Fibonacci Formulas

Definitions:

\[ F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2, \text{ with } F_0 = 0, F_1 = 1 \]
\[ L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 2, \text{ with } L_0 = 2, L_1 = 1 \]
\[ F_n = (-1)^{n-1}F_n, \quad L_n = (-1)^nL_n \]

The Binet Formulas:

\[ F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \]

where \( \alpha = 1 + \sqrt{5}/2 = 1.61803398, \quad \beta = 1 - \sqrt{5}/2 = -0.61803398. \)

Note that \( \alpha + \beta = 1, \quad \alpha \beta = -1, \quad \alpha - \beta = \sqrt{5}, \quad \alpha^2 = \alpha + 1. \)

Symmetry principle: If \( R(x,y) \) is any rational function of \( x \) and \( y \) with rational coefficients such that \( R(\alpha, \beta) = 0 \), then \( R(\beta, \alpha) = 0. \) Thus, for example, from \( \alpha^2 = \alpha + 1 \), we get \( \beta^2 = \beta + 1 \), and from \( 2 = \alpha + \alpha^{-2} \) we get \( 2 = \beta + \beta^{-2}. \)

Recurrences A sequence \((\ldots,g_{-1},g_0,g_1,\ldots)\) satisfying the Fibonacci recurrence \( g_n = g_{n-1} + g_{n-2} \) is called a generalized Fibonacci sequence. The set of generalized Fibonacci sequences is a vector space of dimension 2 over the complex numbers. This implies that for any three such sequences, one can be expressed as a linear combination of the other two. Here are some generalized Fibonacci sequences:

\[ g_n = F_{n+i}, \text{ for any } i \]
\[ g_n = L_{n+j}, \text{ for any } j \]
\[ g_n = \alpha^n \]
\[ g_n = \beta^n \]

Note also that if \( g_n \) is a generalized Fibonacci sequence then so is \( g_{n+i} \) for any \( i \). Note that the Binet formulas express \( L_n \) and \( F_n \) in terms of \( \alpha^n \) and \( \beta^n \). Here are some other formulas that express generalized Fibonacci sequences as linear combinations of other generalized Fibonacci sequences.

\[ L_n = F_{n-1} + F_{n+1} = F_n + 2F_{n-1} = 2F_{n+1} - F_n \]
\[ F_n = \frac{1}{5}(L_{n-1} + L_{n+1}) = \frac{1}{5}(2L_{n+1} - L_n) \]
\[ \alpha^n = \alpha F_n + F_{n-1} = F_{n+1} - \beta F_n = \beta^n + \sqrt{5}F_n = \frac{1}{2}(L_n + \sqrt{5}F_n) = \frac{1}{\sqrt{5}}(L_{n+1} - 5\beta L_n) \]
\[ F_{n+i} = F_{i-1}F_n + F_iF_{n+1} = \frac{1}{5}(L_{i-1}L_n + L_iL_{n+1}) \]
\[ L_{n+i} = F_{i-1}L_n + F_iL_{n+1} \]
\[ F_nF_i = \frac{1}{5}(L_{n+i} - (-1)^iL_{n-i}) \]
\[ L_nL_i = L_{n+i} + (-1)^iL_{n-i} \]
\[ F_nL_i = F_{n+i} + (-1)^iF_{n-i} \]
\[ F_{n+i} = \frac{1}{2}(F_iL_n + L_iF_n) \]
\[ L_{n+i} = \frac{1}{2}(5F_iF_n + L_iL_n) \]

Since any generalized Fibonacci sequence is determined by any two consecutive values, to prove any of these formulas it is sufficient to check that it is true for \( n = 0 \) and \( n = 1. \)
We know that if \((g_n)\) is any generalized Fibonacci sequence then for any \(i, j,\) and \(k,\) there is a linear dependence among \((g_{n+i}), (g_{n+j}),\) and \((g_{n+k});\) and this formula may be written as
\[
(-1)^k F_{j-k} g_{n+i} + (-1)^j F_{k-i} g_{n+j} + (-1)^i F_{i-j} g_{n+k} = 0.
\]
As an important special case, if we take \(j = 0\) and \(k = 2i,\) and use the fact that \(F_{2i} = F_i L_i,\) we get
\[
g_{n+2i} - L_i g_{n+i} + (-1)^i g_n = 0.
\]
Here are some summation formulas, some of which give Zeckendorf representations:
\[
\sum_{i=0}^{n} F_i = F_{n+2} - 1
\]
\[
\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1
\]
\[
\sum_{i=0}^{k} F_{j+4i} = F_{2k+2} F_{2k+j}
\]
\[
\frac{F_{mn}}{F_n} = L_{(m-1)n} + (-1)^n L_{(m-3)n} + L_{(m-5)n} + \cdots + (-1)^\frac{m-2}{2} n L_n, \quad m \text{ even}
\]
The case \(m = 2\) of this formula is especially important: \(F_{2n} = F_n L_n.\)
\[
\frac{F_{mn}}{F_n} = L_{(m-1)n} + (-1)^n L_{(m-3)n} + L_{(m-5)n} \cdots + (-1)^\frac{m-3}{2} n L_{2n} + (-1)^\frac{m-1}{2} n, \quad m \text{ odd}
\]
Can you find similar formulas for \(L_{mn}/L_n\) when \(m\) is odd?
If \(m > n\) and \(m - n\) is odd then
\[
F_m - F_n = F_{m-1} + F_{m-3} + \cdots + F_{n+3} + F_{n+1} + F_{n-1}.
\]
If \(m > n\) and \(m - n\) is even then
\[
F_m - F_n = F_{m-1} + F_{m-3} + \cdots + F_{n+3} + F_{n+1}.
\]
Can you find similar formulas for \(L_m - L_n?\)
If \(m - n\) is divisible by 8 then
\[
\frac{F_m - F_n}{3} = (F_{m-3} + F_{m-5}) + (F_{m-11} + F_{m-13}) + \cdots + (F_{n+5} + F_{n+3}).
\]