

1. Study systematically the identities that can be obtained by expressing various bases for the generalized Fibonacci sequences in terms of each other.
2. Let D be the set of doubly indexed sequences $(a_{m,n})_{m,n}$, where m and n range over the integers, such that for fixed m , $(a_{m,n})_n$ is a generalized Fibonacci sequence and for fixed n , $(a_{m,n})_m$ is a generalized Fibonacci sequence.
 - (a) Show that D is a finite dimensional vector space.
 - (b) What is the dimension of D ?
 - (c) Find some bases for this vector space, and find some identities that express various elements of this vector space in terms of basis elements
 - (d) Show that some of the identities that we have already seen may be viewed as identities of the type considered in (c)
 - (e) If $(a_{m,n})$ is in D , what can you say about the generating function $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m y^n$?
3. Let D be as in problem 1.
 - (a) Study the subspace of D consisting of sequences $(a_{m,n})$ that satisfy $a_{m,n} = a_{n,m}$.
 - (b) Study the subspace of D consisting of sequences $(a_{m,n})$ that satisfy $a_{m,n} = -a_{n,m}$
 - (c) What can you say about the generating functions for the sequences in (a) and (b)?
 - (d) (You may need to consult some linear algebra textbooks for this.) What are tensor powers of a vector space? Symmetric powers? Exterior powers? (You might also look for the terms “tensor algebra” etc.) What is their connection with (a) and (b)?
4. Consider the set of sequences of the form $(g_n h_n)$, where g_n and h_n are generalized Fibonacci sequences.
 - (a) Show that these sequences do not form a vector space.
 - (b) Let V be the span of these sequences (in the vector space of all sequences). What is the dimension of V ?
 - (c) Find some bases for V , and find some identities associated with them as in problem 1.
 - (d) Can V be described as the set of solutions of a recurrence?
 - (e) Is there any connection between this vector space and those considered in the previous two problems?
 - (f) What can you say about generating functions for sequences in V ?
5.
 - (a) Let $\mathbf{g} = (g_n)$ be a generalized Fibonacci sequence, and let $\Theta(\mathbf{g}) = ((-1)^n g_{-n})$. We know from Problem Set 1 that $\Theta(\mathbf{g})$ is also a generalized Fibonacci sequence. Show that Θ is a linear transformation from the vector space of generalized Fibonacci sequences to itself. Find the eigenvalues and eigenvectors of Θ .
 - (b) Can you think of any other linear transformations on the vector space of generalized Fibonacci sequences? What are their eigenvalues and eigenvectors?

6. The *shift operator* E is a linear transformation on the vector space of all sequences, defined by $E(a_n) = (a_{n+1})$. Since we are working with sequences indexed by all integers, E has an inverse, which satisfies $E^{-1}(a_n) = (a_{n-1})$. We may also consider powers of E , and more generally, polynomials in E and E^{-1} . For example, if we let $I = E^0$ be the identity operator, then we have $(E^2 - 2E + I - E^{-1})(a_n) = (a_{n+2} - 2a_{n+1} + a_n - a_{n-1})$. Thus E generates an algebra of linear transformations. We may restrict these linear transformations to generalized Fibonacci sequences, obtaining a homomorphic image of this algebra. (For example, when restricted to generalized Fibonacci sequences, $E^2 - E$ is equal to the identity operator.) Study this algebra and its applications.
7. More generally, study the algebra of operators on generalized Fibonacci sequences generated by E and the linear transformation Θ of problem 5. (As a first step, do Θ and E commute?)
8. Generalize all of these questions to other sequences that satisfy recurrences.