CONTINUED FRACTIONS AND FIBONACCI NUMBERS

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Abstract. The following is a collection of interesting facts about continued fractions and their ties to Fibonacci numbers.

1. Introduction

In this paper we will study some general properties of continued fractions, mainly their formation and properties of their intermediate terms called convergents, and finish by finding continued fraction representations for $\frac{F_{n+1}}{F_n}$, $\frac{L_n}{F_n}$ and other similar continued fractions related to Fibonacci numbers.

Let us first look at the expression

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \ddots + \cfrac{1}{q_n}}}} \tag{1}$$

where $q_1, \ldots, q_n$ are positive integers, and q_0 is a nonnegative integer. The above expression is called a continued fraction and the terms $q_0, q_1, \ldots, q_n$ are called partial quotients. To simplify the notation you can represent (1) as $[q_0, q_1, \ldots, q_n]$.

2. Euclidean Algorithm

Before we go any further, let us look at a process of finding the greatest common divisor for numbers a and b.

First we divide a by b with a remainder, and let the quotient be q_0 and the remainder r_1. Then it is not hard to show that

$$a = bq_0 + r_1 \text{ and } 0 \leq r_1 < b. \tag{2}$$

Note that when a is less than b, $q_0 = 0$. Continuing, if we divide b by r_1 and call the quotient q_1 and the remainder r_2, we will get that $b = r_1q_1 + r_2$ and that $0 \leq r_2 < r_1$. Since $r_1 < b$, q_1 must not equal to 0. Next, by dividing r_1 by r_2 we find some $q_2 \neq 0$ and r_3, such that $r_1 = q_2r_2 + r_3$ and $0 \leq r_3 < r_2$. Sooner or later this process will have to terminate, since all positive integers $r_1, r_2, r_3, \ldots$ are different and each is less than b. Meaning that their number does not exceed b, and that the process must terminate no later than the bth step and always when the remainder becomes 0. This process is known as the Euclidean Algorithm.

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3. Continued Fractions and the Euclidean Algorithm

Now let us look at the way we can turn an ordinary fraction $\frac{a}{b}$ into the continued fraction shown above. Applying the Euclidean algorithm we get the following equalities:

$$\begin{aligned} a &= bq_0 + r_1 \\ b &= r_1 q_1 + r_2 \\ r_1 &= r_2 q_2 + r_3 \\ &\cdots\cdots\cdots \\ r_{n-2} &= r_{n-1} q_{n-1} + r_n \\ r_{n-1} &= r_n q_n. \end{aligned} \tag{3}$$

¿From the first equality in the above system of equations we get that

$$\frac{a}{b} = q_0 + \frac{r_1}{b} = q_0 + \frac{1}{\frac{b}{r_1}}.$$

¿From the second equality we get

$$\frac{b}{r_1} = q_1 + \frac{r_2}{r_1} = q_1 + \frac{1}{\frac{r_1}{r_2}},$$

so that

$$\frac{a}{b} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\frac{r_1}{r_2}}}.$$

And finally from the third equality we get

$$\frac{r_1}{r_2} = q_2 + \frac{r_3}{r_2} = q_2 + \frac{1}{\frac{r_1}{r_2}},$$

and because of that

$$\frac{a}{b} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\frac{r_2}{r_3}}}}.$$

Continuing this process we will get to the final equality

$$\frac{a}{b} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \ddots + \cfrac{1}{q_n}}}. \tag{4}$$

Notice that $q_n > 1$. (If $q_n = 1$ then the whole algorithm would have ended a step earlier.) Also we can rewrite q_n in (4) as $(q_n - 1) + \frac{1}{1}$.

4. Uniqueness of Continued Fraction Representation

¿From the previous section we saw that any rational fraction $\frac{a}{b}$ can be expanded into a continued fraction $[q_0, q_1, \dots, q_n]$. Now let's show that such expansion is unique. For the purposes of proving this let's take two equal continued fractions ω and ω', and let $q_0, q_1, \dots, q_n$ and $q'_0, q'_1, \dots, q'_n$ be their partial quotients.

