Representation theory of $S_n$ via Okounkov-Vershik

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Motivation: symmetry

For any finite group $G$, a natural question is “what can $G$ act on”? In other words, what structures respect the symmetry encoded by $G$? Put another way, we are looking for subgroups of matrices that perform as $G$ would.

The question of enumerating such “$G$-like matrices” turns out to be an interesting question with beautiful answers. This talk answers the question for a superstar actor: $S_n$. 
References and Acknowledgements

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  arxiv.org/abs/math/0503040v3

- Omer Offen Reading Course Spring 2017
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The irreducible representations of $S_n$ are called *Specht Modules*, and have already been constructed through other means:

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- **The Combinatorics of Tableaux**
  - Certain subgroups corresponding to a tableau are used to create representations which turn out to be irreducible.

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  - Duality between $S_n$ and $GL(N)$

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Outline

1. General Representation Theory
   - Representations
   - Permutation representations
   - Group Algebra
   - Irreducibles
   -Multiplicity-free subgroups $H \triangleleft G$

2. $S_{n-1} \triangleleft S_n$ (multiplicity-free)
   - Symmetric Actions
   - Proof that $S_n \times S_{n-1} \curvearrowright S_n$ is symmetric
   - Equivalence to $S_n \times S_{n-1} \curvearrowright S_n$ symmetric action

3. Gelfand-Tsetlin Basis and YJM generators
   - Branching Graph
   - GZ-Algebra and equivalence
   - Generating set for GZ-Algebra
   - Basis Elements $\leftrightarrow$ Eigenvalues
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General Representation Theory

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The representation from $G \rtimes G$ is denoted $L[G]$
Unlike most representations, the regular representation $L[G]$ is an algebra. In fact, every representation can be viewed as a module over $L[G]$:

Example

$\rho_{\text{perm}}((12) + 2(23)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Remark: The center of $L[G]$ consists of central functions (those constant across conjugacy classes).
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where the sum is over irreducible representations with respective vector spaces $V_\rho$. This isomorphism is also called the Fourier Transform map.
- If $V_\rho$ and $V_\gamma$ are irreducible and non-isomorphic, then $\text{Hom}_G(V_\rho, V_\gamma)$ is null. If they are isomorphic, the Hom-set is one-dimensional. (Schur’s Lemma)
Irreducible Representations

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Multiplicity-free Subgroups

A subgroup $H < G$ inherits any representations of $G$ via restriction. If all irreducible reps of $G$ restrict to multiplicity-free reps of $H$, we say $H < G$ and say $H$ is a *multiplicity-free subgroup* of $G$. 
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<td>(S_3)</td>
<td>yes</td>
</tr>
<tr>
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<td>(S_4)</td>
<td>no</td>
</tr>
<tr>
<td>(\mathbb{Z}_n)</td>
<td>(\mathbb{Z}_{mn})</td>
<td>yes</td>
</tr>
<tr>
<td>(\mathbb{Z}_n)</td>
<td>(D_n)</td>
<td>no</td>
</tr>
<tr>
<td>(G_1)</td>
<td>(G_1 \times G_2)</td>
<td>no(^1)</td>
</tr>
</tbody>
</table>

\(^1\)Provided \(G_2\) is not abelian
## Multiplicity-free subgroup examples

Which of these subgroups are multiplicity-free?

<table>
<thead>
<tr>
<th>Subgroup</th>
<th>Group</th>
<th>Multiplicity-free?</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1}</td>
<td>(\mathbb{Z}_n)</td>
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Outline for §2 \( S_{n-1} \triangleleft S_n \) (multiplicity-free)

1. General Representation Theory

2. \( S_{n-1} \triangleleft S_n \) (multiplicity-free)
   - Symmetric Actions
   - Proof that \( S_n \times S_{n-1} \curvearrowright S_n \) is symmetric
   - Equivalence to \( S_n \times S_{n-1} \curvearrowright S_n \) symmetric action

3. Gelfand-Tsetlin Basis and YJM generators
Outline of proof $S_{n-1} \preceq S_n$
Outline of proof $S_{n-1} \preccurlyeq S_n$

This proof is due to Ceccherini et al. The proof consists of three steps (all surprising)
Outline of proof $S_{n-1} \lhd S_n$

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2. $S_n \times S_{n-1} \curvearrowright S_n$ is a symmetric action, so $L[S_n]$ is a multiplicity-free rep of $S_{n-1} \times S_n$. 
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Symmetric Action Examples

Definition

A transitive action \( G \curvearrowright X \) is called *symmetric* if the induced action \( G \curvearrowright X \times X \) has orbits symmetric in the two coordinates.

