

1. Read Richard Stanley's collection of interpretations for the Catalan numbers.
 - (a) Find interesting bijections between some of them.
 - (b) Find analogous interpretations for the Narayana numbers.
 - (c) Find similar interpretations for the Motzkin and Schröder numbers.
2. We have seen that each Dyck path has a unique factorization into prime Dyck paths. This gives the generating function relation $c(x) = 1/(1-p(x))$ where $c(x)$ is the generating function for all Dyck paths and $p(x)$ (which is equal to $xc(x)$) is the generating function for prime Dyck paths.

Similar identities are very common in lattice path enumeration. We often have a set of paths that can be uniquely factored into prime paths. In such a case, if A is the generating function for all such paths and B is the generating function for the prime paths, then A and B are related by $A = 1/(1-B)$. Note that if we know A and want to find B , we can solve this equation to get $B = 1 - A^{-1}$.

Such decompositions exist for the generating function for the Catalan numbers, the Narayana numbers, the Motzkin numbers, or the Schröder numbers. Can you find variations or generalizations of these? Here are some generating functions A to look at—try to identify the corresponding B and find a combinatorial interpretation. Try to come up with your own examples.

- (a) $A(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$.
- (b) $A(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{2} \binom{2n}{n} x^n$.
- (c) $A(x)$ counts paths that end at a point of the form $(2n, 2n)$.
- (d) $A(x) = c(x)^2$
- (e) $A(x) = c(x)^3$
- (f) $A(x) = 1 + xc(x)$

3. Find the number of lattice paths from $(0,0)$ to (a,b) not touching the lines $y = x + r$ or $y = x - s$, where $r, s > 0$ and $-s < b - a < r$. There are many ways to solve this problem and many different ways to express the answer. (This is a generalization of the classical “gambler's ruin” problem.)
4. (a) Show that if $a_1 > c_1$, $a_2 \leq c_2$, $b_1 > d_1$, and $b_2 \leq d_2$, then the number of pairs of nonintersecting paths, one from (a_1, a_2) to (b_1, b_2) and the other from (c_1, c_2) to (d_1, d_2) is

$$\binom{b_1 + b_2 - a_1 - a_2}{b_1 - a_1} \binom{d_1 + d_2 - c_1 - c_2}{d_1 - c_1} - \binom{d_1 + d_2 - a_1 - a_2}{d_1 - a_1} \binom{b_1 + b_2 - c_1 - c_2}{b_1 - c_1}.$$

In this formula any binomial coefficients $\binom{p}{q}$ with $p < 0$ should be interpreted as 0.

5. Show that the number of paths from $(0, 0)$ to $(m + n, m + n - 1)$ which never touch any of the points $(m, m), (m + 1, m + 1), (m + 2, m + 2), \dots$ is

$$\frac{m}{2(m+n)} \binom{2m}{m} \binom{2n}{n}.$$

6. Find the number of paths of length n , starting at $(0, 0)$ that stay strictly below the diagonal. (We don't care where they end.) There are several interesting ways to solve this.
7. (a) How many paths to (m, n) touch the diagonal exactly r times?
 (b) How many paths to (m, n) cross the diagonal exactly r times?
 (c) How many paths to (m, n) touch the diagonal exactly r times and cross the diagonal exactly s times?
 (d) Find analogous results for Motzkin, Schröder, etc. paths.
8. (a) (The Chung-Feller theorem.) Show that the number of paths to (n, n) with exactly $2k$ steps above the diagonal is independent of k as long as $0 \leq k \leq n$. What is this number?
 (b) How many paths to (m, n) , with $m > n$, are there with exactly $2k$ steps above the diagonal?

9. Let

$$G(x, y) = \sum_{m, n=0}^{\infty} \binom{m+n}{m}^2 x^m y^n.$$

- (a) Find a combinatorial interpretation to the power series \bar{f} defined by $G = (1 - \bar{f})^{-1}$.
 (b) Show that $\bar{f} = 2f - x - y$, where f satisfies

$$f = x + y + \frac{xy}{1-f}.$$

- (c) Deduce that

$$f = \frac{1}{2} \left(1 + x + y - \sqrt{(1-x-y)^2 - 4xy} \right)$$

and

$$G = ((1-x-y)^2 - 4xy)^{-1/2}.$$

10. Let $u_{m,n}$ be the number of pairs (P_1, P_2) of paths from $(0, 0)$ to (m, n) such that that the two paths intersect only at their starting and ending points, and at every point in between, P_1 is above P_2 . (We take $u_{m,n}$ to be 0 if m or n is 0.) Let $U(x, y) = \sum_{m,n} u_{m,n} x^m y^n$. Show that $U = U(x, y)$ satisfies $U = (x + U)(y + U)$ and use Lagrange inversion to find an explicit formula for $u_{m,n}$.

11. Count Dyck paths according to the following parameters (or collections of these parameters):
- (a) number of peaks
 - (b) number of peaks at height at least 2
 - (c) number of double rises
 - (d) number of returns to the x -axis
 - (e) number of steps before the first return to the x -axis

12. A binary tree (for the purposes of this problem) is a rooted tree in which every vertex has either a left child, a right child, both, or neither. Find the number of binary trees with l left children and r right children (and thus $l + r + 1$ vertices in all, since the root has no parent).

13. Show that the number of ordered forests of k ordered trees in which i_s vertices have exactly s children for each s is

$$k \frac{(n-1)!}{i_0! i_1! \cdots i_n!},$$

where $n = i_0 + i_1 + \cdots$, if $i_1 + 2i_2 + \cdots = n - k$, and 0 otherwise. (This result is equivalent to the Lagrange inversion formula.)

14. If we count incomplete binary trees according to the number of left and right children, we get Narayana numbers. Study the analogous situation for incomplete ternary and more generally k -ary trees. (In an incomplete ternary tree, every child is a left, middle, or right child.)

15. Study noncrossing partitions. (They form a lattice which has many interesting properties.)

16. We know that the Catalan numbers count triangulations of a polygon. Count other types of decompositions of polygons.

17. Count various types of polyominoes.