GENERATING FUNCTIONS OF F_{an+b} AND L_{an+b}

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1. INTRODUCTION

In this paper, we will look at the sequences F_{an+b} and L_{an+b} in order to determine any interesting consequences of their respective generating functions.

Before we begin, let us state several useful formulas, derived previously:

$$\begin{aligned} F_{n+i} &= F_{n+1}F_i + F_nF_{i-1} && (1)\\ L_{n+i} &= F_iL_n + F_iL_{n+1} && (2)\\ \sum_{n=0}^{\infty} F_nx^n &= \frac{x}{1-x-x^2}, \sum_{n=0}^{\infty} F_{n+1}x^n = \frac{1}{1-x-x^2} && (3)\\ \sum_{n=0}^{\infty} L_nx^n &= \frac{2-x}{1-x-x^2}, \sum_{n=0}^{\infty} L_{n+1}x^n = \frac{1+2x}{1-x-x^2} && (4)\\ F_n &= \frac{\alpha^n-\beta^n}{\sqrt{5}} && (5)\\ L_n &= \alpha^n+\beta^n && (6) \end{aligned}$$

In last two formulae, known as the Binet Formulae, $\alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2$.

Let us analyze F_{an+b} first. A few elementary results follow:

2. SPECIAL CASES

Theorem 1. *If $a = 1$, the generating function of (F_{an+b}) is* $\dfrac{F_b + F_{b-1}x}{1-x-x^2}$.

Proof. Since $a = 1$, (F_{an+b}) simplifies to (F_{n+b}). We can use (1) in order to find the generating function of this sequence:

$$\sum_{n=0}^{\infty} F_{n+b}x^n = \sum_{n=0}^{\infty}(F_{n+1}F_b + F_nF_{b-1})x^n = F_b\sum_{n=0}^{\infty} F_{n+1}x^n + F_{b-1}\sum_{n=0}^{\infty} F_nx^n.$$

Applying (3), this simplifies to

$$\frac{F_b}{1-x-x^2} + \frac{F_{b-1}x}{1-x-x^2} = \frac{F_b+F_{b-1}x}{1-x-x^2}.$$

□

Analogously, we may use this approach with (L_{an+b}):

Theorem 2. *If $a = 1$, the generating function of (L_{an+b}) is* $\dfrac{L_b + L_{b-1}x}{1-x-x^2}$.

Proof. Since $a = 1$, (L_{an+b}) simplifies to (L_{n+b}). We can use (2) in order to find the generating function of this sequence:

$$\sum_{n=0}^{\infty} L_{n+b}x^n = \sum_{n=0}^{\infty}(F_{b-1}L_n + F_bL_{n+1})x^n = F_{b-1}\sum_{n=0}^{\infty} L_nx^n + F_b\sum_{n=0}^{\infty} L_{n+1}x^n).$$

Applying (4) where appropriate, this simplifies to

$$\frac{F_{b-1}(2-x)}{1-x-x^2} + \frac{F_b(1+2x)}{1-x-x^2}) = \frac{2F_{b-1} + F_b + (2F_b - F_{b-1})x}{1-x-x^2}.$$

Using the identity $L_n = F_{n+1} + F_{n-1}$, we may further simplify this to

$$\frac{L_b + L_{b-1}x}{1-x-x^2}.$$

□

3. General Case

With this knowledge in mind, we may now look for a general formula for the generating function of F_{an+b} and L_{an+b}. We have thus far chiefly found generating functions of sequences by finding recurrences that those sequences satisfy. However, in this case, the Binet Formulas will prove to be a useful tool.

Theorem 3.

$$\sum_{n=0}^{\infty} F_{an+b}x^n = \frac{F_b + F_{b-a}x}{1 - L_ax + (-1)^ax^2}.$$

Proof. In accordance with (5), $F_{an+b} = \frac{\alpha^{an+b} - \beta^{an+b}}{\sqrt{5}}$. Hence,

$$\sum_{n=0}^{\infty} F_{an+b}x^n = \sum_{n=0}^{\infty} \frac{\alpha^{an+b} - \beta^{an+b}}{\sqrt{5}}x^n = \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty} \alpha^{an+b}x^n - \sum_{n=0}^{\infty} \beta^{an+b}x^n\right].$$

At this point, we can factor out out α^b and β^b from the two sum expressions, respectively and simplify to obtain

$$\frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty} \alpha^b(\alpha^ax)^n - \sum_{n=0}^{\infty} \beta^b(\beta^ax)^n\right] = \frac{1}{\sqrt{5}}\left[\alpha^b\sum_{n=0}^{\infty} (\alpha^ax)^n - \beta^b\sum_{n=0}^{\infty} (\beta^ax)^n\right].$$

