A NEW APPLICATION TO THE JOSEPHUS PROBLEM

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ABSTRACT. This paper introduces a new way to describe the Josephus Problem. With a circle and an unknown number of sectors drawn from a combination of radii and diameters, we find that the number of sectors of a circle equals the number of radii drawn. Once we know the initial number of sectors, we find that the winning sector is given by a recurrence relation:

$$G(n,k) = (G(n-1,k)+k) \bmod n, \text{ where}$$

$$G(2,k) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 2 & \text{if } k \text{ is odd} \end{cases}$$

where n is the number of sectors, $k-1$ is the number of sectors skipped, and $G(n,k)$ is the winning sector.

If $k = 2$, we obtain the following formula:

$$G(n,2) = 1 + 2n - 2^{\lfloor 1+\log_2 n \rfloor}$$

1. Introduction

The Josephus Problem is named after a historian of the first century, Flavius Josephus. During the Jewish-Roman War, he and 40 other Jewish rebels were trapped in a cave by the Romans. Prefering not to be held prisoner, the 41 men decided to form a circle and go around, killing every third man until all were deceased. Josephus did not want any part of the suicide nonsense. Using his mathematical genius, he strategically placed himself in the spot where the last person would have to commit suicide. Since no one was alive to witness the final suicide, Josephus walked away intact [1, p. 8].

2. A New Way to Describe the Josephus Problem

A circle is drawn, followed by an unknown combination of radii and diameters in any order. Players can place their initials in only one sector until all are filled. One player then chooses a sector and clockwise counts to k. On k, the sector landed on is crossed out. The count continues in the same direction from the next open sector, skipping crossed out sectors. Repeat this process until all but one sector is crossed out. The player whose initials is in that sector wins the game.

3. How Many Sectors are Drawn in the Circle?

Assume a circle with 2 distinct radii are drawn, OA and OB. (see Figure 1.) The circle must be divided into 2 sectors. Suppose a third, distinct radius is drawn, OC. One of the two sectors must be cut into two sectors. Thus, 3 sectors are drawn in the circle, as are 3 radii.

The same can be done for 4, 5, 6, . . . radii.

Date: April 29, 2003.

Therefore, if only radii are drawn, the number of sectors drawn in a circle is equal to the number of radii in the circle.

Since a diameter is equal to the length of 2 radii, it can be counted as 2 radii drawn.

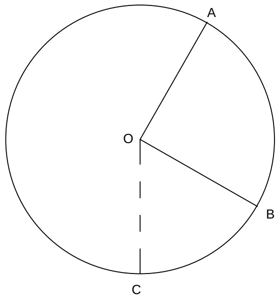

FIGURE 1

4. Finding the Last Sector

Before we begin, we need to be specify a few things:

1. We will be using a modified mod function. By $a \bmod b$, we mean the least positive residue of a modulo b. So, if b divides a, $a \bmod b = b$. We will denote the regular modulo function as mod$'$.

2. We will use the interpretation for the circle as the following sequence: $\overbrace{1}, 2, \ldots, n,$

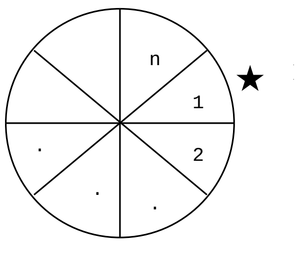

FIGURE 2

where the starred sector is the next spot you count from. From Figure 2, the next spot we count from is one. In the sequence, it will be written as the first element with an overbrace.

3. The winning sector will be denoted as $G(n, k)$, where n is the number of sectors and $k - 1$ is the number of spaces you skip as you go around. The number you count to is k. On k, the sector landed on is crossed out.

4. From the sequence, an answer of 1 means the place you start is the winning sector, an answer of 2 is the space next to the starting point, . . .

Theorem 1. $G(n,k) = (G(n-1,k)+k) \bmod n$, *where*

$$G(2,k) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 2 & \text{if } k \text{ is odd} \end{cases}$$

Proof. Suppose we have a circle drawn with $n-1$ sectors, where b is the winning spot. Written as a sequence from 1 to $n-1$, we have

$$\overbrace{1}, 2, \ldots, b-1, b, b+1, \ldots, n-1. \tag{1}$$

Note: b does not have to lie between 2 and $n-1$. If $b=1$, it would be shown as

$$\overbrace{b}, b+1, \ldots, n-1. \tag{2}$$

Suppose we now look at a circle with n sectors. If k' is the first sector to be crossed out, then

$$k' = k \bmod n. \tag{3}$$

Note: k' is used to show that if $k > n$, we will go around the circle, counting the starting point at least once. The mod function enables us to have values within our numbered sectors.

