

### Generating Function Problems

1. We know that the generating function for the Fibonacci sequence is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$

and the generating function for the Lucas sequence is

$$\sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2}.$$

What can you say about the generating function for an arbitrary generalized Fibonacci sequence?

2. Find the generating functions for the sequences  $(u_n)$  and  $(v_n)$  of problem set 2.
3. (a) Show that  $\sum_{n=0}^{\infty} F_{2n} x^n = x/(1-3x+x^2)$  and  $\sum_{n=0}^{\infty} F_{2n+1} x^n = (1-x)/(1-3x+x^2)$ .  
 (b) More generally show that

$$\sum_{n=0}^{\infty} F_{pn+q} x^n = \frac{F_q + (-1)^q F_{p-q} x}{1 - L_p x + (-1)^p x^2}.$$

4. Evaluate  $\sum_{n=0}^{\infty} F_n^2 x^n$ ,  $\sum_{n=0}^{\infty} L_n^2 x^n$ , etc.

5. (a) Prove that  $F_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k}$ , and for  $n \geq 1$ ,  $L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k}$ .

(b) From the Binet formula,

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right],$$

find another formula for the Fibonacci numbers by expanding in powers of  $\sqrt{5}$  with the binomial theorem, and find a similar formula for Lucas numbers.

6. The *exponential generating function* for the sequence  $(a_0, a_1, \dots)$  is  $\sum_{n=0}^{\infty} a_n x^n / n!$ .

(a) Show that  $\sum_{n=0}^{\infty} F_n \frac{x^n}{n!} = \frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x})$ , and find a similar formula for the Lucas numbers.

(b) By multiplying the identity of (a) by  $e^x = \sum_{n=0}^{\infty} x^n / n!$ , show that  $\sum_{k=0}^n \binom{n}{k} F_k = F_{2n}$ . Generalize.

7. Let  $R(n, k)$  be the number of ways that  $n$  can be expressed as a sum  $F_{i_1} + F_{i_2} + \dots + F_{i_k}$ , where  $i_1 > i_2 > \dots > i_k \geq 2$ . In other words,  $R(n, k)$  is the number of different partitions of  $n$  into  $k$  distinct parts, each of which is a Fibonacci number. Let  $R(n) = \sum_k R(n, k)$ , so that  $R(n)$  is the number of partitions of  $n$  into distinct Fibonacci numbers, and let  $R_n(t)$  be the polynomial  $\sum_k R(n, k)t^k$ , so that  $R(n) = R_n(1)$ . Show that

$$\sum_{n=0}^{\infty} R(n)x^n = \prod_{i=2}^{\infty} (1 + x^{F_i})$$

and that

$$\sum_{n=0}^{\infty} R_n(t)x^n = \prod_{i=2}^{\infty} (1 + tx^{F_i}).$$

What can you say about the numbers  $R(n)$  and the polynomials  $R_n(t)$ ? In particular,

- Is  $R(n)$  equal to 1 for infinitely many values of  $n$ ?
- Does  $R(n)$  grow arbitrarily large? How fast does it grow?
- Can you explain why  $R_n(t)$  often (but not always) factors?
- Why are the first and last coefficients of  $R_n(t)$  always 1?