COUNTING DIRECTED-CONVEX POLYOMINOES ACCORDING TO THEIR PERIMETER

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Abstract. An approach is presented for the enumeration of directed-convex polyominos that are not parallelogram polyominoes and we establish that there are $\binom{2n}{n-2}$ with a perimeter of $2n+4$. Finally using known results we prove that there are $\binom{2n}{n}$ directed-convex polyominos with a perimeter of $2n+4$.

1. Introduction

By a *directed-convex polyomino* we mean a *convex polyomino* which contains the lower left corner of its minimal bounding rectangle. By a *convex polyomino* we mean a connected union of a finite number of unit squares whose vertices are lattice points of the x, y plane such that all intersections with horizontal or vertical lines are connected sets. Finally by a *parallelogram polyomino* we mean a convex polyomino that contains the lower left and upper right corners of its minimal bounding rectangle. Until the last part of this paper we don't consider parallelogram polyominoes to be directed-convex polyominoes (see Figure 1 for a directed-convex polyomino and a parallelogram polyomino).

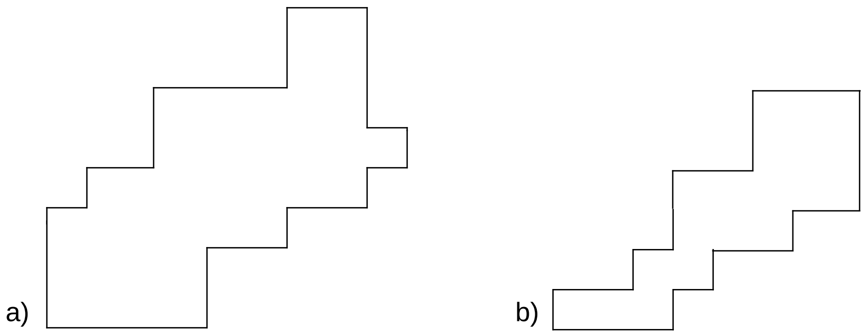

Figure 1. (a) Directed-convex Polyomino (b) Parallelogram Polyomino

In our approach to enumerating the directed-convex polyominos we introduce a new type of polyomino that we call *tail-dc-polyominoes.* A tail-dc-polyomino is composed of a "tail" and a directed-convex polyomino. After computing the generating function for the tail-dc-polyomino and for its tail we are finally able to deduce the generating function for the directed-convex polyominos that we easily identified with the one for $\binom{2n}{n-2}$.

2. Tail-dc-polyominoes and their generating function

A way to define a tail-dc-polyominoes is as a triple of paths. First is a pair of paths composed of north and east steps (respectively $(0,1)$ and $(0,1)$) where one of the paths starts at $(0,0)$ with a north step and ends at $(m,b-1)$ and the other starts also at $(0,0)$ but with an east step and ends at $(a-1,n)$ where $a-1<m$ and $b-1<n$. Then we have a third path which is composed of south and east steps starting at $(m,b-1)$ with an east step followed by a south step and ending at $(a-1,n)$ with the same two steps. (See Figure 2).

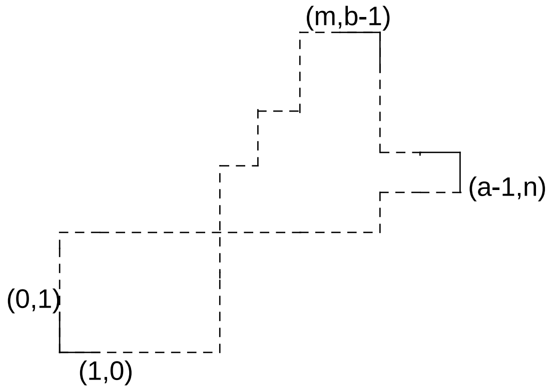

Figure 2. Tail-dc-polyomino

So the generating function for the tail-dc-polyominoes is:

$$T_1(x,y)=\sum_{m,n=0}^{\infty}\sum_{a=0}^{m-2}\sum_{b=0}^{n-2}\binom{n+a-1}{a}\binom{m+b-1}{m-1}\binom{m+n-a-b-4}{n-b-2}x^m y^n. \tag{1}$$

where the coefficient of $x^m y^n$ in $T_1(x,y)$ is the number of tail-dc-polyominoes that have as minimal bounding rectangle a rectangle of width m and height n. But this generating function can be simplified and in order to do so we need to make the following observations.

