WHICH VALUES OF A FIBONACCI FUNCTION ON $\mathbb{Z}^2$ COMPLETELY CHARACTERIZE THAT FUNCTION?

NATHAN DAVIS

Abstract. We will find a general criterion with which to determine if four known values of some real-valued Fibonacci function on $\mathbb{Z}^2$ completely characterize that function.

1. A Generalization of Fibonacci Sequences

Over the course of this discussion, we will need to generalize the notion of a Fibonacci sequence. We will do this by defining Fibonacci functions over arbitrary vector spaces.

Let V be a vector space. Let $\mathcal{F}^n(V)$ be the set of functions $F : \mathbb{Z}^n \to V$ that satisfy the following property:

(1)
$$F(x_1, \ldots, x_i, \ldots, x_n) = F(x_1, \ldots, x_i - 1, \ldots, x_n) + F(x_1, \ldots, x_i - 2, \ldots, x_n)$$

for all $x_j \in \mathbb{Z}$ and $i \in \mathbb{Z}$, $1 \leq i \leq n$.

Given any vector space V over a field U, we can make $\mathcal{F}^n(V)$ a vector space over U in the following manner:

$$(F_1 + F_2)(X) = F_1(X) + F_2(X) \tag{2}$$

$$(uF)(X) = u(F(X)) \tag{3}$$

for all $F, F_1, F_2 \in \mathcal{F}^n(V)$, $X \in \mathbb{Z}^n$, and $u \in U$.

2. The Structure of $\mathcal{F}^2(V)$

We will say that some 4-tuple $(X_1, X_2, X_3, X_4) \in (\mathbb{Z}^2)^4$ is *deterministic* if, given any real vector space V and any 4-tuple $(v_1, v_2, v_3, v_4) \in V^4$, there exists a unique $F \in \mathcal{F}^2(V)$ which satisfies $F(X_i) = v_i$ for $i \in \{1, 2, 3, 4\}$.

Lemma 1. *Let $G \in \mathcal{F}^2(\mathbb{R}^4)$. Then the 4-tuple $(X_1, X_2, X_3, X_4) \in (\mathbb{Z}^2)^4$ is deterministic if the set $\{G(X_1), G(X_2), G(X_3), G(X_4)\}$ is linearly independent.*

Date: March 26, 2000.

Proof. Let V be a real vector space. Let $(v_1, v_2, v_3, v_4) \in V^4$. Because

$$\{G(X_1), G(X_2), G(X_3), G(X_4)\}$$

is linearly independent, we see from linear algebra that there exists a unique linear map $\theta : \mathbb{R}^4 \to V$ such that $\theta(F(X_i)) = v_i$ for $i \in \{1, 2, 3, 4\}$. Let $F = \theta \circ G$. Then $F \in \mathcal{F}^2(V)$ and $F(X_i) = v_i$ for $i \in \{1, 2, 3, 4\}$. Again we see from linear algebra that for any $\phi \in \mathcal{F}^2(V)$, there exists a $\gamma : \mathbb{R}^4 \to V$ such that $\phi = \gamma \circ G$. Therefore, because θ is unique, F is the unique function in $\mathcal{F}^2(V)$ which satisfies the above property. This function F is the function required by our definition of deterministic. Therefore, (X_1, X_2, X_3, X_4) is deterministic. □

We may now continue on to our main theorem.

3. A General Criterion

Theorem 1. *Given any four points (m_1, n_1), (m_2, n_2), (m_3, n_3), and (m_4, n_4) in $\mathbb{Z}^2$ and some $F \in \mathcal{F}^2(\mathbb{R})$, the four values $F(m_1, n_1)$, $F(m_2, n_2)$, $F(m_3, n_3)$, and $F(m_4, n_4)$ uniquely determine F if the matrix*

$$\begin{bmatrix} F_{m_1-1}F_{n_1-1} & F_{m_1-1}F_{n_1} & F_{m_1}F_{n_1-1} & F_{m_1}F_{n_1} \\ F_{m_2-1}F_{n_2-1} & F_{m_2-1}F_{n_2} & F_{m_2}F_{n_2-1} & F_{m_2}F_{n_2} \\ F_{m_3-1}F_{n_3-1} & F_{m_3-1}F_{n_3} & F_{m_3}F_{n_3-1} & F_{m_3}F_{n_3} \\ F_{m_4-1}F_{n_4-1} & F_{m_4-1}F_{n_4} & F_{m_4}F_{n_4-1} & F_{m_4}F_{n_4} \end{bmatrix}$$

is invertible.

Proof. Take G of Lemma 1 to be the function in $\mathcal{F}^2(\mathbb{R}^4)$ defined by

$$\begin{aligned} G(0,0) &= (F_{-1}, F_0, F_0, F_0) \\ G(1,0) &= (F_0, F_{-1}, F_0, F_0) \\ G(0,1) &= (F_0, F_0, F_1, F_0) \\ G(1,1) &= (F_0, F_0, F_0, F_1). \end{aligned}$$

Let $(m, n) \in \mathbb{Z}^2$. Straightfoward computation along the vertical "lines" $x = 0$ and $x = 1$ in $\mathbb{Z}^2$ shows that

$$\begin{aligned} G(0,n) &= (F_{n-1}, F_0, F_n, F_0) \\ G(1,n) &= (F_0, F_{n-1}, F_0, F_n). \end{aligned}$$

Now computation along the line $y = m$ shows that

$$G(m,n) = (F_{m-1}F_{n-1}, F_{m-1}F_n, F_mF_{n-1}, F_mF_n). \tag{4}$$

Therefore, the invertibility of the matrix of Theorem 1 is equivalent to the linear independence of the four vectors of Lemma 1. □

We now have a general criterion with which to determine if four points are deterministic. There are several cases in which it is easy to compute whether or not four points are deterministic. In the following propositions, let D denote the matrix of Theorem 1, and let D_i denote the i^{th} row of D.