Theorem 1. *If ω and ω' are two equivalent continued fractions then $q_0 = q'_0, q_1 = q'_1, q_2 = q'_2$ and so on.*

Proof. It is not hard to see that if q_0 is the greatest integer in ω and q'_0 is the greatest integer in ω' then $q_0 = q'_0$. Having this in mind we can rewrite our continued fraction as

$$\omega = q_0 + \frac{1}{\omega_1}, \quad \omega' = q'_0 + \frac{1}{\omega'_1} \tag{5}$$

where ω and ω' are again continued fractions. ¿From the fact that $\omega = \omega'$ and $q_0 = q'_0$ it is not hard to derive the fact that ω_1 must equal to ω'_1. Similarly, by performing induction on the indices of ω, we will get that q_1 must equal to q'_1, q_2 must equal to q'_2 and so on. With this our proof is complete. □

5. Convergents

Let

$$\omega = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \ddots + \cfrac{1}{q_n}}} \tag{6}$$

And let's look at the following numbers:

$$q_0, \; q_0 + \frac{1}{q_1}, \; q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2}}, \; \dots .$$

The preceding numbers can be represented as ordinary fractions in the following way

$$\begin{aligned} \frac{P_0}{Q_0} &= \frac{q_0}{1}, \\ \frac{P_1}{Q_1} &= q_0 + \frac{1}{q_1}, \\ \frac{P_2}{Q_2} &= q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2}}, \\ &\cdots\cdots\cdots \\ \frac{P_n}{Q_n} &= \omega, \end{aligned}$$

where $\dfrac{P_i}{Q_i}$ are called convergents. Note that going from $\dfrac{P_k}{Q_k}$ to $\dfrac{P_{k+1}}{Q_{k+1}}$ can be done by simply replacing q_k with $q_k + \dfrac{1}{q_{k+1}}$. Similarly going from $\dfrac{P_k}{Q_k}$ to the whole continued fraction is accomplished by replacing q_k with $q_k + \dfrac{1}{\omega_{k+1}}$; that is, replacing q_k with a continued fraction ω_k.

An important role in the study of continued fractions is played by the following theorem.

Theorem 2. *The numbers P_k and Q_k have the following three properties:*

$$P_{k+1} = P_k q_{k+1} + P_{k-1}, \tag{7}$$

$$Q_{k+1} = Q_k q_{k+1} + Q_{k-1}, \tag{8}$$

$$P_{k+1}Q_k - P_k Q_{k+1} = (-1)^k. \tag{9}$$

Proof. Let us prove all of the above by induction on k. First let's see what happens when $k = 1$:

$$\frac{P_1}{Q_1} = q_0 + \frac{1}{q_1} = \frac{q_0 q_1 + 1}{q_1}.$$

Since $q_0 q_1 + 1$ and q_1 are relatively prime, the fraction $\dfrac{q_0 q_1 + 1}{q_1}$ can not be reduced. The left side of the equation, $\dfrac{P_1}{Q_1}$, is not reducible by construction. This means, since the two fractions are equal and irreducible, their numerators and denominators are equal. Specifically, $P_1 = q_0 q_1 + 1$ and $Q_1 = q_1$. Further more,

$$\frac{P_2}{Q_2} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2}} = \frac{q_0(q_1 q_2 + 1) + q_2}{q_1 q_2 + 1}. \tag{10}$$

The greatest common divisor of the numbers $q_0(q_1q_2 + 1) + q_2$ and $q_1q_2 + 1$ is equal to $(q_2, q_1q_2 + 1)$ which is in turn equal to $(q_2, 1)$, or simply 1. This means that the right side of equation (10) is irreducible, and therefore

$$\begin{gathered} P_2 = q_0(q_1q_2 + 1) + q_2 = (q_0q_1 + 1)q_2 + q_0 = P_1 q_2 + P_0, \\ Q_2 = q_1 q_2 + 1 = Q_1 q_2 + Q_0, \end{gathered}$$

and

$$\begin{aligned} P_2Q_1 - P_1Q_2 &= ((q_0q_1 + 1)q_2 + q_0)(q_1) - (q_0q_1 + 1)(q_1q_2 + 1) \\ &= q_0q_1^2q_2 + q_2q_1 + q_0q_1 - q_0q_1^2q_2 - q_1q_2 - q_0q_1 - 1 = (-1)^1 \end{aligned}$$

With this we have proved the base case of our induction proof.

