

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas numbers are defined by the same recurrence with the initial conditions $L_0 = 2$, $L_1 = 1$. Equivalently, $L_n = F_{n-1} + F_{n+1}$ for $n \geq 1$. Here is a table of the initial values of the Fibonacci and Lucas numbers:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364

Let n be a positive integer. A *Zeckendorf representation* of n is an expression of n as a sum of Fibonacci numbers in the form $n = F_{i_1} + F_{i_2} + \cdots + F_{i_k}$, where $i_1 - i_2 \geq 2$, $i_2 - i_3 \geq 2$, \dots , $i_{k-1} - i_k \geq 2$, and $i_k \geq 2$. For example, a Zeckendorf representation of 100 is $F_{11} + F_6 + F_4$.

1. Prove that every positive integer has a unique Zeckendorf representation.
2. Compute the Zeckendorf representations for F_n^2 for $n = 1, 2, \dots$. Try to figure out the pattern and prove that it holds in general. Similarly look at $F_n F_{n+1}$, L_n^2 , $L_n F_n$, F_n^3 , and so on. For which of these can you find a general formula for the Zeckendorf representation?
3. (a) Show that F_{mn}/F_m is always an integer for $m > 0$.
 (b) Compute the Zeckendorf representation of F_{mn}/F_m for different values of m and n . Can you find any patterns?
4. Compute the Zeckendorf representations for $(F_{8n+1} - 1)/3$, $(F_{8n+2} - 1)/3$, and $(F_{8n+3} - 2)/3$. More generally, choose integers m and i , and let $P(m)$ be the period of the Fibonacci sequence modulo m . What can you say about the patterns in the Zeckendorf representations of $(F_{P(m)n+i} - F_i)/m$?
5. Can you find patterns for the Zeckendorf representation of any other numbers defined by formulas involving the Fibonacci and Lucas numbers?
6. Many results for Fibonacci numbers have analogues for Lucas numbers. In particular, there is an analog of Zeckendorf's theorem for the Lucas numbers: Every positive integer has a unique representation as a sum of Lucas numbers in the form $n = L_{i_1} + L_{i_2} + \cdots + L_{i_k}$, where $i_1 - i_2 \geq 2$, $i_2 - i_3 \geq 2$, \dots , $i_{k-1} - i_k \geq 2$, and $i_k \geq 0$, with the additional condition that L_0 and L_2 may not both be used. (Some restriction is necessary since $L_0 + L_2 = L_1 + L_3$.) Can you find analogous patterns for Lucas number representations?
7. Can you find analogues of Zeckendorf's theorem for other generalized Fibonacci sequences?

8. (a) It is easy to determine the last digit of a number n when expressed in base b —this is simply the remainder when n is divided by b . Is there any reasonable way to compute the last “digit” in the Zeckendorf representation of n without computing all the others first?
- (b) For any positive integer n , we define $\Theta(n)$ as follows: Let $F_{i_1} + F_{i_2} + \cdots + F_{i_k}$ be the Zeckendorf representation of n . Then $\Theta(n)$ is $F_{i_1+1} + F_{i_2+1} + \cdots + F_{i_k+1}$. Can you say anything interesting about Θ ? In particular, since F_{i+1} is close to αF_i , you might compare $\Theta(n)$ with αn .
9. The number $\alpha = (1 + \sqrt{5})/2$, which satisfies $\alpha^2 = \alpha + 1$, is closely connected to the Fibonacci and Lucas numbers. In particular, there are explicit formulas for F_n and L_n in terms of α and $\beta = (1 - \sqrt{5})/2$: $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$. It follows that F_n is the nearest integer to $\alpha^n/\sqrt{5}$. Show that every real number has an “almost unique” representation in the form $a_k\alpha^k + a_{k-1}\alpha^{k-1} + \cdots + a_0\alpha^0 + a_{-1}\alpha^{-1} + a_{-2}\alpha^{-2} + \cdots$, where each a_i is 0 or 1 and at least one of a_i and a_{i+1} must be 0, and clarify what “almost unique” means. (For example, α has two different representations, α itself and $1 + \alpha^{-2} + \alpha^{-4} + \cdots$. This nonuniqueness is analogous to the fact that 1 has two decimal representations, 1 itself and .9999...) Prove that every positive integer has a unique representation as a *finite* sum of nonconsecutive powers of α . For example, $1 = \alpha^0$, $2 = \alpha^1 + \alpha^{-2}$, and $3 = \alpha^2 - \alpha^{-2}$. Can you find any patterns in α -representations analogous to those in Lucas and Zeckendorf representations?
10. Show that every integer, positive or integer, has a unique expression as a sum of nonconsecutive Fibonacci numbers with negative subscripts. For example, we have $-3 = F_{-4}$, $-2 = F_{-4} + F_{-1}$, $-1 = F_{-2}$, $0 = 0$, $1 = F_{-1}$, $2 = F_{-3}$, $3 = F_{-3} + F_{-1}$, and $4 = F_{-5} + F_{-2}$. (You can prove this either by induction or using lexicographic ranking.) All the questions we’ve asked about Zeckendorf representations (patterns, Lucas and α versions, etc.) can be asked about these representations.
11. Here are two possible ways to generalize Zeckendorf’s theorem:
- (a) Take any set A of finite sequences of integers of some fixed length and find an algorithm for lexicographic ranking of the elements of A . (For example, if we take A to be the set of all binary words of length n , we get the binary representation of integers from 0 to $2^n - 1$.) Of course, in order to get something interesting, your set A must not be chosen carefully. Can you get Lucas representations in this way?
- (b) Let $t_0 = 1 < t_1 < t_2 < \cdots$ be positive integers. We define the t -representation of a nonnegative integer n to be the representation