Written as a sequence from 1 to n, we have

$$\overbrace{1}, 2, \ldots, k'-1, k', k'+1, \ldots, n-1, n. \tag{4}$$

We are now left with $n-1$ sectors. The next sector we count from is $k'+1$, shown as

$$\overbrace{k'+1}, \ldots, n-1, n, 1, 2, \ldots, k'-1. \tag{5}$$

Because sequences (1) and (5) contain the same number of sectors, the last sector to be crossed out will be the bth sector in the sequence (5), which is $(b+k') \bmod n$. The fact that $k' = k \bmod n$ implies that this is equal to

$$(b+k) \bmod n. \tag{6}$$

Therefore, the winning sector is

$$G(n,k) = (G(n-1,k)+k) \bmod n. \tag{7}$$

We have found a recurrence relation and we need to find the initial value for G. You can't have one sector uncrossed; the game would be over. Thus, the least number of uncrossed sectors you can have is two. Suppose $k=2$. We will count "1, 2". (See Figure 3.) Two will be crossed out, leaving $G(2,2)=1$. Suppose $k=4$. We will count "1, 2, 1, 2". Two will be crossed out, leaving $G(2,4)=1$. One can clearly see that for any even k, $G(2,k)=1$. Likewise, for any odd k, one will be crossed out, leaving two as the answer. Thus, $G(2,k)=2$ for any odd k. □

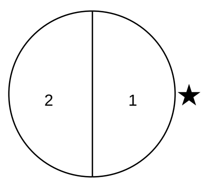

FIGURE 3

5. Run Lengths

The formula $G(n,k)$ has an interesting phenomena when $n > k$ (see note before Theorem 2) and $k \geq 2$. Recall that $G(n,k) = (G(n-1,k)+k) \bmod n$.

Suppose $G(n,k) = 1$. Then, $G(n+1,k) = 1+k$. Since we are adding a small number, k, to 1, when we take the modulo of this number to a large number, n, we have an answer of $1+k$. For that matter, unless $G(n-1,k)+k > n$, we will continue to add k to the answer.

Suppose $G(n,k)$ has a value in $\{1,\ldots,k-1\}$. We will continue to add sectors into the circle and find the values of $G(n+i,k)$ for $i = 1,2,\ldots$. The next time $G(n+i,k)$ has value in $\{1,\ldots,k-1\}$, i will be equal to the run length.

Note: By $n > k$, we are looking at values of $G(n,k)$ that after some point, there is a clear indication that all values are the sum of the previous term to the left of it and k. Refer to the table on the page 9.

Theorem 2. *A circle starting with n sectors will have a run length of approximately (off by 1 or 2)* $\frac{n}{k-1}$*. Each one after that will have a run length multiplied by* $\left(\frac{k}{k-1}\right)^{x-1}$ *times the previous length.*

In other words, starting from n, the xth will be $\frac{n}{k-1}\left(\frac{k}{k-1}\right)^{x-1}$.

Proof. Suppose $G(n,k) = d$, where d has a value in $\{1,\ldots,k-1\}$. Then $G(n+i,k) = d+ki$ if $d+ki \leq n+i$. Isolating i leaves $i \leq \frac{n-d}{k-1}$.
Note: After this point, all values in this proof are approximate values.

Approximately, $G(n+i,k) \approx ki$ for $i \leq \frac{n}{k-1}$. Thus, a run starting at n will have a run length of $\frac{n}{k-1}$.

The next run will begin at appproximately $n+\frac{n}{k-1} = \frac{nk}{k-1}$. Then $G\left(\left(\frac{nk}{k-1}\right)+i,k\right) \approx ki$ for $ki \leq \frac{nk}{k-1}+i$. Isolating i leaves $i \leq \frac{nk}{(k-1)^2}$. Thus, the second run will have length $\frac{nk}{(k-1)^2}$.

The next run will begin at approximately $\frac{nk}{k-1} + \frac{nk}{(k-1)(k-1)} = \frac{nk^2}{(k-1)^2}$. Then $G\left(\left(\frac{nk^2}{(k-1)^2}\right)+i,k\right) \approx ki$ for $ki \leq \frac{nk^2}{(k-1)^2}+i$. Isolating i leaves $i \leq \frac{nk^2}{(k-1)^3}$. The third run will have a length of $\frac{nk^2}{(k-1)^3}$.

Similarly, each new run length will be multiplied by $\frac{k}{k-1}$ from the previous run length. □

6. A Special Case I: When k=2

When you look at $G(n,2)$ for $n \geq 2$, a nice pattern is made. The first 17 results for $G(n,2)$ are displayed in the table below.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$G(n,2)$	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5

From these results the following observations can be made about $G(n,2)$:

Theorem 3. $G(n,2) = 1+2n-2^{[1+\log_2 n]}$.

