.

Let’s begin with $B^{\text{red}}$. Let $M$ be an arbitrary finitely generated $\Lambda$-module. We will use the spectral sequence of Proposition 2.2.1

$\text{Ext}_\Lambda^p(\text{H}_3^{(-q)}(\Lambda^\#(2)), M) \Rightarrow \text{H}_p^{+q}(M^{(-1)}(-1))$.

From [WWE15, Corollary 6.1.5], we have

$H^0(\Lambda^\#(2)) \cong 0, \quad H^1(\Lambda^\#(2)) \cong 0, \quad H^2(\Lambda^\#(2)) \cong X_\chi(1)$.

The spectral sequence degenerates to yield a functorial isomorphism

$\text{Hom}_\Lambda(X_\chi(1), M) \xrightarrow{\sim} \text{H}_1^c(M^{(-1)}(-1))$.

From this and (4.1.4), we see that $X_\chi(1)$ and $B^{\text{red}}$ represent the same functor $M \mapsto \text{H}_1^c(M^{(-1)}(-1))$. Then the Yoneda lemma implies that $X_\chi(1) \xrightarrow{\sim} B^{\text{red}}$.

To calculate $C^{\text{red}}$, we first use the spectral sequence

$\text{Ext}_\Lambda^p(\text{H}_3^{(-q)}(\Lambda^\#(1)), \Lambda) \Rightarrow \text{H}_p^{+q}(\Lambda^{(-)})$.

The following cohomology groups are computed in [WWE15, Corollary 6.1.5].

$H^0(\Lambda^\#(1)) \cong 0, \quad H^1(\Lambda^\#(1)) \cong \mathcal{E}_\theta, \quad H^2(\Lambda^\#(1)) \cong X_\theta$.

By Proposition 2.2.5, the assumption $X_\theta = 0$ makes this first spectral sequence degenerate, and we compute

$H^1_c(\Lambda^{(-)}) \cong 0, \quad H^2_c(\Lambda^{(-)}) \cong \mathcal{X}_\chi^{-1}(1), \quad H^3_c(\Lambda^{(-)}) \cong 0$.

Now let $M$ be any finitely generated $\Lambda$-module, and apply the spectral sequence

$\text{Ext}_\Lambda^p(\text{H}_3^{(-q)}(\Lambda^{(-)}), M) \Rightarrow \text{H}_p^{+q}(M^{#}(1))$,

which degenerates to yield $\text{Hom}_\Lambda(\mathcal{X}_\chi^{-1}(1), M) \cong \text{H}_1^c(M^{#}(1))$. As above, using (4.1.5) and the Yoneda lemma, we obtain $\mathcal{X}_\chi^{-1}(1) \xrightarrow{\sim} C^{\text{red}}$. □

4.2. A version of Wiles’s numerical criterion. Having controlled $\mathcal{J}/\mathcal{J}^2$ in terms of Iwasawa class groups, we will now make use of a version of Wiles’s numerical criterion [Wil95, Appendix] to prove our $R = T$-theorem. We thank Eric Urban for suggesting that the numerical criterion might be used to improve an earlier version. We follow the exposition of [OSRS97].

Consider the diagram

$\begin{array}{ccc}
R_D^{\text{red}} & \xrightarrow{\phi} & \mathcal{J} \\
\downarrow{\pi_{R_D^{\text{red}}}} & & \downarrow{\pi} \\
\Lambda & & \\
\end{array}$
Lemma 4.2.3. We have algebra. Then Fitt

Theorem 4.2.1. [dSRS97, Section 3 Theorem (pg. 9)] The map φ is an isomorphism of complete intersections if and only if φ(Fitt_B^red(J)) ⋭ \mathfrak{m}_A\mathfrak{g}.

We will use this theorem to show that φ is an isomorphism under the assumption that X_\theta = 0. The proof is inspired by the proof of Criteria I in [dSRS97]. We first prove some preliminary results. The notation ‘Fitt’ that appears in the theorem refers to Fitting ideals, which are reviewed in [dSRS97, Section 1]. We will make frequent use of the following well known property of Fitting ideals (see [dSRS97, Proposition 1.1]).

Lemma 4.2.2. Let A be a ring, M a finitely presented A-module, and B an A-algebra. Then Fitt_B(M \otimes_A B) = Fitt_A(M) \cdot B.

Lemma 4.2.3. We have π(Fitt_\mathfrak{g}(I)) = Fitt_\Lambda(I/I^2) \subset (\xi) as ideals of \Lambda.

Proof. By Lemma 4.2.2 we have

π(Fitt_\mathfrak{g}(I)) = Fitt_\Lambda(I \otimes_\mathfrak{g} \Lambda).

Since I \otimes_\mathfrak{g} \Lambda = I/I^2 \cong I/I^2, we have π(Fitt_\mathfrak{g}(I)) = Fitt_\Lambda(I/I^2).