$$n = \sum_{i=0}^{\infty} e_i t_i$$

defined inductively as follows, using a greedy algorithm. If $n = 0$ then we take e_i to be 0 for all i . Otherwise, let k be the largest integer such that $t_k \leq n$ and define e_k to be $\lfloor n/t_k \rfloor$ and define e_i to be 0 for $i > k$. Finally, choose e_0, e_1, \dots, e_{k-1} so that $\sum_{i=0}^{k-1} e_i t_i$ is the t -representation of $n - \lfloor n/t_k \rfloor t_k$. Note that if $u_i = b^i$, where b is an integer greater than

2, we get the base b representation and if $u_i = F_{i+2}$ we get the Zeckendorf representation. Can we get the Lucas representation this way? Note: you may find it convenient to define the t -representation to be the sequence (e_0, e_1, \dots) .

It's not hard to show that any sequence (t_n) gives a unique way of expressing each positive integer, but in order to get something interesting, we need some more conditions on (t_n) . For example, we have an "interesting" representation if we describe (t_n) explicitly and we can also describe explicitly which sequence (e_n) are t -representations. Can you do this for the case in which t_n is $2^{n+1} - 1$? Can you find any general conditions on (t_n) that will give a reasonable characterization of the t -representations?

Suppose that we allow the t_n to be positive real numbers that aren't necessarily integers, or even arbitrary real numbers. Is there anything interesting that can be said? Of course, you may put some conditions on (t_n) .

12. (a) A *dual Zeckendorf representation* of n is an expression of n as a sum of Fibonacci numbers in the form $n = F_{i_1} + F_{i_2} + \dots + F_{i_k}$, where $i_1 > i_2 > \dots > i_k \geq 2$, and for each j from 1 to $k - 1$, $i_j - i_{j+1}$ is always 1 or 2. In other words, two consecutive Fibonacci numbers are never skipped. Prove that every integer has a unique dual Zeckendorf representation.
- (b) More generally, let us call an expression for n as $F_{i_1} + F_{i_2} + \dots + F_{i_k}$, where $i_1 > i_2 > \dots > i_k \geq 2$, a *Fibonacci representation* of n . Show that among all Fibonacci representations of n , the Zeckendorf representation has the greatest number of summands and the dual Zeckendorf representation has the least number of summands.
- (c) Let $R(n)$ be the number of Fibonacci representations of n . What can you say about $R(n)$?
- (d) More generally, let $R(n, k)$ be the number of Fibonacci representations of n with k summands. What can you say about $R(n, k)$?
- (e) We may define a partial order on the set of Fibonacci representations of n in which we go up by combining two summands F_i and F_{i-1} into a summand F_{i+1} , as long as F_{i+1} was not already a summand. What can you say about this partial order? For example, is it a lattice?
13. We may ask the same questions as in the previous problem if we modify the definition of Fibonacci representation by allowing summands to be repeated, so, for example, the Fibonacci representations of 4 would be $F_4 + F_2$, $F_3 + F_3$, $F_3 + F_2 + F_2$, and $F_2 + F_2 + F_2 + F_2$.