Proof. 1. All values are odd. From the sequence $\overbrace{\;1\;}, 2, 3, 4, 5, \ldots, n$, and because $k = 2$, our count will go "1, $\not{2}$, 3, $\not{4}$, 5, ..." Another way to look at our circle as numbered odd (O) and even (E) sectors: $\overbrace{\;O\;}, E, O, E, O, \ldots, n$. Since $k = 2$, we will land on every even sector the first time around the circle, leaving only odd numbered sectors. Thus, the winning sector must be odd.

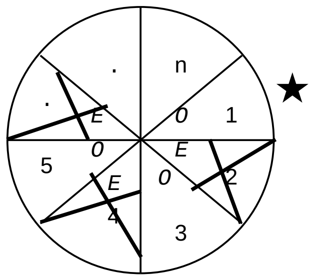

FIGURE 4

2. Except when $n = G(n,2)$, when you add one sector, the location of the winning sector goes up by 2.

3. If n is a power of 2, $G(n,2) = 1$. Suppose $n = G(n,2)$. If one sector is added, we have $n+1$ sectors. By definition of $G(n,k)$, we get $(G((n+1)-1,2)+2) \bmod (n+1)$. Then, $(G(n,2)+2) \bmod (n+1) = (n+2) \bmod (n+1) = 1$.

4. Results return to 1 in steps of powers of 2. For example, $G(4,2) = 1$ and $G(8,2) = 1$. It takes 4 steps, a power of 2, to reach 1 again. From our definition of a run length, if $G(n,2) = 1$, $G(n+i,2) = 1+2i$ for $1+2i \leq n+i$. Isolating i leaves $i = \frac{n-1}{k-1} \approx n$. Thus, it will have run length n, and we know from the previous point n is a power of 2. Likewise, $2n$, the number of sectors at the end of the run, is also a power of 2. □

7. A Special Case II: When n=3

When $n = 3$ the following pattern occurs: $3, 3, 2, 2, 1, 1, 3, 3, 2, 2, 1, 1, \ldots$. This pattern is generated by the following formula:

$$G(3,k) = 3 - ([\frac{k-1}{2}] \bmod '3). \tag{8}$$

Note: If $k = 1$, then $G(n,1) = n$ and the value of $G(3,1) = 3$. There are no violations to the Josephus Problem when $k = 1$. Each sector you pass is crossed out and by the time you reach the last sector, that is the only sector remaining.

8. A Special Case III: When n=k

Theorem 4. *If $n = k$, then $G(n,k) = G(n-1,k)$ for $k \geq 3$.*

Proof. Suppose $n = k$. Let's also assume we have a circle drawn with n sectors. Written as a sequence from 1 to n, we have

$$\overbrace{\;1\;}, 2, \ldots, n-1, n. \tag{9}$$

Since $n = k$, the first sector to be crossed out will be n and we are left with the following circle:

$$\widehat{1}, 2, \ldots, n-1. \tag{10}$$

If we look at another circle drawn with $n-1$ sectors, it will be written as

$$\widehat{1}, 2, \ldots, n-1. \tag{11}$$

Because sequences (10) and (11) contain the same number of sectors and starting position, a circle with n sectors and $n-1$ sectors will follow the same order of sectors crossed out. Therefore, if $n = k$, then

$$G(n,k) = G(n-1,k). \tag{12}$$

Remember, it won't work for $k = 2$. By definition, $G(2,2) = G(1,2)$, but no circle can have 1 sector uncrossed.

□

9. The Palindrome Principle

The rectangular grid on page 9 plots all the winning sectors for any n, k. Imagine two parallel lines drawn in such that they enclose all values $G(k,k)$ and $G(k-1,k)$ for $k \geq 4$.

Observe the following sequences near the parallel lines:
When $k = 4$, the sequence $1, 2, 2, 1$ occurs. The same holds for $k = 5$.
When $k = 6$, the sequence $1, 3, 4, 4, 3, 1$ occurs.
When $k = 7$, the sequence $2, 4, 5, 5, 4, 2$ occurs.

All these sequences form palindromes. The number of elements in the palindrome is always even. Since the middle of the palindrome contains the two elements $G(k,k)$ and $G(k-1,k)$, an additional two elements must be added outside these to have a palindrome property.