Now lets suppose that equations (7), (8) and (9) are true, and examine the equality

$$\frac{P_{k+1}}{Q_{k+1}} = \frac{P_k q_{k+1} + P_{k-1}}{Q_k q_k + 1 + Q_{k-1}}. \tag{11}$$

By the previously mentioned fact the transition from $\dfrac{P_{k+1}}{Q_{k+1}}$ to $\dfrac{P_{k+2}}{Q_{k+2}}$ is accomplished by replacing q_{k+1} in $\dfrac{P_{k+1}}{Q_{k+1}}$ by $q_{k+1}+\dfrac{1}{q_k+2}$. Because P_k, Q_k, P_{k-1} and Q_{k-1} do not contain q_{k+1} we have

$$\frac{P_{k+2}}{Q_{k+2}}=\frac{P_k(q_{k+1}+\dfrac{1}{q_{k+2}})+P_{k-1}}{Q_k(q_{k+1}+\dfrac{1}{q_{k+2}})+Q_{k-1}},$$

or using the inductive properties of equation (7) and (8),

$$\frac{P_{k+2}}{Q_{k+2}}=\frac{P_{k+1}q_{k+2}+P_k}{Q_{k+1}q_{k+2}+Q_k}. \tag{12}$$

Now we need to prove the irreducibility of the right side of equation (12). For this it is enough to show that the numerator and the denominator of this fraction are relatively prime.

Lets suppose that $P_{k+1}q_{k+2}+P_k$ and $Q_{k+1}q_{k+2}+Q_k$ have some common divisor $d>1$. Then

$$(P_{k+1}q_{k+2}+P_k)Q_{k+1}-(Q_{k+1}q_{k+2}+Q_k)P_{k+1}$$

must also be divisible by d. But by our inductive hypothesis (9) this expression equals to $(-1)^{k+1}$ and cannot be divisible by d.

So, the right side of (12) is irreducible, and therefore equation (12) is a composed of two irreducible fractions. Therefore,

$$P_{k+2}=P_{k+1}q_{k+2}+P_k$$

and

$$Q_{k+2}=Q_{k+1}q_{k+2}+Q_k.$$

To complete our proof by induction we need only to show that

$$P_{k+2}Q_{k+1}-P_{k+1}Q_{k+2}=(-1)^{k+1}.$$

This is done very simply because

$$P_{k+2}Q_{k+1}-P_{k+1}Q_{k+2}=P_{k+1}q_{k+2}Q_{k+1}+P_kQ_{k+1}-P_{k+1}q_{k+2}Q_{k+1}-P_{k+1}Q_k,$$

which simplifies to $(-1)^{k+1}$ with the use of the two expressions we have just proven. With this the proof is complete. □

6. Some Continued Fractions and Fibonacci Numbers

Theorem 3. *If a continued fraction has n partial quotients, each of which equals to one, then the fraction is* $\dfrac{F_{n+1}}{F_n}$.

Proof. Let's call a continued fraction with n partial quotients α_n. Then,

$$\alpha_1,\alpha_2,\dots,\alpha_n$$

are the convergents of the above defined continued fraction.

Let $\alpha_k = \dfrac{P_k}{Q_k}$. Since

$$\alpha_1 = 1 = \frac{1}{1}$$

and

$$\alpha_2 = 1 + \frac{1}{1},$$

then $P_1 = 1$ and $P_2 = 2$. Furthermore, using equation (7) we have $P_{n+1} = P_n q_{n+1} + P_{n-1} = P_n + P_{n-1}$. Using the the fact that this is a generalized Fibonacci sequence we conclude that $P_n = F_{n+1}$.

Similarly $Q_1 = 1$, $Q_2 = 1$ and $Q_{n+1} = Q_n q_{n+1} + Q_{n-1} = Q_n + Q_{n-1}$, so that $Q_n = F_n$. This means that,

$$\alpha_n = \frac{F_{n+1}}{F_n}.$$

□

Theorem 4. *If* ω *contains* $q_0, q_1, q_2, \dots, q_n$, *where*

$$q_0 = q_1 = q_2 = \cdots = q_{i-1} = q_{i+1} = \cdots = q_n = 1,$$
$$q_i = 2 \ (i \neq 0),$$

then

$$\omega = \frac{F_{i+1}F_{n-i+3} + F_i F_{n-i+1}}{F_i F_{n-i+3} + F_{i-1}F_{n-i+1}}.$$

Proof. Let's consider the function $m_i(x)$, where i are nonnegative integers, defined by the following,

$$\begin{aligned}
m_0(x) &= x \\
m_1(x) &= 1 + \frac{1}{x} = \frac{x+1}{x} \\
m_2(x) &= 1 + \frac{1}{1 + \frac{1}{x}} = \frac{2x+1}{x+1} \\
&\vdots \\
m_i(x) &= 1 + \frac{1}{m_{i-1}(x)}
\end{aligned}$$