Now, since b/I = \Lambda/\xi, we see that I \otimes_b Q(b) \cong Q(b), and hence Fitt_Q(b)(I \otimes_b Q(b)) = 0. Applying the same lemma many times, we have Fitt_b(I) = 0 and so Fitt_{b/I}(I/I^2) = Fitt_\Lambda/\xi(I/I^2) = 0, and so finally Fitt_\Lambda(I/I^2) \subset (\xi).

Proposition 4.2.4. Assume that X_\theta = 0. Then X_χ(1) = J/J^2 \cong I/I^2 \cong I/I^2. In particular, Fitt_\Lambda(I/I^2) = (\xi).

Remark 4.2.5. Sharifi has studied a map from class groups to I/I^2 similar to the one that appears in the following proof; see the map of [Sha07, Theorem 5.2].

Proof. From (4.1.1), we have a surjection

(4.2.6) \quad B^{\text{red}} \otimes_\Lambda C^{\text{red}} \twoheadrightarrow J/J^2 \twoheadrightarrow I/I^2 \twoheadrightarrow I/I^2.

By Proposition 4.1.2 we have B^{\text{red}} \cong X_\chi(1) and C^{\text{red}} \cong X_\chi^{#-1}(1), and so by Proposition 2.2.5 we have C^{\text{red}} \cong \Lambda. Hence we have a surjection

(4.2.6) \quad \Theta : X_\chi(1) \twoheadrightarrow I/I^2.

But by the previous lemma, Fitt_\Lambda(I/I^2) \subset (\xi) = char_\Lambda(X_\chi(1)). This implies that the ker(\Theta) is finite (see [Wak15a, Lemma A.7] for example), and hence ker(\Theta) = 0, since X_\chi(1) has no finite submodule by Ferrero-Washington [FW79]. Thus \Theta is an isomorphism, and therefore so are all of the maps in (4.2.6).

For the final statement, notice that, by Ferrero-Washington and Lemma 2.2.4 X_\chi(1) has projective dimension 1. Therefore the ideal Fitt_\Lambda(X_\chi(1)) is principal. This implies Fitt_\Lambda(X_\chi(1)) = char_\Lambda(X_\chi(1)) (see [Wak15a, Lemma A.6], for example).

Lemma 4.2.7. We have Ann_\mathfrak{g}(I) = ker(\mathfrak{g} \rightarrow b), and, in particular, Ann_\mathfrak{g}(I) \nsubseteq \mathfrak{m}_A\mathfrak{g}. The restriction of π to Ann_\mathfrak{g}(I) induces an isomorphism

π|_{Ann} : Ann_\mathfrak{g}(I) \simto (\xi).
Proof. That \( \text{Ann}_\Lambda(I) = \ker(\delta \to h) \) and the fact that \( \pi|_{\text{Ann}} \) is an isomorphism follow from [WWE15 Proposition 3.2.5]. To see that \( \text{Ann}_\Lambda(I) \not\subset m_A\delta \), note that \( \delta \to h \) is a surjection of free \( \Lambda \) modules of distinct rank, and so \( \text{Ann}_\Lambda(I) = \ker(\delta \to h) \) is a non-zero \( \Lambda \)-free direct summand of \( \delta \).

**Theorem 4.2.8.** Assume that \( X_\theta = 0 \). Then \( \phi : P^\text{ord}_D \to \delta \) is an isomorphism of complete intersections.

**Proof.** By Theorem 4.2.1 it suffices to show that 
\[
\phi(Fitt_{P^\text{ord}_D}(J)) \not\subset m_A\delta.
\]

By Lemma 4.2.3 and Proposition 4.2.4, we have
\[
\pi_{P^\text{ord}_D}(Fitt_{P^\text{ord}_D}(J)) = \pi(Fitt_\delta(I)) = (\xi).
\]
Since \( \phi(Fitt_{P^\text{ord}_D}(J)) \subset \text{Ann}_\Lambda(I) \) (\[3SRS97 Proposition 1.1 (i)]), we have
\[
\pi_{P^\text{ord}_D}(Fitt_{P^\text{ord}_D}(J)) = \pi(\phi(Fitt_{P^\text{ord}_D}(J))) = \pi(\text{Ann}(\phi(Fitt_{P^\text{ord}_D}(J))).
\]
But we know that \( \pi_{P^\text{ord}_D}(Fitt_{P^\text{ord}_D}(J)) = (\xi) \), and \( \pi(\text{Ann}_\Lambda(I)) = (\xi) \), so we have
\[
\pi_{\text{Ann}}(\phi(Fitt_{P^\text{ord}_D}(J))) = \pi_{\text{Ann}}(\text{Ann}_\Lambda(I))
\]
Since \( \pi_{\text{Ann}} \) is an isomorphism by Lemma 4.2.7, we conclude that \( \phi(Fitt_{P^\text{ord}_D}(J)) = \text{Ann}_\Lambda(I) \). It then follows from Lemma 4.2.7 that \( \phi(Fitt_{P^\text{ord}_D}(J)) \not\subset m_A\delta \). \( \square \)