Remark. By $\mathrm{CT}_x\, f(x)$ we mean the constant term of x in $f(x)$. For example if $f(x)=x+y+3$ then $\mathrm{CT}_x\, f(x)=y+3$.

Lemma 2.1.
$$\binom{n}{k} = \operatorname*{CT}_{\alpha} \frac{(1+\alpha)^n}{\alpha^k}$$

Proof. By the binomial theorem we have
$$\binom{n}{k} = \operatorname*{CT}_{\alpha} \sum_{i=0}^{\infty} \binom{n}{i} \alpha^{i-k}$$
and since
$$\sum_{i=0}^{\infty} \binom{n}{i} \alpha^{i} = (1+\alpha)^n$$
we can conclude that
$$\binom{n}{k} = \operatorname*{CT}_{\alpha} \frac{(1+\alpha)^n}{\alpha^k}.$$
□

Lemma 2.2. *Let $Q(\alpha; x, y) \in \mathbb{Q}[\alpha][[x, y]]$ be a formal power series in x and y. If $u \in \mathbb{Q}[[x, y]]$ with constant term zero, then*
$$\operatorname*{CT}_{\alpha} \frac{\alpha}{\alpha - u} Q(\alpha; x, y) = Q(u; x, y).$$
where $\frac{\alpha}{\alpha-u} Q(\alpha; x, y)$ is expanded in $\mathbb{Q}((\alpha))[[x, y]]$.

Proof. We know that
$$\frac{\alpha}{\alpha - u} = \frac{1}{1 - u/\alpha} = \sum_{n \geq 0}^{\infty} u^n/\alpha^n.$$
So for $k \geq 0$
$$\operatorname*{CT}_{\alpha} \frac{\alpha}{\alpha - u} \alpha^k = u^k.$$

This shows that the lemma holds when $Q(\alpha; x, y) = \alpha^k$. In the general case $Q(\alpha; x, y) = \sum_{k=0}^{\infty} q_k(x, y)\alpha^k$ where $q_k(x, y)$ is a formal series in x and y. Therefore by linearity we extend the lemma to all $Q(\alpha; x, y)$. □

We can use Lemma 2.1 to convert all the binomial coefficients in T_1 so
$$T_1(x, y) = \sum_{m,n=0}^{\infty} \sum_{a=0}^{m-2} \sum_{b=0}^{n-2} \operatorname*{CT}_{\alpha} \frac{(1+\alpha)^{a+n-1}}{\alpha^a} \operatorname*{CT}_{\beta} \frac{(1+\beta)^{m+b-1}}{\beta^b} \times \operatorname*{CT}_{\gamma} \frac{(1+\gamma)^{m+n-a-b-4}}{\gamma^{n-b-2}} x^m y^n.$$

This can also be written as

$$T_1(x,y) = \sum_{m,n=0}^{\infty} \sum_{a=0}^{m-2} \sum_{b=0}^{n-2} \operatorname*{CT}_{\alpha,\beta,\gamma} \frac{\alpha^{-a}(1+\alpha)^a}{(1+\gamma)^a} \frac{\beta^{-b}(1+\beta)^b\gamma^b}{(1+\gamma)^b} \times (1+\beta)^m(1+\gamma)^m x^m \frac{(1+\alpha)^n(1+\gamma)^n y^n}{\gamma^n} \frac{\gamma^2}{(1-\alpha)(1+\beta)(1+\gamma)^4}.$$

Then since $a-1 < m$ and $b-1 < n$ if we change the order of the summations we obtain the following result:

$$T_1(x,y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{m=a+2}^{\infty} \sum_{n=b+2}^{\infty} \operatorname*{CT}_{\alpha,\beta,\gamma} \frac{\alpha^{-a}(1+\alpha)^a}{(1+\gamma)^a} \frac{\beta^{-b}(1+\beta)^b\gamma^b}{(1+\gamma)^b} \times (1+\beta)^m(1+\gamma)^m x^m \frac{(1+\alpha)^n(1+\gamma)^n y^n}{\gamma^n} \frac{\gamma^2}{(1-\alpha)(1+\beta)(1+\gamma)^4}.$$