We will prove that

$$m_i(x) = 1 + \frac{1}{m_{i-1}} = \frac{F_{i+1}x + F_i}{F_i x + F_{i-1}} \tag{13}$$

Let's prove equation (13) by induction on i. First let's show that the base case is true. When $i = 1$ we have $m_1(x) = \dfrac{x+1}{x} = \dfrac{F_2 x + F_1}{F_1 x + F_0}$. Now let's suppose that this equation is true for

$i = k \geq 1$. Then if we prove that it is true for $k+1$ we have proven our hypothesis. We have

$$m_{k+1}(x) = 1 + \frac{1}{m_k(x)} = 1 + \frac{1}{\dfrac{F_{k+1}x + F_k}{F_kx + F_{k-1}}} = \frac{F_{k+1}x + F_k + F_kx + F_{k-1}}{F_{k+1}x + F_k} = \frac{F_{k+2}x + F_{k+1}}{F_{k+1}x + F_k}.$$

For convenience let's rewrite the function $m_i(x)$ as a continued fraction $[1^i, x]$.

Returning to the expression $[1^i2, 1^{n-i}]$ where n is the last term of the continued fraction, let $x = [2, 1^{n-i}]$ to give

$$\left[1^i, 2, 1^{n-i}\right] = \left[1^i, x\right].$$

By Theorem 3, $x = 2 + \dfrac{F_{n-i}}{F_{n+1-i}}$. Plugging this back into equation (13) we get

$$\begin{aligned}\left[1^i, 2, 1\right] = \frac{F_{i+1}\left(2 + \dfrac{F_{n-i}}{F_{n+1-i}}\right) + F_i}{F_i\left(2 + \dfrac{F_{n-i}}{F_{n+1-i}}\right) + F_{i-1}} &= \frac{F_{i+1}[2F_{n+1-i} + F_{n-i}] + F_{n+1-i}F_i}{F_i\left(2F_{n+1-i} + F_{n-i}\right) + F_{n+1-i}F_i - 1} \\ &= \frac{F_{i+1}F_{n+3-i} + F_iF_{n+1-i}}{F_iF_{n+3-i} + F_{i-1}F_{n+1-i}}.\end{aligned}$$

Our theorem has been proven. ☐

Theorem 5. *For any integer $n \geq 1$ and integer $m \geq 0$,*

$$\frac{F_{n+3}}{F_n} = \begin{cases} [4^{m-1}, 3] & \text{if } n = 3m+1, \\ [4^{m-1}, 5] & \text{if } n = 3m+2, \\ [4^{m-1}, 4] & \text{if } n = 3m+3. \end{cases}$$

Proof. To prove the above theorem let us first examine the expression $p_i(x) = [4^i, x]$ defined for all nonnegative integers n. We know that

$$\begin{aligned} p_0(x) &= x \\ p_1(x) &= 4 + \frac{1}{x} = \frac{4x+1}{x} \\ p_2(x) &= 4 + \frac{1}{4 + \dfrac{1}{x}} = \frac{17x+4}{4x+1} \\ &\ \vdots \\ p_i(x) &= 4 + \frac{1}{p_{i-1}(x)} \end{aligned}$$

Further examination of the above expression reveals that

$$p_i(x) = \left[4^i, x\right] = \frac{U_{i+1}x + U_i}{U_ix + U_{i-1}}, \tag{14}$$

where $U_0 = 0$, $U_1 = 1$ and $U_i = 4U_{i-1} + U_{i-2}$.

Let's prove this by induction on i. The base case is $i = 1$. We have $p_1(x) = \dfrac{4x+1}{x} = \dfrac{U_2x+U_1}{U_1x+U_0}$, since $U_2 = 4$, $U_1 = 1$ and $U_0 = 0$. Now let's suppose that for some nonnegative integer k

$$p_k(x) = 4 + \frac{1}{p_{k-1}(x)} = \frac{U_{k+1}x + U_k}{U_kx + U_{k-1}}.$$

We only need to show that the above holds for $\left[4^{k+1}, x\right]$ to prove our hypothesis. We have

$$4 + \frac{1}{p_k(x)} = 4 + \frac{U_kx + U_{k-1}}{U_{k+1}x + U_k} = \frac{4U_{k+1}x + 4U_k + U_kx + U_{k-1}}{U_{k+1}x + U_k} = \frac{U_{k+2}x + U_{k+1}}{U_{k+1}x + U_k}.$$