**Remark 4.2.9.** This answers in the affirmative a question of Sharifi [Sha09 Section 5] whether \( \delta \) is Gorenstein when \( X_\theta = 0 \). As noted in loc. cit., it follows from work of Ohta that \( \delta \) is Gorenstein when \( x_\theta = 0 \), and so our result improves on Ohta’s. It was proven in [Wak15b Theorem 1.2] that \( \delta \) is Gorenstein when \( X_\theta = 0 \) under the additional assumption of Sharifi’s conjecture.

### 4.3. Applications

**Corollary 4.3.1.** Assume that \( X_\theta = 0 \) and that \( X_\chi(1) \) is cyclic. Then the ideals \( J \subset P^\text{ord}_D \), \( I \subset \delta \) and \( I \subset h \) are all principal, and both \( \delta \) and \( h \) are complete intersections.

**Proof.** By Proposition 4.2.4, we have \( X_\chi(1) \cong J/J^2 \cong I/I^2 \cong \Lambda/I \), and so, if \( X_\chi(1) \) is cyclic, then each ideal is principal by Nakayama’s lemma. The fact that \( \delta \) and \( h \) are complete intersection follows by general commutative algebra (see [WWE15 Lemma 4.1.2], for example).

These results have applications to Sharifi’s conjecture. Recall that Sharifi’s conjecture states that two maps \( \Upsilon \) and \( \varpi \) are isomorphisms [Sha11]. In [FK12, EKS14] this conjecture was refined to state that \( \Upsilon \) and \( \varpi \) are mutually inverse. (See also [WWE15 Section 7.3] for a review of Sharifi’s conjecture using the same notations as this paper.)

**Corollary 4.3.2.** Consider the maps 
\[
\Upsilon : X_\chi(1) \to H^-/IH^- \quad \text{and} \quad \varpi : H^-/IH^- \to X_\chi(1)
\]
defined by Sharifi. If \( X_\theta = 0 \), then \( \Upsilon \) is an isomorphism. If, in addition, \( X_\chi(1) \) is cyclic, then \( \varpi \) is an isomorphism as well. If, in further addition, \( \xi_\chi \) has no multiple root, then they are mutual inverses.
Proof. By Theorem 4.2.8 if $X_\theta = 0$, then $\mathcal{H}$ is Gorenstein. It is known that if $\mathcal{H}$ is Gorenstein, then $\Upsilon$ is an isomorphism (see [Sha11, Proposition 4.10]). If $X_\chi(1)$ is cyclic, then the previous corollary implies that $\mathcal{H}$ and $\mathfrak{h}$ are complete intersection, and hence Gorenstein. The result now follows by work of Fukaya-Kato.

Indeed, since $\mathfrak{h}$ is Gorenstein and $H^-$ is a dualizing module over $\mathfrak{h}$ (see [WWE15, Corollary 3.3.2]), we see that $H^-/IH^- \simeq \mathfrak{h}/I \cong \Lambda/\xi$, which has no $p$-torsion. Moreover, the fact that $\mathfrak{h}$ is Gorenstein implies the condition $C(\mathfrak{h})$ of Fukaya-Kato by [FK12, Section 7.2.10]. Then the final two claims follow from Theorems 7.2.8 and 7.2.7 of [FK12], respectively. □

This $R = T$-isomorphism provides a new proof of a modularity result of Skinner and Wiles. One of the main results of [SW99] is the following

**Theorem 4.3.3 (Skinner-Wiles).** Suppose that $\rho : G_\mathbb{Q} \to \text{GL}_2(F)$ is continuous, irreducible, and ramified at finitely many primes, where $F/\mathbb{Q}_p$ is a finite extension and $p$ is odd. Suppose that $\bar{\rho}_{ss} \simeq \omega^{-1} \oplus \theta^{-1}$, and that

1. $\chi|\sigma_p \neq 1$,
2. $\rho|_{I_p} \simeq \begin{pmatrix} \kappa_{\text{cyc}}|t_p & 0 \\ 0 & * \end{pmatrix}$,
3. $\det \rho = \psi k^{-3}$ is odd, where $\psi$ is a finite order character and $k \geq 2$.

Then $\rho$ comes from a modular form.

We can give a new proof of this result in certain cases, following directly from Theorem 4.2.8. The main conditions are that Vandiver’s conjecture holds for the relevant isotypic parts of the class group, and that $\rho$ is a lift of “minimal level.” We write characters of $\text{Gal}(\mathbb{Q}(\zeta_Np)/\mathbb{Q})$ and $(\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ interchangeably, and write $\chi = \theta \omega^{-1}$ as usual.