Consider the following chart:

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$G(k,k)$	2	2	4	5	4	8	8	7	11	8	13	4	11	12	8	12	2
no. of elements/2	2	2	3	3	3	4	4	4	5	4	5	3	5	5	4	5	2

Not only does $G(k,k)$ determine how many elements there will be, it also determines the whole palindromic sequence. $G(k,k)$ relates to a certain triangular number displayed below. (See Figure 5.) Locate $G(k,k)$ and find the corresponding row that contains the value $G(k,k)$. The row number doubled will give you the number of elements in the palindromic sequence.

For example, when $k = 12$, $G(12,12) = 11$. Eleven is located on the 5th row, indicating there are $5 \cdot 2 = 10$ elements in the palindromic sequence. Eleven also tells us the elements of the of the palindromic sequence:

$$1, 5, 8, 10, 11, 11, 10, 8, 5, 1. \tag{13}$$

The difference between $G(k,k)$ and $G(k-1,k)$ is zero. Going to the left or right from that point, the difference between each two numbers increases by one. For example, when $k = 12$, $11-11 = 0$, $11-10 = 1$, $10-8 = 2$, $8-5 = 3$, and $5-1 = 4$.

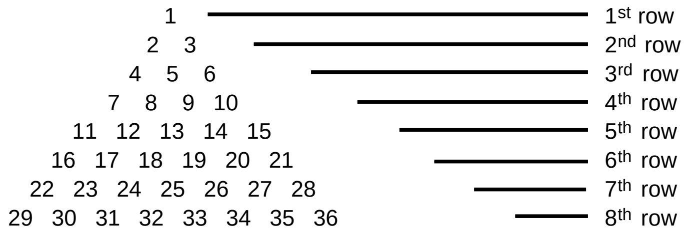

FIGURE 5

Theorem 5. *Let r be the index of the row in Figure 5 containing $G(k,k)$. Then for $i = 1, 2, \ldots, r$, $G(k+i,k) = G(k-i-1,k)$ and $G(k,k) - G(k+i,k) = G(k-1,k) - G(k-i-1) = i$.*

Proof. Let's assume b is the value of $G(k,k)$. Then by equation (12), $G(k-1,k) = b$. The sequences written out with $k-1$ and k sectors are written below.

$$\overbrace{1}, 2, \ldots, b, \ldots, k-2, k-1. \tag{14}$$

$$\overbrace{1}, 2, \ldots, b, \ldots, k-1, k. \tag{15}$$

From sequence (14), if we get rid of the next sector, it will be the one in the starting position, leaving us with $k-2$ sectors:

$$\overbrace{2}, 3, \ldots, b, \ldots, k-2, k-1. \tag{16}$$

By definition of $G(n,k)$ and the assumption $G(k-1,k) = b$, we know that $G((k-1),k) = (G(k-2,k)+k) \bmod (k-1)$, or $b = (G(k-2,k)+k) \bmod (k-1) = (G(k-2,k)+1) \bmod (k-1)$. This implies $b-1 = G(k-2,k)$ since $G(k-2,k) < k-1$.

Let's look at a circle with $k+1$ sectors:

$$\overbrace{1}, 2, \ldots, b, \ldots, k, k+1. \tag{17}$$

By definition of $G(n,k)$ and the assumption $G(k,k) = b$, $G(k+1,k) = (b + k) \bmod (k+1) = (b-1) \bmod (k+1)$. This means $G(k+1,k) = b-1$ unless $b = 1$. Therefore, $G(k+1,k) = G(k-2,k)$ from the previous paragraph.

Let's cross off the next sector from sequence (16). It will be the one in the second position, leaving us with $k-3$ sectors:

$$\overbrace{4}, \ldots, b, \ldots, k-2, k-1, 2. \tag{18}$$

By definition of $G(n,k)$ and that $G(k-2,k) = b-1$, we know that $G((k-2),k) = (G(k-3,k)+k) \bmod (k-2)$, or $b-1 = (G(k-2,k)+k) \bmod (k-2) = (G(k-3,k)+1) \bmod (k-2)$. This implies $b-3 = G(k-3,k)$ since $G(k-3,k) < k-2$.

Let's look at a circle with $k+2$ sectors:

$$\overbrace{1}, 2, \ldots, b, \ldots, k, k+1, k+2. \tag{19}$$

By definition of $G(n,k)$ and the assumption $G(k+1,k) = b-1$, $G(k+2,k) = (b-1+k) \bmod (k+2) = (b-3) \bmod (k+2)$. This means $G(k+2,k) = b-3$ unless $b = 3$. Therefore, $G(k+2,k) = G(k-3,k)$ from the previous paragraph. Similarly, for any $G(k+i,k) = G(k-i-1,k) = b - T(i)$, where $T(i)$ is the ith triangular number defined as $T(i) = \frac{i(i+1)}{2}$.

