Formulas and tables

Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

$$\begin{aligned} c(x) &= 1 + xc(x)^2 \\ &= \frac{1}{1-xc(x)} \end{aligned}$$

Let $C(x) = c(x)^2 = \big(c(x)-1\big)/x = \sum_{n=0}^{\infty} C_{n+1}x^n$. Then

$$\begin{aligned} C(x) &= \big(1 + xC(x)\big)^2 \\ &= \frac{1}{1-2x-x^2C(x)} \end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012

Motzkin numbers:

$$M_n = \sum_{0\le k\le n/2} \binom{n}{2k} C_k$$

$$\begin{aligned} m(x) = \sum_{n=0}^{\infty} M_n x^n &= \frac{1}{1-x} c\left(\frac{x^2}{(1-x)^2}\right) \\ &= \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2} \end{aligned}$$

$$\begin{aligned} m(x) &= 1 + xm(x) + x^2m(x)^2 \\ &= \frac{1}{1-x-x^2m(x)} \end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M_n	1	1	2	4	9	21	51	127	323	835	2188	5798	15511	41835	113634

Big Schröder numbers:

$$R_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k$$

$$\begin{aligned} r(x) = \sum_{n=0}^{\infty} R_n x^n &= \frac{1}{1-x} c\left(\frac{x}{(1-x)^2}\right) \\ &= \frac{1-x-\sqrt{1-6x+x^2}}{2x} \end{aligned}$$

$$\begin{aligned} r(x) &= 1 + xr(x) + xr(x)^2 \\ &= \frac{1}{1-x-xr(x)} = \frac{1}{1-2xs(x)} \\ 1 + xr(x) &= \frac{1}{1-xs(x)} \end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
R_n	1	2	6	22	90	394	1806	8558	41586	206098	1037718	5293446	27297738

Little Schröder numbers:

$$S_0 = 1; \quad S_n = R_n/2 \quad \text{for } n \geq 1$$

$$s(x) = \sum_{n=0}^{\infty} S_n x^n = \frac{1+x-\sqrt{1-6x+x^2}}{4x}$$

$$\begin{aligned} r(x) &= 2s(x) - 1 \\ s(x) &= 1 + xs(x)r(x) \\ &= \frac{1}{1-xr(x)} \\ \sum_{n=0}^{\infty} S_{n+1} x^n &= \left(1 - 3x - \sum_{n=2}^{\infty} R_{n-1} x^n\right)^{-1} \\ &= \frac{s(x)^2}{1-xs(x)} \end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
S_n	1	1	3	11	45	197	903	4279	20793	103049	518859	2646723	13648869

Ballot numbers:

$$\begin{aligned} B(0,0) &= 0 \\ B(m,n) &= \binom{m+n-1}{n} - \binom{m+n-1}{n-1} \\ &= \frac{m-n}{m+n}\binom{m+n}{n} \text{ for } (m,n) \neq (0,0) \end{aligned}$$

$$\begin{aligned} C_n &= B(n+1,n) \\ \sum_{m,n=0}^{\infty} B(m,n)x^m y^n &= \frac{x-y}{1-x-y} \\ 1 + \sum_{m,n=0}^{\infty} |B(m,n)|x^m y^n &= \frac{\sqrt{1-4xy}}{1-x-y} \\ 1 + \sum_{m>n\geq 0} B(m,n)x^m y^n &= \frac{1}{1-xc(xy)} \\ \sum_{m>n\geq 0} B(m,n)x^m y^n &= \frac{x-xyc(xy)}{1-x-y} \\ \sum_{n=0}^{\infty} B(n+k,n)z^n &= \sum_{n=0}^{\infty} \frac{k}{2n+k}\binom{2n+k}{n} z^n = c(z)^k \\ \sum_{n=0}^{\infty} \binom{2n+k}{n} z^n &= \frac{c(z)^k}{\sqrt{1-4z}} \end{aligned}$$

$B(m,n)$:

$m\backslash n$	0	1	2	3	4	5	6	7	8
0	0	−1	−1	−1	−1	−1	−1	−1	−1
1	1	0	−1	−2	−3	−4	−5	−6	−7
2	1	1	0	−2	−5	−9	−14	−20	−27
3	1	2	2	0	−5	−14	−28	−48	−75
4	1	3	5	5	0	−14	−42	−90	−165
5	1	4	9	14	14	0	−42	−132	−297
6	1	5	14	28	42	42	0	−132	−429
7	1	6	20	48	90	132	132	0	−429
8	1	7	27	75	165	297	429	429	0