Since these are two geometric series, we may further simplify the expression to

$$\frac{1}{\sqrt{5}}\left[\frac{\alpha^b}{1-\alpha^ax} - \frac{\beta^b}{1-\beta^ax}\right] = \frac{\alpha^b - \alpha^b\beta^ax - \beta^b + \beta^b\alpha^ax}{\sqrt{5}(1 - \alpha^ax - \beta^ax + \alpha^a\beta^ax^2)}.$$

In accordance with the properties of α and β, we may further simplify this to

$$\frac{\alpha^b - \beta^b + (\alpha^{a-b} - \alpha^{b-a})x}{\sqrt{5}(1 - L_ax + (-1)^ax^2)} = \frac{F_b + F_{b-a}x}{1 - L_ax + (-1)^ax^2}.$$

□

Theorem 4.

$$\sum_{n=0}^{\infty} L_{an+b}x^n = \frac{L_b - L_{b-a}x}{1 - L_a x + (-1)^a x^2}.$$

Proof. In accordance with (6), $L_{an+b} = \alpha^{an+b} + \beta^{an+b}$. Hence,

$$\sum_{n=0}^{\infty} L_{an+b}x^n = \sum_{n=0}^{\infty} (\alpha^{an+b} + \beta^{an+b})x^n = \sum_{n=0}^{\infty} \alpha^{an+b}x^n + \sum_{n=0}^{\infty} \beta^{an+b}x^n.$$

Factoring out out α^b and β^b from the two sum expressions, respectively and simplifying, we obtain

$$\sum_{n=0}^{\infty} \alpha^b(\alpha^a x)^n + \sum_{n=0}^{\infty} \beta^b(\beta^a x)^n = \alpha^b \sum_{n=0}^{\infty} (\alpha^a x)^n + \beta^b \sum_{n=0}^{\infty} (\beta^a x)^n.$$

Since these are geometric series, they can be simplified to

$$\frac{\alpha^b}{1 - \alpha^a x} + \frac{\beta^b}{1 - \beta^a x}.$$

We may now repeat the simplification process employed in the previous theorem in order to arrive at our final result of

$$\frac{L_b - L_{b-a}x}{1 - L_a x + (-1)^a x^2}.$$

□

The validity of these formulae can easily be checked by means of comparison to our special cases from theorems 1 and 2. For example, taking the formula from theorem 3 and letting $a = 1$, we obtain

$$\frac{F_b + F_{b-1}x}{1 - L_1 x + (-1)x^2} = \frac{F_b + F_{b-1}x}{1 - x - x^2}.$$

4. Further Generating Functions

We shall now investigate another Fibonacci-related generating function, for the sake of completeness. Before we do so, let us state the following well-known theorem:

Theorem 5. *Let* $(a_n)_0^\infty$ *be a sequence satisfying the recurrence* $a_{n+k} + c_1 a_{n+k-1} + \cdots + c_k a_n = 0$ *for* $n \geq 0$*. Then*

$$\sum_{n=0}^{\infty} a_n x^n = \frac{p(x)}{1 + c_1 x + c_2 x^2 + \cdots + c_k x^k}$$

where $p(x)$ *is a polynomial of degree* $\leq k - 1$.

Now, we may proceed to analyze the generating function of the sequence F_n^2.

Theorem 6.

$$\sum_{n=0}^{\infty} F_n^2 x^n = \frac{x - x^2}{1 - 2x - 2x^2 + x^3}.$$

Proof. We need to find some linear relation among $F_n^2, F_{n+1}^2, \ldots$

Note the following set of identities:

$$\begin{aligned}
F_n^2 &= F_n^2 \\
F_{n+1}^2 &= F_{n+1}^2 \\
F_{n+2}^2 &= (F_n + F_{n+1})^2 = F_n^2 + 2F_nF_{n+1} + F_{n+1}^2 \\
F_{n+3}^2 &= (F_{n+1}F_3 + F_nF_2)^2 = (2F_{n+1} + F_n)^2 = 4F_{n+1}^2 + 4F_nF_{n+1} + F_n^2
\end{aligned}$$

These identities imply that

$$F_{n+3}^2 - 2F_{n+2}^2 = -F_n^2 + 2F_{n+1}^2.$$

So

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0.$$

Hence, by theorem 3, we may conclude that

$$\sum_{n=0}^{\infty} F_n^2 x^n = \frac{p(x)}{x^3 - 2x^2 - 2x + 1}$$

where $p(x)$ is a polynomial of degree ≤ 2. We may now show that $p(x) = -x^2 + x$. Suppose that $p(x) = p_0 + p_1x + p_2x^2$. Then

$$(1 - 2x - 2x^2 + x^3) \sum_{n=0}^{\infty} F_n^2 x^n = p_0 + p_1x + p_2x^2.$$

Equating coefficients of x^m for $m \leq 2$, we obtain $p(x) = x - x^2$ and thus arrive at our result,

$$\sum_{n=0}^{\infty} F_n^2 x^n = \frac{x - x^2}{1 - 2x - 2x^2 + x^3}.$$

$\square$