

This problem set is due on Friday, February 7.

Do at least four problems, not all of the same kind. Do more if you can, but don't try to do all of them!

Difficulty ratings: 1 (easy), 2 (moderate), 3 (challenging, but possible), 4 (difficult).

Except stated otherwise, all paths are in the plane with steps $(1, 0)$ and $(0, 1)$.

1. [2] Compute the number of paths from $(0, 0)$ to (n, n) with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$ that never go above the diagonal for small values of n . (You should find a more efficient approach than writing down all the paths—either a recurrence or a generating function.) Look up the numbers you obtain in the Online Encyclopedia of Integer Sequences. What are they called?

2. (a) [2–] If k is a nonnegative integer and a is arbitrary, the binomial coefficient $\binom{a}{k}$ is defined to be $a(a-1)(a-2)\cdots(a-k+1)/k!$. Prove that $\binom{2n}{n} = (-4)^n \binom{-1/2}{n}$ and that the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$ is given by $C_n = (-1)^n 2^{2n+1} \binom{1/2}{n+1}$. (Note: if you've already done this problem in a previous course, you don't need to do it again.)

- (b) [2–] Using the results of (a) and the binomial theorem $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$, prove that

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

and

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

3. [2] How many paths from $(0, 0)$ to (m, n) are there

(a) with j horizontal segments?

(b) with j horizontal and k vertical segments?

4. [2–] Using the reflection principle or the method of images count paths from (a, b) to (m, n) , where $a > b$ and $m > n$, that never touch the line $x = y$.

5. (a) [3–] Let s and t be positive integers. Use the reflection principle to count paths from $(0, 0)$ to (m, n) that never touch the lines $y = x + s$ and $y = x - t$. You may assume that (m, n) lies between these lines. (Your answer will be a sum.)

- (b) [2] Study the numbers obtained from the problem of part (a) for small values of s and t . Can you identify any of them? (You don't need to do part (a) to do this problem.)

6. (a) [3+] (The three-candidate ballot problem.) Suppose that $a > b > c > 0$ and $l > m > n > 0$. Use the reflection principle to count paths in \mathbf{R}^3 from (a, b, c) to (l, m, n) , with steps $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, that stay in the region $\{(x, y, z) \mid x > y > z > 0\}$.

7. (a) [2+] Let $P_k(m, n)$ be the number of paths in the plane of length k , with steps $(\pm 1, 0)$ and $(0, \pm 1)$, from $(0, 0)$ to (m, n) . Find a simple formula for $P_k(m, n)$.
- (b) [3–] Using the reflection principle, find a formula in terms of the numbers of part (a) for the number of paths in the plane of length k , with steps $(\pm 1, 0)$ and $(0, \pm 1)$, from (a, b) to (m, n) , where a, b, m , and n are positive integers, that stay strictly in the first quadrant (i.e., they never touch the lines $x = 0$ or $y = 0$).
8. (a) [3–] Using a modification of the reflection principle, show that the number of paths from (a, b) to (m, n) , where $a > b$ and $m > n$, that never touch the line $x = y$, with k left turns, is $\binom{m-a}{k} \binom{n-b}{k} - \binom{m-b}{k} \binom{n-a}{k}$.
- (b) [2–] Use the result of (a) to derive the formula for the Narayana numbers, $N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$.
- (c) [3+] Find a formula for counting paths from (a, b) to (m, n) , where $a > b$ and $m > n$, that never touch the line $x = y$, with k right turns.
9. (a) [2–] Using the method of images, prove that the number of paths from $(1, 0)$ to (m, n) , where $m > 2n$, that never touch the line $x = 2y$ is $\binom{m+n-1}{m-1} - 2\binom{m+n-1}{m} = \frac{m-2n}{m+n} \binom{m+n}{m}$. (If you do part (c), you don't need to do (a).)
- (b) [2–] Use the result of (a) to count complete ternary trees (ordered trees in which every vertex has either 0 or 3 children).
- (c) [2+] Generalizing the result of (a), let r be a positive integer. Use the method of images to count paths from $(1, 0)$ to (m, n) , where where $m > rn$, that never touch the line $x = ry$.
- (d) [2] Use the result of (c) to generalize (b).
10. (a) [1] Prove that $\binom{2n}{n} = 2\binom{2n-1}{n-1}$ and that $\binom{3n}{n} = 3\binom{3n-1}{n-1}$.
- (b) [1] Give a combinatorial proof that $\binom{2n}{n} = 2\binom{2n-1}{n-1}$.
- (c) [?] Give a combinatorial proof that $\binom{3n}{n} = 3\binom{3n-1}{n-1}$.
- (d) [2–] What is the connection between these formulas and the method of images?
11. [2+] Use the method of images to count paths from $(3, 0)$ to (m, n) , where $m > 2n$, that never touch the line $x = 2y$. (Your answer should involve several binomial coefficients; it doesn't simplify completely.)
12. [2] Use the cycle lemma to solve part (a) or (c) of problem 9.
13. [3] Use the cycle lemma to derive the formula for the Narayana numbers.
14. Let k be an integer greater than 1. A (complete) k -ary tree is an ordered tree in which every vertex has either 0 or k children.
- (a) [1+] Prove that a k -ary tree with n internal vertices has $(k-1)n + 1$ leaves. (Hint: Every internal vertex has k children.)

- (b) [2–] Let $t_k(n)$ be the number of k -ary trees with n internal vertices. Find a functional equation that the generating function $\sum_{n=0}^{\infty} t_k(n)x^n$ satisfies.
15. [2] Find a functional equation for counting ordered trees (by the number of vertices) in which every vertex has either 0, 1, or 2 children. Solve it using the quadratic formula (or have Maple do it), making sure that you select the right solution to the quadratic equation. Compute some of the coefficients using Maple, and look them up in the Online Encyclopedia of Integer Sequences.
16. [2] As in problem 15, but for ordered trees in which no vertex has exactly 1 child.
17. [2] As in problem 16, but count by the number of leaves, rather than the total number of vertices.
18. [2] Convert the previous three problems into lattice path counting problems.
19. (a) [2] Find a functional equation for the generating function (in two variables) for the Narayana numbers using one of the combinatorial interpretations, and solve it.
- (b) [2+] Use the Lagrange inversion formula and the functional equation of part (a) to get a formula for the Narayana numbers.