If we take the difference between $G(k,k)$ and $G(k+1,k)$, $b-(b-1) = 1$. Likewise, the difference between $G(k-1,k)$ and $G(k-2,k)$ is 1. The difference between $G(k+1,k)$ and $G(k+2,k)$ is 2, $(b-1)-(b-3) = 2$. Similarly, the difference between $G(k+i-1,k)$ and $G(k+i,k)$ is i. From the previous paragraph, it can also be deduced that $G(k-i,k) - G(k-i-1,k) = i$.

Now it will be explained why $G(k,k)$ determines the number of elements in the palindromic sequence. First, $G(k,k) \geq 2$. If $G(k,k) = 1$, $b-1 = 0$, and there is no zero, nor any negative number displayed as a position of one of the sectors. Thus, $G(k,k)$ determines the maximum value of i. In addition, depending on what $G(k,k)$ is, we subtract 1 from it, then 2 from that, and so forth until we reach zero or a negative number. $G(k,k)$ is the value of the middle elements of the palindromic sequence, and each one after that is determined by the previous element.

□

Note: The proof above implies that $G(k,k)$ can never have the value of a triangular number. The sectors crossed out in sequences (14), (16), and (18), are 1, 3, and 6, respectively. If the sequence after (18) is written out, the next sectors crossed out will be the 10, then 15, and so on as long as $n \geq T(i)$.

The proof above only guarantees values within the palindromic sequence, but the subtraction continues outside of the palindrome. Consider $G(7,12) = 1$, the outers member of the palindrome. The next whole number in the pattern is 5, the next number needed to be subtracted from 1. If we instead went back 5 spaces from sector #1, we would land on 3, the value of $G(6,12)$.

$$3, 4, 5, 6, 7, \widehat{\;1\;}. \tag{20}$$

The same holds true at the other end. $G(16,12) = 1$ and 5 needs to be subtracted. Going back 5 spaces lands us on 13, the value of $G(17,12)$.

$$13, 14, 15, 16, 17, \widehat{\;1\;} \tag{21}$$

Note: When going to the left, we use the number of sectors before going backward. For example, we were finding the value of $G(6,12)$ and we used 7 sectors to count backward.
Note: When going to the right, we use the number of sectors used after going backward. For example, we were finding the value of $G(17,12)$ and we used 17 sectors to count backward.

$k\backslash n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5	7	9	11
3	2	2	1	4	1	4	7	1	4	7	10	13	2	5	8	11	14	17	20	2
4	1	2	2	1	5	2	6	1	5	9	1	5	9	13	1	5	9	13	17	21
5	2	1	2	2	1	6	3	8	3	8	1	6	11	1	6	11	16	2	7	12
6	1	1	3	4	4	3	1	7	3	9	3	9	1	7	13	2	8	14	20	5
7	2	3	2	4	5	5	4	2	9	5	12	6	13	5	12	2	9	16	3	10
8	1	3	3	1	3	4	4	3	1	9	5	13	7	15	7	15	5	13	1	9
9	2	2	3	2	5	7	8	8	7	5	2	11	6	15	8	17	8	17	6	15
10	1	2	4	4	2	5	7	8	8	7	5	2	12	7	1	11	3	13	3	13
11	2	1	4	5	4	1	4	6	7	7	6	4	1	12	7	1	12	4	15	5
12	1	1	1	3	3	1	5	8	10	11	11	10	8	5	1	13	7	19	11	2
13	2	3	4	2	3	2	7	2	5	7	8	8	7	5	2	15	10	4	17	9
14	1	3	1	5	1	1	7	3	7	10	12	13	13	12	10	7	3	17	11	4
15	2	2	1	1	4	5	4	1	6	10	1	3	4	4	3	1	16	12	7	1
16	1	2	2	3	1	3	3	1	7	1	5	8	10	11	11	10	8	5	1	17
17	2	1	2	4	3	6	7	6	3	9	2	6	9	11	12	12	11	9	6	2
18	1	1	3	1	1	5	7	7	5	1	7	12	2	5	7	8	8	7	5	2
19	2	3	2	1	2	7	2	3	2	10	5	11	2	6	9	11	12	12	11	9
20	1	3	3	3	5	4	8	1	1	10	6	13	5	10	14	17	1	2	2	1
21	2	2	3	4	1	1	6	9	10	9	6	1	8	14	3	7	10	12	13	13

Table 1

References

[1] Graham, R.L., Knuth, D. E., and Patashnik, O. (1998). *Concrete Mathematics: A Foundation for Computer Science,* Reading, MA: Addison-Wesley.