I.e. for all \((x, y) \in X \times X\) there exists \(g \in G\) such that \((gx, gy) = (y, x)\).
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$\mathbb{Z}_6 \acts [6]$
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Proof: Schur’s Lemma
Proof that $S_n \times S_{n-1} \curvearrowright S_n$ is symmetric

Claim: This action is symmetric

Proof: Unpacking the definition results in trying to solve these simultaneous equations:

1. $\sigma \pi_1 \upsilon^{-1} = \pi_2$ (1)
2. $\sigma \pi_2 \upsilon^{-1} = \pi_1$ (2)

Each equation has $|S_n| - 1$ degrees of freedom ($\upsilon$ is free), and there is a simultaneous solution by a conjugacy fact: (It is equivalent to finding $\upsilon \in S_{n-1}$ such that $\pi_2^{-1} \pi_1 = \upsilon \pi_2^{-1} \upsilon$, which is always solvable because you can conjugate any element of $S_n$ to its inverse by an element of $S_{n-1}$. )
$S_n \times S_{n-1} \curvearrowright S_n$ is symmetric

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given $\pi_1, \pi_2 \in S_n$ find $\sigma \in S_n$ and $\nu \in S_{n-1}$ such that

\[ \sigma \pi_1 \nu^{-1} = \pi_2 \]  
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\begin{align*}
\sigma \pi_1 \nu^{-1} &= \pi_2 \\
\sigma \pi_2 \nu^{-1} &= \pi_1
\end{align*}

(1) (2)

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(It is equivalent to finding $\nu \in S_{n-1}$ such that $\pi_2^{-1} \pi_1 = \nu \pi_1^{-1} \pi_2 \nu^{-1}$, which is always solvable because you can conjugate any element of $S_n$ to its inverse by an element of $S_{n-1}$).
$L[S_n]$ is multiplicity-free rep of $S_n \times S_{n-1}$ means...

---

3. The irreducible representations of a product of groups are completely given by tensor products of the irreducibles of the individual groups.

4. The dual of an irreducible representation is also irreducible.
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In formulas, if $H < G$ with this permutation rep of $G \times H$ on $L[G]$ denoted by $\eta$, then for any $\sigma \in Irrep(G)$ and $\rho \in Irrep(H)$ there is an isomorphism of vector spaces

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The map is $T \mapsto (v \otimes w \mapsto (g \mapsto (T(v)(\sigma(g^{-1})w))))$

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**Corollary** If $\eta$ is multiplicity-free, then $H < G$ (by facts$^{34}$)

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In formulas, if $H \triangleleft G$ with this permutation rep of $G \times H$ on $L[G]$ denoted by $\eta$, then for any $\sigma \in \text{Irrep}(G)$ and $\rho \in \text{Irrep}(H)$ there is an isomorphism of vector spaces

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**Corollary** If $\eta$ is multiplicity-free, then $H \triangleleft G$ (by facts\(^3\))
This completes our argument $S_{n-1} \triangleleft S_n$

---

\(^3\)The irreducible representations of a product of groups are completely given by tensor products of the irreducibles of the individual groups.

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Outline for §3 Gelfand-Tsetlin Basis and YJM generators

1. General Representation Theory

2. $S_{n-1} \triangleleft S_n$ (multiplicity-free)

3. Gelfand-Tsetlin Basis and YJM generators
   - Branching Graph
   - GZ-Algebra and equivalence
   - Generating set for GZ-Algebra
   - Basis Elements ↔ Eigenvalues
Given $n$, consider the directed graph with vertex set the irreducibles of $S_i$ for $i = 1, \ldots, n$ and an edge goes from a rep of $S_{i+1}$ to one of $S_i$ if the target appears in the restriction of the source.
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This “works” because $S_{n-1} \preceq S_n$. 
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Gelfand-Tsetlin basis

Using the branching graph, we canonically deconstruct each irreducible rep of $S_n$ into one-dimensional subspaces (sub-representations). Choosing a vector from each such subspace (not canonical) defines a basis called the GT basis. (They form a basis by the direct sum decomposition)
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Remark: “The GT basis” refers to many different bases; one for each irreducible rep of \( S_n \).
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Example: Branching graph of $S_4$
GZ-Algebra

Definition

\[ GZ(n) := \{ f \in L[S_n] \mid \text{all Fourier Transforms of } f \text{ are diagonal in GT bases} \} \]
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\( GZ(n) \) is a commutative sub-algebra of \( L[S_n] \), and represents those elements of the group algebra who “play well” with the GT basis. In particular, those operators with the GT basis as eigenvectors.
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Surprising Theorem