Narayana numbers:

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

$$N(n,k) = N(n, n-k+1)$$

$$\sum_{k=1}^{n} N(n,k) = C_n \text{ for } n \geq 1.$$

$$\sum_{k=0}^{n} (-1)^k N(n+1, k+1) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} C_{n/2} & \text{if } n \text{ is even} \end{cases}$$

Let

$$F = F(x,y) = \sum_{i,j=0}^{\infty} N(i+j+1, j+1) x^i y^j.$$

Then

$$\begin{aligned} F(x,y) &= F(y,x) \\ F(x,x) &= c(x)^2 \\ F(x,y) &= \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2xy} \\ &= \frac{1}{1-x-y} c\left(\frac{xy}{(1-x-y)^2}\right) \end{aligned}$$

$$\begin{aligned} F &= (1+xF)(1+yF) \\ &= 1 + (x+y+xyF)F \\ &= \frac{1}{1-x-y-xyF} \end{aligned}$$

Let $G = G(x,y) = 1 + xF$. Then

$$\begin{aligned} G(x,x) &= c(x) \\ G &= 1 + xG + yG(G-1) \\ &= \frac{1}{1-x-y(G-1)} \end{aligned}$$

$N(n,k)$:

$n \backslash k$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	1	0	0	0	0	0
4	1	6	6	1	0	0	0	0
5	1	10	20	10	1	0	0	0
6	1	15	50	50	15	1	0	0
7	1	21	105	175	105	21	1	0
8	1	28	196	490	490	196	28	1

Riordan numbers (related to Motzkin numbers):

$$J_n = \frac{1}{n+1} \sum_{1 \le k \le n/2} \binom{n+1}{k} \binom{n-k-1}{k-1}, \quad n > 0$$

$$j(x) = \sum_{n=0}^{\infty} J_n x^n = \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x(1+x)}$$

$$= \frac{2}{1 + x + \sqrt{1 - 2x - 3x^2}}$$

$$j(x) = \frac{1}{1+x} + xj(x)^2$$

$$j(x) = \frac{1}{1 - x^2 m(x)}$$

$$1 + xm(x) = \frac{1}{1 - xj(x)}$$

$$\sum_{n=0}^{\infty} J_{n+2} x^n = \left(1 - x - 2x^2 - \sum_{n=3}^{\infty} M_{n-2} x^n\right)^{-1}$$

$$\sum_{n=0}^{\infty} M_{n+1} x^n = \left(1 - 2x - \sum_{n=3}^{\infty} J_{n-1} x^n\right)^{-1}$$

n	0	1	2	3	4	5	6	7	8	9	10	11
J_n	1	0	1	1	3	6	15	36	91	232	603	1585

Fine numbers (related to Catalan numbers):

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{1 + 2x - \sqrt{1 - 4x}}{2x(2+x)}$$

$$= \frac{1 - \sqrt{1 - 4x}}{x(3 - \sqrt{1 - 4x})} = \frac{2}{1 + 2x + \sqrt{1 - 4x}}$$

$$f(x) = \frac{1}{1 - x^2 c(x)^2}$$

$$1 + xc(x) = \frac{1}{1 - xf(x)}$$

$$1 + 2xc(x)^2 = \frac{1}{1 - 2xf(x)}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
F_n	1	0	1	2	6	18	57	186	622	2120	7338	25724	91144

Unknown numbers (related to Schröder numbers)

$$u(x) = \sum_{n=0}^{\infty} U_n x^n = \frac{1+3x-\sqrt{1-6x+x^2}}{2x(3+2x)}$$
$$= \frac{2}{1+3x+\sqrt{1-6x+x^2}}$$

$$u(x) = \left(1 - \sum_{n=2}^{\infty} R_{n-1} x^n\right)^{-1}$$
$$1 + xs(x) = \frac{1}{1-xu(x)}$$
$$\sum_{n=0}^{\infty} \tfrac{1}{2} U_{n+2} x^n = \left(1 - 3x - 4x^2 - \sum_{n=3}^{\infty} R_{n-1} x^n\right)^{-1}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
u_n	1	0	2	6	26	114	526	2502	12194	60570	305526	1560798	8058714