A GENERALIZATION OF ZECKENDORF'S THEOREM

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ABSTRACT. We define generalized Fibonacci sequences of recurrence k and discuss an analogue of Zeckendorf's Theorem.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

Recall that a *generalized Fibonacci sequence* is any sequence (g_n) that satisfies the recurrence $g_{n-1} + g_{n-2} = g_n$. Zeckendorf's theorem states that every non-negative integer has a unique representation as the sum of non-consecutive Fibonacci numbers. Since $F_1 = F_2$ we additionally require that the terms in the Zeckendorf representation of an integer n be F_n where $n \geq 2$. A *generalized Fibonacci sequence of order* k satisfies the recurrence $g_n = g_{n-1} + g_{n-2} + \cdots + g_{n-k}$ for all n.

We will now define an analogue of a property of generalized Fibonacci sequences for such sequences of an arbitrary order k.

Definition 1. *Let (g_n) be a generalized Fibonacci sequence of order k. A k-Zeckendorf representation of a non-negative integer x with respect to (g_n) is a sum $x = \sum_{i=1} \epsilon_i g_i$ where each ϵ_i is 0 or 1 and no k consecutive ϵ_i are equal to 1.*

2. ZECKENDORF'S THEOREM FOR GENERALIZED FIBONACCI SEQUENCES OF ORDER k

We will begin by proving two lemmas.

Lemma 1. *In general, $\sum_{r=0}^{k} 2^r = 2^{k+1} - 1$.*

Proof. We proceed using weak induction. The formula holds for $k = 0$. Let us assume that this condition holds for $r = k - 1$. We now show that the formula works for $r = k$:

$$\begin{aligned}(2^0 + 2^1 + \cdots + 2^{k-1}) + 2^k &= (2^k - 1) + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1.\end{aligned}$$

□

Lemma 2. *Let (g_n) be a generalized Fibonacci seuence of order k. Then for all integers i, $g_{i+1} < 2g_i$.*

Proof. By the Fibonacci recurrence,

$$2g_i = g_i + g_{i-1} + g_{i-2} + \cdots + g_{i-(k-1)} + g_{i-k}.$$

This implies that

$$2g_i = g_{i+1} + g_{i-k}.$$

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Obviously, $g_{i+1} < g_{i+1} + g_{i-k}$. This proves the lemma. □

We will now prove an analogue of Zeckendorf's theorem for generalized Fibonacci sequences (g_n) of order k.

Theorem 1. *Consider the set of generalized Fibonacci sequences (g_n) of order k such that $g_0 < g_1 < \cdots < g_{k-2} < g_{k-1}$. Within this set there exists a unique generalized Fibonacci sequence (g_n) of order k such that every $x \in \mathbf{Z}_{\geq 0}$ has a unique k-Zeckendorf representation.*

Before we present a formal proof of this theorem, let us consider two examples. First, we recall the truncated Fibonacci sequence of Zeckendorf's theorem, $1, 2, 3, 5, 8, \ldots$ which has the above property. Some experimentation reveals that the sequence of order three $1, 2, 4, 7, 13, \ldots$ which also has the above property. This is easily proven via induction. Observe that these each of these sequences begins $2^0, 2^1, \ldots, 2^{k-1}$ where k is the order of the sequence. We will now show the generalized Fibonacci sequence (g_n) beginning $2^0, 2^1, \ldots, 2^{k-1}$ has this property.

Proof. First we will demonstrate by induction the existence of a unique k-Zeckendorf representation for each non-negative integer x. Clearly, there exist representations for all integers m such that $0 \leq m \leq 2^k - 1$. The representations of the integers m such that $0 \leq m \leq 2^k - 2$ correspond to the binary representations of these numbers. By Lemma 1 $2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1$, so the representation of $2^k - 1$ is g_k. So we have a surplus of base cases. Assume that for all integers j such that $0 \leq j < x$ there exists a unique k-Zeckendorf representation. We now demonstrate that there exists a unique k-Zeckendorf represetation of x. Let i be the greatest integer such that $g_i \leq x$. We know that such an i exists because the terms of (g_n) are increasing. Let $x - g_i = g_{a_1} + g_{a_2} + \cdots + g_{a_r}$ be the unique representation of $x - g_i$. Then there exists a representation of $x = g_i + g_{a_1} + g_{a_2} + \cdots + g_{a_r}$. This is a k-Zeckendorf representation if $r < k - 1$. Now suppose $r \geq k - 1$. We can assume that $a_1 > a_2 > \cdots > a_r$. We now want to show that $i > a_1$. If $i < a_1$ then $a_1 > i$ is the greatest integer such that $g_{a_1} \leq x$. But this is a contradiction since i is the greatest integer such that $g_i \leq x$. Now suppose $i = a_1$. Then $x = g_i + g_i + g_{a_2} + \cdots + g_{a_r}$. Lemma 2 tells us that this sum is greater than g_i. But then $g_{i+1} > x$ which is a contradiction since i is the greatest integer such that $g_i \leq x$. So $g_i + g_{a_1} + g_{a_2} + \cdots + g_{a_r}$ is a k-Zeckendorf representation of x.

Now let us prove the uniqueness of this representation. Suppose there exists another k-Zeckendorf representation of x that does not contain g_i. Then the maximum value of this representation is

$$F(i) = \sum_{\substack{1 \leq j \leq i \\ k \nmid j}} g_{i-j}.$$

We now use induction on i to demonstrate $F(i) = g_i - 1 < x$ for all i. From Lemma 1, we know that this is true for the base cases $i \leq k - 1$. Now we assume $i \geq k$ and that $F(l) = g_l - 1$ for all $l < i$. In particular, $F_{i-k} = g_{i-k} - 1$. Then

$$\begin{aligned}
(F_i &= g_{i-1} + g_{i-2} + \cdots + g_{i-k+1} + F(i-k) \\
&= g_i - g_{i-k} + (g_{i-k} - 1 \\
&= g_i - 1 < x
\end{aligned}$$

So the k-Zeckendorf representation we saw earlier is unique.

We now demonstrate the uniqueness of (g_n). Suppose for some arbitrary order k there exists a second generalized Fibonacci sequence (h_n) with $h_0 < h_1 < \cdots < h_{k-1}$ such that every non-negative integer x has a unique k-Zeckendorf representation with respect to (h_n). Then there is at least one integer q in the interval [0, k-1] such that $g_q \neq h_q$. If $h_q < g_q$ then h_q has more than one k-Zeckendorf representation in (h_n). The two representations are h_q itself and the binary representation of h_q in $h_0, h_1, \ldots, h_{q-1}$. If $h_q > g_q$ then 2^q does not have a k-Zeckendorf representation in (h_n). The first q terms provide representations for all integers up to $2^q - 1$. Since $h_q > g_q = 2^q$, 2^q has no representation. So our conjecture that the sequence (h_n) existed was false. So the sequence (g_n) is unique. □

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