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\(^5\text{RHS} \subset \text{LHS}: \text{The Fourier Transform distributes over an element of RHS, and } \rho(\text{central}) \text{ is always a scalar. Normally that is not enough, because the } Z(i) \text{ are not nested, but the inductive basis solves this problem. For the reverse, construct a naïve basis for RHS as convolutions of elements of } GZ(n).} \)
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If \( Z(n) := Z(L[S_n]) \), then considering all \( Z(i) \) as sub-algebras of \( L[S_n] \), we have

\[ GZ(n) = \langle Z(1), \ldots, Z(n) \rangle \]

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**Proof:** Omitted, but this only works for the GT basis!\(^5\)

---

\(^5\) RHS \( \subset \) LHS: The Fourier Transform distributes over an element of RHS, and \( \rho(\text{central}) \) is always a scalar. Normally that is not enough, because the \( Z(i) \) are not nested, but the inductive basis solves this problem. For the reverse, construct a naïve basis for RHS as convolutions of elements of \( GZ(n) \).
Generating set

Surprising Theorem

\{\text{operators who like GT basis}\} = \langle Z(1), \ldots, Z(n) \rangle
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What does RHS look like as a subset of \( L[S_n] \)? Polynomials in all conjugacy classes, including lower conjugacy classes.
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Is there a “smarter” generating set? I.e. is there redundancy among these conjugacy classes?
Generating set for GZ Algebra

Lynchpin Theorem (Okounkov-Vershik '04)
Generating set for GZ Algebra

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The GZ Algebra $\langle Z(1), \ldots, Z(n) \rangle$ is generated by the elements $X_i$ for $i = 1, \ldots, n$ where

$$X_i := (1i) + (2i) + \cdots + (i - 1, i)$$

In other words, every $S_n$ conjugacy class can be built from lower ones with the “new transpositions” (the $X_i$ are called Young-Jucys-Murphy elements; $X_1 = 0$, $X_2 = (12)$, etc.).
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Proof:

Obtain $k$-cycles: use induction and multiply all $k-1$ cycles by $X_n$. You get what you want, plus an extra term with two cycles. Use a “lower-terms” argument to cancel the extra term.

Once you have all $k$-cycles, multiply them together and use a “lower-terms” argument to get an arbitrary class. This last step turns out to be exactly the statement that power sum symmetric functions form a multiplicative basis.

Sage example for $n = 4$. 
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Eigenvalues for the generators

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There is an injective map from \{paths in branching graph\} to $\mathbb{C}^n$ defined by

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We denote the image of this map by Spec($n$).
Remarks about Spec(n)

- There is a bijective map from \{paths in branching graph\} to Spec(n) defined by
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- Define an equivalence relation on \( \text{Spec}(n) \) by

  \[
  (a_1, \ldots, a_n) \approx (b_1, \ldots, b_n)
  \]

  if the corresponding paths have the same start (the same irreducible representation).

- There are \( p(n) \) equivalence classes.

- \( \text{Spec}(n) \) is a finite set. If we understood \( \text{Spec}(n) \) (with combinatorics) we would understand the branching graph.
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Properties of Spec(n)

Lemma; $L[S_n]$ identity

Proof: immediate from conjugating $X_i$ by $s_i$. 

Let $a = (a_1, \ldots, a_n) \in \text{Spec}(n)$ with corresponding basis vector $v_a$ and representation $\rho_a$. If $\alpha$ is the coefficient of $v_a$ in $\rho_a(s_i)v_a$, then $\alpha$ satisfies 

$1 = (a_i + 1 - a_i)\alpha$ 

Proof: Apply the above identity to $v_a$ and equate coefficients on $v_a$. 

Duncan Levear
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If $s_i := (i - 1, i)$ then

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$$X_is_i = s_iX_{i+1} - 1$$  \hspace{1cm} (3)

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Lemma
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Proof: Apply the above identity to $\nu$ and equate coefficients on $\nu$. 
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- in fact, the converse is also true, with a bit more work\(^6\)
- We also know \( a_1 = 0 \)

---

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Because $S_{n-1} \lhd S_n$, we can define the branching graph, and paths of this graph index a special basis called the GT basis.
Summary

- Because $S_{n-1} \triangleleft S_n$, we can define the branching graph, and paths of this graph index a special basis called the GT basis.

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The answer to this question is a sub-algebra of $L[S_n]$ which happens to be generated by the “new transposition” elements $X_i$. 
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- Therefore, we consider the set of possible eigenvalues of the $X_i$ applied to this basis.
- This $n$-tuple encodes a lot— you can deduce the basis vector just from its eigenvalues (because the $X_i$ generate all diagonal operators for that basis). So Spec($n$) encodes the entire branching graph.
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We currently have a few small results about elements of Spec($n$), but this is far from a characterization.
We want to find some structure which encodes Spec(n). I.e. describe the $n$-tuples of complex numbers combinatorially. This will be accomplished through Standard Young Tableaux.