By using well known results on geometric series we can simplify this generating function even more. Using the fact that

$$\sum_{m=a+2}^{\infty} (1+\beta)^m(1+\gamma)^m x^m = \frac{((1+\beta)(1+\gamma)x)^{a+2}}{1-(1+\beta)(1+\gamma)x}$$

and

$$\sum_{n=b+2}^{\infty} \frac{(1+\alpha)^n(1+\gamma)^n y^n}{\gamma^n} = \frac{(\frac{(1+\alpha)(1+\gamma)y}{\gamma})^{b+2}}{1-\frac{(1+\alpha)(1+\gamma)y}{\gamma}},$$

we obtain

$$T_1(x,y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \operatorname*{CT}_{\alpha,\beta,\gamma} \frac{\alpha^{-a}(1+\alpha)^a}{(1+\gamma)^a} \frac{\beta^{-b}(1+\beta)^b\gamma^b}{(1+\gamma)^b} \frac{((1+\beta)(1+\gamma)x)^{a+2}}{1-(1+\beta)(1+\gamma)x} \times \frac{(\frac{(1+\alpha)(1+\gamma)y}{\gamma})^{b+2}}{1-\frac{(1+\alpha)(1+\gamma)y}{\gamma}} \frac{\gamma^2}{(1-\alpha)(1+\beta)(1+\gamma)^4}.$$

Finally by repeating the same procedure we get

$$T_1(x,y) = \operatorname*{CT}_{\alpha,\beta,\gamma} \frac{1}{1-(1+\beta)(1+\gamma)x} \frac{1}{1-\frac{(1+\alpha)(1+\gamma)}{\gamma}y} \frac{1}{1-\frac{(1+\alpha)(1+\beta)}{\alpha}x} \times \frac{1}{1-\frac{(1+\alpha)(1+\beta)}{\beta}y}(1+\alpha)(1+\beta)x^2y^2.$$

In order to simplify even more the generating function we are going to use Lemma 2.1. Define $Q(x,y,\gamma)$ by

$$Q(x,y,\gamma)=\frac{1}{1-(1+\beta)(1+\gamma)x}\frac{1}{1-\frac{(1+\alpha)(1+\beta)}{\alpha}x}\frac{1}{1-\frac{(1+\alpha)(1+\beta)}{\beta}y}(1+\alpha)(1+\beta)x^2y^2$$

so

$$\begin{aligned}T_1(x,y)&=\operatorname*{CT}_{\alpha,\beta}\operatorname*{CT}_{\gamma}\frac{1}{1-\frac{(1+\alpha)(1+\gamma)}{\gamma}y}Q(x,y,\gamma)\\&=\operatorname*{CT}_{\alpha,\beta}\operatorname*{CT}_{\gamma}\frac{\gamma}{\gamma-(1+\alpha)y-\gamma(1+\alpha)y}Q(x,y,\gamma)\\&=\operatorname*{CT}_{\alpha,\beta}\operatorname*{CT}_{\gamma}\frac{\gamma}{\gamma(1-(1+\alpha)y)-(1+\alpha)y}Q(x,y,\gamma)\\&=\operatorname*{CT}_{\alpha,\beta}\operatorname*{CT}_{\gamma}\frac{\gamma}{\gamma-\frac{(1+\alpha)y}{1-(1+\alpha)y}}\frac{Q(x,y,\gamma)}{1-(1+\alpha)y}.\end{aligned}$$

Then by Lemma 2.2,

$$T_1(x,y)=\operatorname*{CT}_{\alpha,\beta}\frac{Q\left(x,y,\frac{(1+\alpha)y}{1-(1+\alpha)y}\right)}{1-(1+\alpha)y},$$

which after simplifying is equal to

$$\begin{aligned}T_1(x,y)=\operatorname*{CT}_{\alpha,\beta}\frac{1-y-\alpha y}{1-x-\beta x-y-\alpha y}\frac{1}{1-y(1+\alpha)}\frac{\alpha}{\alpha-(1+\alpha)(1+\beta)x}&\\\times\frac{\beta}{\beta-(1+\alpha)(1+\beta)y}(1+\alpha)(1+\beta)x^2y^2&\end{aligned}$$