Proposition 1. *If three of four points lie on the same vertical line ($m_1 = m_2 = m_3$) or horizontal line ($n_1 = n_2 = n_3$) then the four points are not deterministic.*

Proof. We will only prove the proposition in the vertical case, as the horizontal case is virtually identical. The system of equations

$$xF_{n_1-1} + yF_{n_2-1} = F_{n_3-1}$$
$$xF_{n_1} + yF_{n_2} = F_{n_3}$$

has a real solution. Therefore the equation

$$xD_1 + yD_2 = D_3$$

has a real solution, and D is singular. Therefore, the four points are not deterministic. □

Proposition 2. *If all four points lie on the same diagonal of slope 1 ($m_2 - m_1 = n_2 - n_1$, $m_3 - m_1 = n_3 - n_1$, and $m_4 - m_1 = n_4 - n_1$) then the four points are not deterministic.*

Proof. We will first state the following easily-proven identity for any $G \in \mathcal{F}^2(\mathbb{R})$ and any $(x, y) \in \mathbb{Z}^2$:

$$-G(x, y) + 2(G(x + 1, y + 1) + G(x + 2, y + 2)) = G(x + 3, y + 3) \tag{5}$$

It follows that the four points $(m_1, n_1), (m_2, n_2), (m_3, n_3)$, and (m_4, n_4) are not deterministic becuase $G(m_4, n_4)$ is determined by $G(m_1, n_1), G(m_2, n_2)$, and $G(m_3, n_3)$. □

Proposition 3. *If one pair of points lies on a vertical line and the other pair lies on a horizontal line ($m_1 = m_2$ and $n_3 = n_4$) then the four points are not deterministic.*

Proof. In this case, $D_1 + D_2 + D_3 = D_4$, so D is singular and the points are not deterministic. □

Proposition 4. *If one pair of points (X_1 and X_2) lies on a vertical line ($m_1 = m_2$), neither of the other two points (X_3 or X_4) lie on this line ($m_3 \neq m_1$ and $m_4 \neq m_1$), and the other two points (X_3 and X_4) do not lie the same horizontal line ($n_3 \neq n_4$) then the four points are deterministic.*

Proof. Since two entries of a generalized Fibonacci sequence uniquely determine that sequence, we know that F is determined along the "vertical line" $x = m_1$. The points (m_1, n_3) and (m_1, n_4) lie on this line. Therefore, by the same reasoning, we see that F is determined along the two "horizontal lines" $y = n_3$ and $y = n_4$. We now know that F is determined on at least two points of any vertical line in $\mathbb{Z}^2$. Therefore, F is determined on all $\mathbb{Z}^2$. □

Proposition 5. *If one pair of points (X_1 and X_2) lies on a horizontal line ($n_1 = n_2$), neither of the other two points (X_3 or X_4) lie on this line ($n_3 \neq n_1$ and $n_4 \neq n_1$), and the other two points (X_3 and X_4) do not lie the same vertical line ($m_3 \neq m_4$) then the four points are deterministic.*

The proof of this proposition is virtually identical to that of Proposition 4.

4. Concluding Remarks

Note that we can assume without loss of generality that one of the points is the origin. Using a Maple program written by Ira Gessel, we see that there are exactly three 4-tuples that include the origin and whose coordinates range from 0 to 4 that are not deterministic and that do not fall into one of the above categories:

$$\begin{array}{llll} (0,0) & (1,2) & (2,1) & (3,3) \\ (0,0) & (1,2) & (2,3) & (4,4) \\ (0,0) & (2,1) & (3,2) & (4,4). \end{array}$$

Each of these sets of points is contained within two diagonals. Using Gessel's program and searching through higher coordinate ranges, we see that many (perhaps most) non-deterministic sets of four points are contained within two diagonals. I have not, however, been able to prove a general theorem regarding whether or not sets of points contained in two diagonals are deterministic. Here are some non-deterministic sets of points that fall into categories which we have not discussed:

$$\begin{array}{llll} (0,0) & (3,2) & (5,5) & (6,3) \\ (0,0) & (3,1) & (4,4) & (6,3) \\ (0,0) & (2,1) & (3,4) & (6,3) \\ (0,0) & (2,1) & (3,6) & (5,3) \\ (0,0) & (2,1) & (3,4) & (6,3) \\ (0,0) & (1,2) & (3,5) & (6,3) \\ (0,0) & (2,1) & (3,6) & (5,3) \\ (0,0) & (3,2) & (5,5) & (6,3) \\ (0,0) & (1,2) & (3,6) & (4,3) \\ (0,0) & (1,3) & (3,6) & (4,4) \\ (0,0) & (3,1) & (4,4) & (6,3). \end{array}$$

Though these data might point to a solution, it remains to be seen if there is a general geometric criterion with which to determine if four points are deterministic.

Department of Mathematics, Brandeis University, Waltham, MA 02454-9110
E-mail address: nated@brandeis.edu