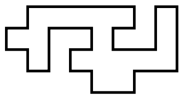
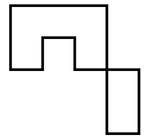
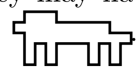
Math 47a **Problems on Polyominoes** **March 11, 2003**

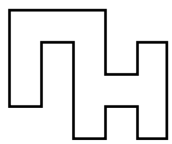
A *polyomino* is a union of squares in the plane, with lattice points as corners, with the property that its interior is connected.

Example:  is a polyomino, but  is not.

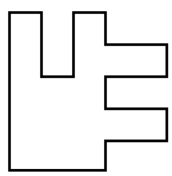
Note that polyominoes need not be simply connected. (In other words they may have holes in them.) Polyominoes are sometimes (especially by physicists) called *animals*: 

Polyominoes can be counted by various parameters. The most common are the perimeter, the area, and the size (horizontal and vertical) of the smallest bounded rectangle. Counting arbitrary polyominoes is a difficult unsolved problem, so most researchers have studied restricted classes.

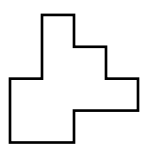
A polyomino is called *vertically convex* if the intersection of any vertical line with it is either empty or is a single line segment. *Horizontally convex* and *diagonally convex* polyominos are defined similarly. A polyomino is called *convex* if it is both vertically and horizontally convex. Thus



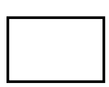
is vertically convex, but not horizontally convex. A polyomino is *directed* if any point of the polyomino can be be reached from the lower left corner by traveling east and north without leaving the polyomino. For example, neither of the polyominoes shown above is directed, but the polyomino

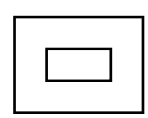


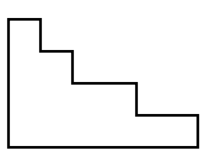
is directed. A particularly nice class of polyominoes are the *directed convex polyominoes*, such as

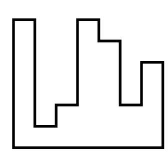


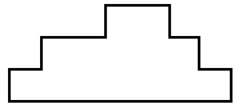
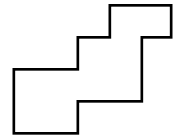
Here are some other especially simple classes of polyominoes that can be counted. I won't give formal definitions here, but you can probably figure them out from the pictures.

Rectangle polyomino: 

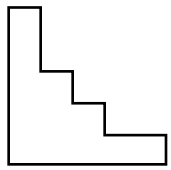
Rectangle polyomino with a hole: 

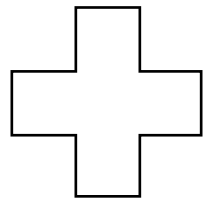
Ferrers or partition polyomino: 

Skyline polyomino: 

Stack or unimodal polyomino:  Parallelogram polyomino: 

Interesting enumeration problems (sometimes easier than the originals) are obtained by adding symmetry conditions. Symmetry can be through vertical, horizontal, or diagonal reflections, or by rotation, or any combination. For example, here is a Ferrers polyomino with diagonal symmetry and a convex polyomino with all possible symmetry:





In general, counting polyominoes by perimeter or minimal bounding rectangle is easier than counting by area, but this is not always true.

Here are some easy polynomial counting problems: rectangles by perimeter or area, Ferrers polynomials (possibly symmetric) by perimeter or bounding rectangle, skyline polynomials by area, parallelogram polynomials (possibly symmetric) by perimeter or bounding rectangle (we have in effect done this in class).

Here are some more challenging, but not too difficult, counting problems (by perimeter or bounding rectangle unless otherwise stated): rectangles with a hole, stacks, Ferrers polyominoes by area, skyline polyominoes, directed column-convex polynomials by area, directed convex polyominoes with diagonal symmetry.

Here are some even more challenging, but not impossible, counting problems: directed convex polyominoes (there is a very nice formula), vertically convex polyominoes by area, stacks by area, parallelogram polyominoes by area.

There are also many variations involving symmetry, not all of which have been studied. In many cases, symmetry makes the problem easier. For example, symmetric convex polyominoes are much easier to count than general convex polyominoes.

It is possible to consider more general polyominoes in which we allow diagonal steps as edges. (We might allow both northeast and northwest edges, or just one kind.) For most, if not all, of the types of polyominoes described above that can be counted, it is also possible to count the diagonal step generalizations. For example, we might consider the generalized parallelogram polyominoes that look like this: