AN INTRODUCTION TO SCHRÖDER AND UNKNOWN NUMBERS

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Abstract. In this article we will introduce two types of lattice paths, Schröder paths and Unknown paths. We will examine different properties of each, and attempt to relate the two sets in a few different ways.

1. Introduction to the Schröder and Unknown numbers

Definition 1. A Schröder path is a lattice path that starts on the x-axis, ends on the x-axis, never goes below the axis and is constructed by the steps $(1,1)$, $(1,-1)$ and $(2,0)$. We will let R_n denote the number of Schröder paths of length $2n$.

Definition 2. A peak is an up-step, $(1,1)$, followed immediately by a down-step, $(1,-1)$. We will say that the height of a peak is k if the top of the peak is at height k in the path.

Definition 3. A set of numbers, which we will call the Unknown numbers, count the number of Schröder paths which have no peaks at height one, and no horizontal edges at height zero. We will let U_n denote the *nth* Unknown number, which counts such Schröder paths of length $2n$. These types of paths will be referred to as Unknown paths.

Definition 4. We will call a non-empty path *prime* in some set of paths S if it cannot be factored into smaller paths p_1 and p_2 such that both p_1 and p_2 are in S.

2. Some generating functions

It is known (see [1]) that if $a(x)$ is a generating function that counts some set of paths S that can all be uniquely factored into primes, and if $p(x)$ is the generating function that counts the prime paths in S then

$$a(x) = \frac{1}{1-p(x)}.$$

To determine some formulas for the generating function of the Schröder numbers, $r(x) = \sum_{n=0}^{\infty} R_n x^n$, lets look at the prime Schröder paths. We will consider the primes of the Schröder paths to be any path which starts and ends on the x-axis and never touches it in-between. It is clear that a path of this type can not be factored further. We will weight each step in the generating function for Schröder paths by x. We know that a single horizontal step has length 2 and is prime. So the horizontal step is counted by x. Also, it is clear that an up-step followed by an arbitrary Schröder path of length $2n-2$ followed by a down-step is prime path of length $2n$. It should be easy to convince yourself that these are the only prime paths. This interpretation leads us to the generating function of the prime Schröder paths $p(x) = x + xs(x)$. And finally this gives us that

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$$r(x) = \frac{1}{1 - x - xr(x)}. \tag{2.1}$$

We can rearrange this to get that

$$r(x) = 1 + xr(x) + xr(x)^2. \tag{2.2}$$

Solving (2.2) for $r(x)$ using the quadratic formula and selecting the solution that gives a power series we get that

$$r(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}. \tag{2.3}$$

Using the same approach as for the Schröder numbers we can discover some formulas for the generating function for the Unknown numbers $u(x) = \sum_{n=0}^{\infty} U_n x^n$. We know that the prime Unknown paths cannot have any horizontal steps on the x-axis or any peaks of height 1. It is not too difficult to see that the only type of prime Unknown path is of the form up-step followed by a non-empty Schröder path followed by a down-step. That is, a prime path of length $2n$ is an up-step followed by a Schröder path of length $2n-2$ followed by a down-step, which is counted by $R_{n-1}x^n$ since there is a unique prime for every Schröder path of length $2n - 2$. This gives us that the generating function for the primes is $p(x) = \sum_{n=2}^{\infty} R_{n-1}x^n$, or equivalently $p(x) = x(r(x) - 1)$. Using the second version we see that this gives us

$$u(x) = \frac{1}{1 - x(r(x) - 1)}. \tag{2.4}$$

If we substitute (2.3) in for $r(x)$ we find that

$$u(x) = \frac{1 + 3x - \sqrt{1 - 6x + x^2}}{2x(3 + 2x)}. \tag{2.5}$$

3. Paths with an even number of peaks

In this section we will see that half of the Unknown paths of length $2n$ have an even number of peaks. To prove this lets consider the subset S of the paths counted by U_n where we consider only paths with an even number of peaks. Now we need to show that S is counted by $\frac{1}{2}U_n$. To do this lets consider any path p in S. Lets take p' to be the path of p where all peaks in p are replaced by horizontal steps. We know that p' is still a path counted by U_n, because if any of the new horizontal steps were at height zero that would imply that p had a peak at height 1. Now, lets say that p' has n horizontal steps. We can replace any number of horizontal steps in p' with peaks (an up-step followed by a down-step) and the result is a path in U_n. This tells us that there are $\binom{n}{k}$ paths with k peaks resulting from p'. So if n is even the number of paths with an odd number of peaks resulting from p' is $\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1}$ which is equal to $\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n}$ which is the number of paths resulting from p' with an even number of peaks. And if n is odd similarly the number of odd peaks resulting from p' is $\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-1}$ which is again the number of paths with an even number of peaks resulting from p'. Since every path counted by U_n can be reduced into a path with only horizontal steps, and every path of this type results in an equal number of paths with and odd number of peaks and even number of peaks, we have that S is counted by $\frac{1}{2}U_n$.

Since the Schröder paths have the same structure as the Unknown paths the same argument applies for them. So we have that the number of Schröder paths of length $2n$ with an even number of peaks is $\frac{1}{2}R_n$.

4. Paths with an initial run of up-steps of even length

Definition 5. A *run* is a series of consecutive steps all of the same type.

Similar to the previous section, we can show that the number of Unknown paths with an initial run of even length is counted by $\frac{1}{2}U_n$. To see this lets consider the following function f from Unknown paths with an even number of initial up-steps U_{even} to Unknown paths with an odd number of initial up-steps U_{odd}. Take u to be some path in U_{even} with an initial run of length $2k$. We know that the step following the initial run must be either a down-step or a horizontal step (otherwise the step would be included in the intitial run). If the step following the initial run is a horizontal step, we replace it with an up-step followed by a down-step (a peak) to get u'. Clearly this leaves u' with an initial run of length $2k+1$ so it is in U_{odd}. If the step after the initial run is a down-step we take u' to be u where we replace the previous up-step and the down-step with a horizontal step. Note that this will not place a horizontal step on the x-axis since the initial run must have length at least 2. This will leave u' with an initial run of length $2k-1$ so it will be in U_{odd}. This process is unique so it is one-to-one. Also we can see that applying f to $f(u)$ gives back u, so for any u' in U_{odd} we have $f(u')$ is in U_{even} (by a similar argument as before) and $f(f(u')) = u'$. So f is onto. This implies that we have found a bijection between Unknown paths with an initial run of even length and Unknown paths with an initial run of odd length. So half of U_n counts U_{even}.

Again, because the Schröder paths have a similar structure to the Unknown paths this argument holds for them also. So $\frac{1}{2}R_n$ also counts Schröder paths with an initial run of even length.

5. A Recurrence Relation for The Schröder Numbers

From (2.3) we know that

$$\begin{aligned} r(x) &= \sum_{n=0}^{\infty} R_n x^n \\ &= \frac{1-x-\sqrt{1-6x+x^2}}{2x}. \end{aligned}$$

For ease of calculation let $a(x) = 2xr(x)$. Then

$$\begin{aligned} a'(x) &= \frac{\partial(a(x))}{\partial x} \\ &= -\frac{1}{2} + \frac{(3-x)\sqrt{1-6x+x^2}}{1-6x+x^2} \end{aligned}$$

Multiplying $a'(x)$ by it denominator we see we are left with something similar to $a(x)$. We find that

$$(1-6x+x^2)a'(x) + (3-x)a(x) = 2+2x. \tag{5.1}$$

From this differential equation we can derive a simple recurrence relation for the Schröder numbers. To do this we should note that the derivative of the generating function $r(x)$ is

$$r'(x) = \sum_{n=0}^{\infty} nR_n x^{n-1}.$$

Let us define R_n to be zero for n less than zero. Using this fact and that $a(x) = \sum_{n=0}^{\infty} 2R_n x^{n+1}$ and $a'(x) = \sum_{n=0}^{\infty} 2(n+1)R_n x^n$ we can now write equation (5.1) to be

$$\begin{aligned}
2x + 2 &= (1 - 6x + x^2)\sum_{n=0}^{\infty} 2(n+1)R_n x^n + (3 - x)\sum_{n=0}^{\infty} 2R_n x^{n+1} \\
&= \sum_{n=0}^{\infty} 2(n+1)R_n x^n - \sum_{n=0}^{\infty} 12(n+1)R_n x^{n+1} \\
&\quad + \sum_{n=0}^{\infty} 2(n+1)R_n x^{n+2} + \sum_{n=0}^{\infty} 6R_n x^{n+1} - \sum_{n=0}^{\infty} 2R_n x^{n+2} \\
&= \sum_{n=0}^{\infty} 2(n+1)R_n x^n - \sum_{n=0}^{\infty} 12nR_{n-1} x^n \\
&\quad + \sum_{n=0}^{\infty} 2(n-1)R_{n-2} x^n + \sum_{n=0}^{\infty} 6R_{n-1} x^n - \sum_{n=0}^{\infty} 2R_{n-2} x^n
\end{aligned}$$

Now looking at the coefficients of x^n (where $n > 1$) in this equation we see that on the left side we get zero, and on the right side we get

$$\begin{aligned}
&[x^n]\sum_{n=0}^{\infty} 2(n+1)R_n x^n - [x^n]\sum_{n=0}^{\infty} 12nR_{n-1} x^n + [x^n]\sum_{n=0}^{\infty} 2(n-1)R_{n-2} x^n \\
&\quad + [x^n]\sum_{n=0}^{\infty} 6R_{n-1} x^n - [x^n]\sum_{n=0}^{\infty} 2R_{n-2} x^n \\
&= (2n+2)R_n - 12nR_{n-1} + (2n-2)R_{n-2} + 6R_{n-1} - 2R_{n-2}
\end{aligned}$$

