Fibonacci Formulas

Definitions:

$$F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2, \text{ with } F_0 = 0, F_1 = 1$$
$$L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 2, \text{ with } L_0 = 2, L_1 = 1$$
$$F_{-n} = (-1)^{n-1}F_n, \quad L_{-n} = (-1)^n L_n$$

The Binet Formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n,$$

where

$$\alpha = 1 + \sqrt{5}/2 = 1.61803398, \quad \beta = 1 - \sqrt{5}/2 = -.61803398.$$

Note that

$$\alpha + \beta = 1, \quad \alpha\beta = -1, \quad \alpha - \beta = \sqrt{5}, \quad \alpha^2 = \alpha + 1.$$

Symmetry principle: If $R(x, y)$ is any rational function of x and y with rational coefficients such that $R(\alpha, \beta) = 0$, then $R(\beta, \alpha) = 0$. Thus, for example, from $\alpha^2 = \alpha + 1$, we get $\beta^2 = \beta + 1$, and from $2 = \alpha + \alpha^{-2}$ we get $2 = \beta + \beta^{-2}$.

Recurrences A sequence $(\ldots, g_{-1}, g_0, g_1, \ldots)$ satisfying the *Fibonacci recurrence* $g_n = g_{n-1} + g_{n-2}$ is called a *generalized Fibonacci sequence*. The set of generalized Fibonacci sequences is a vector space of dimension 2 over the complex numbers. This implies that for any three such sequences, one can be expressed as a linear combination of the other two. Here are some generalized Fibonacci sequences:

$$g_n = F_{n+i}, \text{ for any } i$$
$$g_n = L_{n+j}, \text{ for any } j$$
$$g_n = \alpha^n$$
$$g_n = \beta^n$$

Note also that if g_n is a generalized Fibonacci sequence then so is g_{n+i} for any i. Note that the Binet formulas express L_n and F_n in terms of α^n and β^n. Here are some other formulas that express generalized Fibonacci sequences as linear combinations of other generalized Fibonacci sequences.

$$L_n = F_{n-1} + F_{n+1} = F_n + 2F_{n-1} = 2F_{n+1} - F_n$$
$$F_n = \frac{1}{5}(L_{n-1} + L_{n+1}) = \frac{1}{5}(2L_{n+1} - L_n)$$
$$\alpha^n = \alpha F_n + F_{n-1} = F_{n+1} - \beta F_n = \beta^n + \sqrt{5}F_n = \frac{1}{2}(L_n + \sqrt{5}F_n) = \frac{1}{\sqrt{5}}(L_{n+1} - 5\beta L_n)$$

$$F_{n+i} = F_{i-1}F_n + F_iF_{n+1} = \frac{1}{5}(L_{i-1}L_n + L_iL_{n+1})$$
$$L_{n+i} = F_{i-1}L_n + F_iL_{n+1}$$
$$F_nF_i = \frac{1}{5}(L_{n+i} - (-1)^i L_{n-i})$$
$$L_nL_i = L_{n+i} + (-1)^i L_{n-i}$$
$$F_nL_i = F_{n+i} + (-1)^i F_{n-i}$$
$$F_{n+i} = \frac{1}{2}(F_iL_n + L_iF_n)$$
$$L_{n+i} = \frac{1}{2}(5F_iF_n + L_iL_n)$$

Since any generalized Fibonacci sequence is determined by any two consecutive values, to prove any of these formulas it is sufficient to check that it is true for $n = 0$ and $n = 1$.

We know that if (g_n) is any generalized Fibonacci sequence then for any i, j, and k, there is a linear dependece among (g_{n+i}), (g_{n+j}), and (g_{n+k}); and this formula may be written as

$$(-1)^k F_{j-k} g_{n+i} + (-1)^i F_{k-i} g_{n+j} + (-1)^j F_{i-j} g_{n+k} = 0.$$

As an important special case, if we take $j = 0$ and $k = 2i$, and use the fact that $F_{2i} = F_i L_i$, we get

$$g_{n+2i} - L_i g_{n+i} + (-1)^i g_n = 0.$$

Here are some summation formulas, some of which give Zeckendorf representations:

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$
$$\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1$$
$$\sum_{i=0}^{k} F_{j+4i} = F_{2k+2} F_{2k+j}$$

$$\frac{F_{mn}}{F_n} = L_{(m-1)n} + (-1)^n L_{(m-3)n} + L_{(m-5)n} + \cdots + (-1)^{\frac{m-2}{2}n} L_n, \quad m \text{ even}$$

The case $m = 2$ of this formula is especially important: $F_{2n} = F_n L_n$.

$$\frac{F_{mn}}{F_n} = L_{(m-1)n} + (-1)^n L_{(m-3)n} + L_{(m-5)n} \cdots + (-1)^{\frac{m-3}{2}n} L_{2n} + (-1)^{\frac{m-1}{2}n}, \quad m \text{ odd}$$

Can you find similar formulas for L_{mn}/L_n when m is odd?
If $m > n$ and $m - n$ is odd then

$$F_m - F_n = F_{m-1} + F_{m-3} + \cdots + F_{n+4} + F_{n+2} + F_{n-1}.$$

If $m > n$ and $m - n$ is even then

$$F_m - F_n = F_{m-1} + F_{m-3} + \cdots + F_{n+3} + F_{n+1}.$$

Can you find similar formulas for $L_m - L_n$?
If $m - n$ is divisible by 8 then

$$\frac{F_m - F_n}{3} = (F_{m-3} + F_{m-5}) + (F_{m-11} + F_{m-13}) + \cdots + (F_{n+5} + F_{n+3}).$$