We can further simplify the expression using by the fact that $U_i = \frac{1}{2}F_{3i}$. To prove this let's first prove that $F_t = 4F_{t-3} + F_{t-6}$, where t is any integer. Both sides of this equation are generalized Fibonacci sequences. Recall that a sequence g_n is called a generalized Fibonacci sequence if it satisfies the Fibonacci recurrence $g_n = g_{n-1} + g_{n-2}$. We also know that if two generalized Fibonacci sequences are equal for two values then they are equal for all values. Plugging in $t = 6$ and $t = 7$ we see that the above expression is true. So $F_{3i} = 4F_{3i-3} + F_{3i-6}$ is also true. We see that both U_i and $\frac{1}{2}F_{3i}$ satisfy the same recurrence and they satisfy the same initial conditions since $U_0 = \frac{1}{2}F_0 = 0$ and $U_1 = \frac{1}{2}F_3 = 1$. Therefore $U_i = \frac{1}{2}F_{3i}$.

Now we are ready to prove our theorem. First let's prove the case $n = 3m + 1$. Plugging $x = 3$ into equation (14) and using the fact that $F_{q+p} = F_{p-1}F_q + F_pF_{q+1}$ we get

$$\begin{aligned}[4^i, 3] &= \frac{3U_{i+1} + U_i}{3U_i + U_{i-1}} = \frac{\frac{1}{2}\left(3F_{3i+3} + F_{3i}\right)}{\frac{1}{2}\left(3F_{3i} + F_{3i-3}\right)} = \frac{\frac{1}{2}\left(6F_{3i+1} + 4F_{3i}\right)}{\frac{1}{2}\left(4F_{3i-1} + 2F_{3i-2}\right)} \\ &= \frac{3F_{3i+1} + 2F_{3i}}{2F_{3i} + F_{3i-2}} = \frac{F_4F_{3i+1} + F_3F_{3i}}{F_3F_{3i-1} + F_2F_{3i-2}} = \frac{F_{3i+4}}{F_{3i+1}}.\end{aligned}$$

where $i = m - 2$.

The other two cases are similar. For $n = 3m+2$ we get $[4^i, 5] = \dfrac{F_{3i+5}}{F_{3i+2}}$ and for $n = 3m+3$ we get $[4^i, 4] = \dfrac{F_{3i+6}}{F_{3i+3}}$. □

Theorem 6. *For any integer $n > 3$ and integer $m \geq 1$,*

$$\frac{L_n}{F_n} = \begin{cases} [2, 4^{m-1}, 3] & \text{if } n = 3m+1, \\ [2, 4^{m-1}, 5] & \text{if } n = 3m+2, \\ [2, 4^{m-1}, 4] & \text{if } n = 3m+3. \end{cases}$$

Proof. We already know the expressions for $[4^i, 3]$, $[4^i, 5]$ and $[4^i, 4]$ Then, since what we have is $[2, 4^i, 3]$, using $L_q = F_{q-1} + F_{q+1}$, our final expression becomes

$$\begin{aligned}\left[2, 4^i, 3\right] &= 2 + \frac{F_{3i+1}}{F_{3i+4}} = \frac{2F_{3i+4} + F_{3i+1}}{F_{3i+4}} = \frac{2\left(F_{3i+3} + F_{3i+2}\right) + F_{3i+1}}{F_{3i+4}} \\ &= \frac{2\left(F_{3i+3} + F_{3i+2}\right) + F_{3i+1}}{F_{3i+4}} = \frac{2F_{3i+3} + F_{3i+2} + F_{3i+3}}{F_{3i+4}} \\ &= \frac{2F_{3i+3} + F_{3i+4}}{F_{3i+4}} = \frac{F_{3i+5} + F_{3i+3}}{F_{3i+4}} = \frac{L_{3i+4}}{F_{3i+4}},\end{aligned}$$

where $i = m - 2$.

Again the other two cases are similar. For $n = 3m + 2$ we get $[2, 4^i, 5] = \dfrac{L_{3i+5}}{F_{3i+5}}$ and for $n = 3m + 3$ we get $[2, 4^i, 4] = \dfrac{L_{3i+6}}{F_{3i+6}}$. □