Now equating both sides, dividing by 2 and simplifying we find that

$$(n+1)R_n = (6n-3)R_{n-1} - (n-2)R_{n-2} \tag{5.2}$$

6. A Recurrence Relation for The Unknown Numbers

From (2.5) we have

$$\begin{aligned}
u(x) &= \sum_{n=0}^{\infty} U_n x^n \\
&= \frac{1 + 3x - \sqrt{1 - 6x + x^2}}{6x + 4x^2}.
\end{aligned}$$

Again, to make the calculation a little easier we will take $b(x) = (6x + 4x^2)u(x)$. Note that this makes

$$b(x) = \sum_{n=1}^{\infty} 6U_{n-1}x^n + \sum_{n=2}^{\infty} 4U_{n-2}x^n \tag{6.1}$$

and

$$b'(x) = \sum_{n=0}^{\infty} 6(n+1)U_n x^n + \sum_{n=1}^{\infty} 4(n+1)U_{n-1}x^n \tag{6.2}$$

From the closed form of the generating function $u(x)$ we see that

$$b'(x) = 3 - 18x + 3x^2 + (3 - x)\sqrt{1 - 6x + x^2}$$

This looks very similar to what we found previously for the Schröder numbers. We find that $b(x)$ satisfies differential equation

$$(1 - 6x + x^2)b'(x) + (3 - x)b(x) = 6 - 10x.$$

We will define U_n to be zero for n less than zero. Next substituting in equations (6.1) and (6.2) we find that

$$\begin{aligned} 6 - 10x = {} & \sum_{n=0}^{\infty} 6(n+1)U_n x^n + \sum_{n=0}^{\infty} 4(n+1)U_{n-1}x^n - \sum_{n=0}^{\infty} 36nU_{n-1}x^n \\ & - \sum_{n=0}^{\infty} 24nU_{n-2}x^n + \sum_{n=0}^{\infty} 6(n-1)U_{n-2}x^n + \sum_{n=0}^{\infty} 4(n-1)U_{n-3}x^n. \end{aligned}$$

Now equating the coefficients of x^n in this equation we find that

$$(3n + 3)U_n = (16n - 11)U_{n-1} + 9nU_{n-2} - (2n - 4)U_{n-3}. \tag{6.3}$$

7. Combining the Schroder and Unknown Numbers

We would like to be able to express the Schröder numbers in relation to the Unknown numbers. Using (2.3) and (2.5) we see that

$$(3 + 2x)u(x) = r(x) + 2$$

Using the same method in the previous two sections to express relations between coefficients of x^n in both sides of the equation we find that

$$R_n = 3U_n + 2U_{n-1} \tag{7.1}$$

8. Combinatorial proof of Equation (7.1)

To see that $R_n = 3U_n + 2U_{n-1}$ we will find it easier to prove that $R_n - U_n = 2U_n + 2U_{n-1}$. Let us take S to be the set of paths counted by $R_n - U_n$. We know that the Unknown numbers count Schröder paths with no horizontal steps at height 0 and no peaks at height 1. This would imply that S counts that number of Schroder paths of length $2n$ with at least one horizontal step at height 0 or peak at height 1. Consider the last occurrence of either in the path. Let S^- be the paths that have the last occurrence as a horizontal step on the x-axis. Let S^+ be the paths that have the last occurrence as a peak at height 1. Lets let T be the set of paths counted by $2U_n + 2U_{n-1}$. And let T^- be one half of the paths counted by $U_n + U_{n-1}$, and T^+ be the other half counted by $U_n + U_{n-1}$ (making each path distinct).

Now consider the following function f. We take a path s in S. If s is in S^- then we remove the last horizontal step at height 0 in s and insert an up-step at the start of the path and a down-step where we remove the horizontal step to get $f(s)$.