**Corollary 4.3.4.** Choose some $\theta$ and $\rho$ as in Theorem 4.3.3, and write $N$ for the tame modulus of $\theta$. Assume in addition that

(i) $p \nmid \phi(N)$ and $p \geq 5$;
(ii) $\theta$ is ramified at $p$ when $N > 1$;
(iii) $\rho$ is ramified only at primes dividing $pN$; and,
(iv) $X_\theta = 0$ and $X_{\theta^{-1}, \omega^2} = 0$.

Then $\rho$ comes from a modular form.

We will see in the proof that for each $\rho$, only one of the two groups $X_\theta$ and $X_{\theta^{-1}, \omega^2}$ must be zero. (Also see Remark 4.3.5)

**Proof.** First, we verify the running assumptions of §2.1.

Suppose that $N = 1$. We claim that the existence of $\rho$ implies that $\theta$ cannot be 1 or $\omega^2$. Indeed, in these cases, $\chi = \omega^{-1}$, and Stickelberger’s theorem implies that $X_\chi = 0$ (see e.g. [Was82, Prop. 6.11, Thm. 6.17]). When $X_\chi = 0$, the surjection (4.2.6) shows that the reducibility ideal $\mathcal{J}$ of $R_{\text{ord}}^D$ vanishes. This means that all ordinary pseudodeformations of $\bar{D}$ are reducible, contradicting the existence of the irreducible $\rho$. Hence, if $N = 1$, then $\theta$ is primitive and $\theta \neq \omega^2$, so that the assumptions of §2.1 hold.

Now suppose $N > 1$. Assumption (i) implies that our running assumption that $p \nmid N\phi(N)$ is satisfied. We have that $\theta$ is primitive of modulus either $N$ or $Np$ by definition of $N$. Assumption (ii) rules out the case that $\theta$ is primitive of modulus $N$,
so that assumption (a) of §2.1 is satisfied. Condition (1) is equivalent to assumption (b) of §2.1.

We emphasize that we will use the word ordinary to refer to Definition 3.4.1. In particular, whether $\rho$ is ordinary may depend upon the choice of $\theta$.

If $\chi$ is ramified, i.e. $\theta|_{f_p} \neq \omega|_{f_p}$, condition (2) ensures that $\rho$ is ordinary. This is the case because the quotient representation $\kappa_{\text{cyc}}^{-1}\chi |_{G_p}$ of $\rho|_{G_p}$ must correspond to the idempotent associated to the factor $\theta^{-1}$ in Definition 3.4.1.

In the case that $\chi$ is unramified, either idempotent can correspond to this quotient. In the case that the $\theta^{-1}$ idempotent corresponds to the quotient, then $\rho$ is ordinary. Otherwise, $\rho$ is ordinary when $\theta$ is replaced by $\theta' = \theta^{-1}\omega^2$. Notice that the assumptions of the corollary remain true after replacing $\theta$ by $\theta'$, when $\chi$ is unramified.

Write $\theta_*$ for whichever of $\theta$ or $\theta'$ that $\rho$ is ordinary for; denote the other one by $\theta'_*$. Likewise, write $\chi_* = \theta_*\omega^{-1}$ and $\chi'_* = \theta'_*\omega^{-1}$. By the minimal level assumption (iii), the ordinary pseudorepresentation $\psi(\rho)$ induces a map $R^{\text{ord}}_{\rho|_{G_p}} \rightarrow F$. Applying Theorem 4.2.8, we have the isomorphism $R^{\ast}_{\rho|_{G_p}} \cong A_\rho$, and the resulting map $A_\rho \rightarrow F$ corresponds to an ordinary $p$-adic modular eigenform $f$ with coefficients in $F$ and associated Galois representation $\rho_f \simeq \rho$. Condition (3) implies that this modular form has weight $k \in \mathbb{Z}_{\geq 2}$; consequently, $f$ is classical by [Hid86, Theorem I], and $\rho$ is modular.

Remark 4.3.5. If $\chi$ is ramified, i.e. $\theta|_{f_p} \neq \omega|_{f_p}$, then condition (iv) can be loosened to "$X_0 = 0$.” If $\chi$ is unramified, this weaker assumption implies the modularity of $\rho$ whose twist-unramified quotient corresponds to the idempotent associated to $\omega^{-1}$. Skinner and Wiles [SW97] proved the modularity of such $\rho$ under slightly different assumptions. Among their assumptions is that $X_{\chi^{-1}} = 0$. This is equivalent to $X_\theta = 0$ by the reflection principle, and so it is a much stronger assumption than our assumption that $X_\theta = 0$.

References


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