This problem set is due on Friday, January 24. You don't need to do all the problems, but you should at least think about them all. Try to do at least four of them.

1. Recall that a *Dyck* word is a word D made up of n x's and n y's for some n, with the additional property that if D is factored as D_1D_2 then D_1 has at least as many x's as y's. Note that the empty word is a Dyck word.

   (a) Let D_1 and D_2 be Dyck words. Prove that xD_1yD_2 is a Dyck word.

   (b) Let D be a nonempty Dyck word. Prove that there exist unique Dyck words D_1 and D_2, not necessarily nonempty, such that $D = xD_1yD_2$.

   Hint: It may be easier to work with paths than with Dyck words.

2. Exercise 1 implies that any nonempty Dyck word can be decomposed into two Dyck words. Describe the corresponding decompositions for binary trees and for ordered trees.

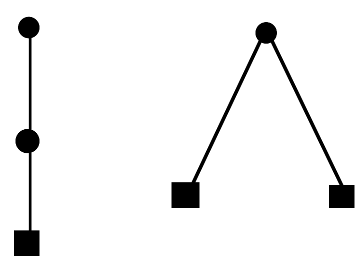
3. A *leaf* of a tree is a vertex with no children. The number of ordered trees with n vertices and k leaves is called a *Narayana number*, and is denoted by $N(n-1,k)$. Later on, we will have a formula for $N(n,k)$. (If you're really ambitious you can try to discover it yourself.)

   (a) What is $\sum_{k=1}^{n} N(n,k)$? (easy)

   (b) Make a table of the values of $N(n,k)$ for $1 \le k \le n \le 3$, and if you're ambitious, for $n = 4$. What symmetry do you observe? Can you prove it? (Hint: See part (d).)

   (c) We have bijections from ordered trees to Dyck words and to binary trees (complete and incomplete). Under these bijections, what do leaves of ordered trees correspond to?

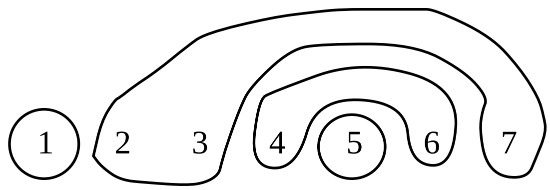
4. Among all ordered trees with three vertices, there are three leaves and three non-leave vertices (these are sometimes called *internal vertices*):



   Prove Shapiro's theorem: among all C_{n-1} ordered trees with n vertices, there are as many leaves as internal vertices. (Hint: See problem 3(c) and (d).)

5. A partition of a finite set A is a set $\{A_1, A_2, \cdots, A_k\}$ of nonempty subsets of A such that every element of A is in exactly one of the A_i. The sets A_i are called the *blocks* of the partition. A partition of the set $[n] = \{1, 2, \ldots, n\}$ is called *noncrossing* if there do not exist $i < j < k < l$ in $[n]$ such that i and k are in one block and j and l are in another block. Thus, all partitions

of [3] are noncrossing, and the only partition of [4] that is "crossing" is $\{\{1,3\},\{2,4\}\}$. We can draw a "noncrossing picture" of a noncrossing partition in the following way (this is a picture of the noncrossing partition $\{\{1\},\{2,3,7\},\{4,6\},\{5\}\}$):



(a) How many noncrossing partitions of $[n]$ are there?

(b) How many noncrossing partitions of $[n]$ are there with k blocks?

6. In class we saw two bijections from binary trees to Dyck words. (The first one was based on the preorder (Łukasiewicz) code of the tree, and the second one was based on a depth-first traversal of the tree in which write an x when we go down a left edge and we write a y when we go up a left edge (and we write nothing for right edges). Prove that these two bijections are the same. (Hint: Use induction.)

7. Show that the Catalan number C_n is the number of sequences $a_1a_2\cdots a_n$ of nonnegative integers such that $a_1 = 0$ and $0 \le a_{i+1} \le a_i + 1$. For example, when $n = 3$, these sequences are

$$000 \qquad 001 \qquad 010 \qquad 011 \qquad 012.$$

Hint: Find a bijection from these sequences to objects that we already know are counted by the Catalan numbers.