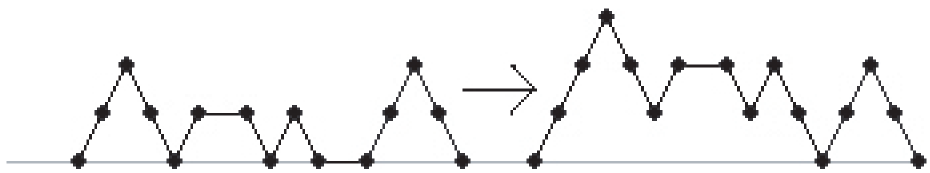

Figure 1. Example $s \in S^-$ to $s' \in T$

This leaves no more horizontal steps at height 0 or peaks at height 1 since they will all be moved up one level. So $f(s)$ will be a path counted by U_n since it is of length $2n$ still unless the path before the horizontal step was empty, in which case we are left with a peak of height 1 at the start. We can chop the peak off and the remaining path is counted by U_{n-1} since it has no peaks at height 1 or horizontal steps at height 0 and has length $2n-2$. So $f(s)$ is in T. Now if s is in S^+ we remove the last peak and place the up-step of the peak at the begining of s and the down-step remains where it is to get $f(s)$. If s is non-empty before the peak then this will be counted by U_n and if not it will be counted by U_{n-1}. Again s is in T. We see that any path $f(s)$ in T is uniquely determined by s since it is s uniquely rearranged. So this tells us that f is one-to-one. We already have seen that U_{n-1} counts Unknown paths of length $2n-2$ with a peak attatched to the front. So for q in T, we just take the first up-step and down-step that touch the x-axis and we remove them. Where the down-step was, we put either a horizontal step which corresponds to a path in S^- if q is in T^- or we put a peak of height 1 which corresponds to a path in S^+ if q is in T^+. So we know every path in T comes from some path in S. So f is onto. This bijection tells us that $R_n - U_n = 2U_n + 2U_{n-1}$.

9. A final equation relating the Unknown and Schröder numbers

Using what we found $r(x)$ to be in (2.3) and $u(x)$ to be in (2.5) we can verify that

$$\frac{1}{2}\frac{u(x)-1}{x^2}(1-2x-2x^2-xr(x)) = 1. \tag{9.1}$$

We can rewrite this to be

$$\frac{1}{2}\frac{u(x)-1}{x^2} = \frac{1}{1-2x-2x^2-xr(x)}. \tag{9.2}$$

Using (9.2) and defining R_n to be zero for n less than zero we find that

$$\sum_{n=0}^{\infty} \frac{1}{2} U_{n+2} x^n = \frac{1}{1 - 2x - 2x^2 - \sum_{n=0}^{\infty} R_{n-1} x^n}. \tag{9.3}$$

We can now find an interpretation of the paths that $\sum_{n=0}^{\infty} \frac{1}{2} U_{n+2} x^n$ counts based on the primes given on the right side of equation (9.3). We see that the prime paths are very similar to the prime paths of the Schröder numbers except we have and additional x and an additional $2x^2$. Let u denote an up-step and d denote a down-step and h denote a horizontal step. We can take the prime Schröder paths and add in an additional prime of length 2, du, and two additional primes of length 4, $dduu$ and dhu.

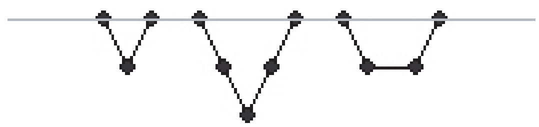

FIGURE 2. Additional Primes

Now we see that any path s constructed of these primes will never go below 2 below the x-axis. By adding on two up-steps at the beginning of such a path and two down-steps at the end to get s' we have a path of length $2n + 4$ that will be an Unknown path of length $2n + 4$. We see this is true since if there is a horizontal step on the x-axis in s' then there is a horizontal step at 2 below the x-axis in s, which is impossible from the primes we have chosen. Also a peak at height 1 in s' is impossible from the primes.

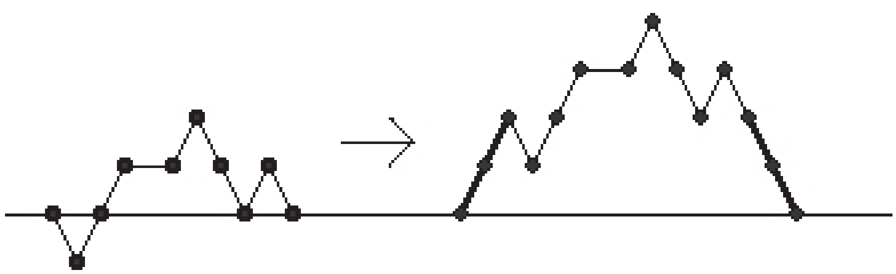

FIGURE 3. Example s_1 to s_1'

The path s' has the property that any horizontal step at height 1 must be preceeded by a down step and followed by an up-step caused by the prime of the form dhu. Also, if s' touches the axis at any point (not the starting or ending point) then the point must be preceeded by two down steps and followed by two up steps, which is caused by the prime $dduu$.

What (9.3) tells us is that Unknown paths with these properties of length $2n + 4$ are exactly half of the paths counted by U_{n+2}.

REFERENCES

[1] Nick Dufresne, *Generating Functions for Lattice Paths in a Free Monoid,* March, 2003