Similarly we can use again Lemma 2.2 on

$$\begin{aligned}T_1(x,y)=\operatorname*{CT}_{\alpha}\operatorname*{CT}_{\beta}\frac{\beta}{\beta-(1+\alpha)(1+\beta)y}\frac{1-y-\alpha y}{1-x-\beta x-y-\alpha y}\frac{1}{1-y(1+\alpha)}&\\\times\frac{\alpha}{\alpha-(1+\alpha)(1+\beta)x}(1+\alpha)(1+\beta)x^2y^2,&\end{aligned}$$

which gives us

$$T_1(x,y)\quad=\quad\operatorname*{CT}_{\alpha}\frac{\alpha}{\alpha-\left(\frac{1+\alpha}{1-y-\alpha y}\right)x}\frac{(1-y-\alpha y)^2}{(1-y-\alpha y)^2-x}\frac{1}{1-y(1+\alpha)}\frac{1+\alpha}{1-y-\alpha y}x^2y^2.$$

Now define $P(x,y,\alpha)$ by

$$P(x,y,\alpha)=\frac{(1-y-\alpha y)^2}{(1-y-\alpha y)^2-x}\frac{1}{1-y(1+\alpha)}\frac{1+\alpha}{1-y-\alpha y}x^2y^2$$

so

$$T_1(x,y) = \operatorname*{CT}_{\alpha} \frac{\alpha}{\alpha - \left(\frac{1+\alpha}{1-y-\alpha y}\right)x} P(x,y,\alpha).$$

Let us factor the denominator of $\frac{\alpha}{\alpha-\left(\frac{1+\alpha}{1-y-\alpha y}\right)x}$ as following

$$\alpha - \left(\frac{1+\alpha}{1-y-\alpha y}\right)x = \frac{y}{y+\alpha y-1}\left(\alpha - \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2y}\right) \times \left(\alpha - \frac{1-x-y+\sqrt{(1-x-y)^2-4xy}}{2y}\right)$$

and let us define r_1 and r_2 as

$$r_1 = \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2y}$$

and

$$r_2 = \frac{1-x-y+\sqrt{(1-x-y)^2-4xy}}{2y}.$$

Then we notice that

$$r_1 r_2 = \frac{x}{y}$$

and $\frac{r_1}{x}$ is a power series in x and y with constant term 1. So now we can rewrite $T_1(x,y)$ as

$$\begin{aligned} T_1(x,y) &= \operatorname*{CT}_{\alpha} \frac{\alpha}{\frac{y}{y+\alpha y-1}(\alpha-r_1)(\alpha-\frac{x}{yr_1})} P(x,y,\alpha) \\ &= \operatorname*{CT}_{\alpha} \frac{\alpha}{\alpha-r_1} \frac{(y+\alpha y-1)\frac{yr_1}{x}}{y(\alpha\frac{yr_1}{x}-1)} \\ &= \operatorname*{CT}_{\alpha} \frac{\alpha}{\alpha-r_1} \frac{(1-y-\alpha y)\frac{r_1}{x}}{1-\alpha\frac{yr_1}{x}}. \end{aligned}$$

We notice that $\frac{(1-y-\alpha y)\frac{r_1}{x}}{1-\alpha\frac{yr_1}{x}}$ is a power series in x, y, and α thus we can use Lemma 2.2 on $T_1(x,y)$ and obtain after simplification the following result,

$$T_1(x,y) = \frac{1-x+y-\sqrt{(1-x-y)^2-4xy}}{y(x^2+(1-y)^2-2x(1+y))(1+x-y+\sqrt{(1-x-y)^2-4xy})}.$$

3. Decomposition of the Tail-dc-polyominos

From the definition of the tail-dc-polyominoes in section 2 it is not clear what is the exact relation between the directed-convex polyominoes and the tail-dc-polyominoes. Therefore in this section we are going to explicitly describe the relation between these polyominoes and give a decomposition of the tail-dc-polyominoes that involves directed convex polyominoes.

As we already know, a tail-dc-polyomino is composed of a pair of paths that starts at $(0, 0)$. If these two paths don't touch we can clearly see that the tail-dc-polyomino is actually a directed-convex polyomino. However if the two paths touch then we could decompose the tail-dc-polyomino into two parts: first the "tail" which is the pair of paths starting at $(0, 0)$ and ending at the last intersection of those paths and the "head" which is the rest of the tail-dc-polyomino (see Figure 3). The head is actually a directed-convex polyomino thus if we find an explicit formula for the generating function of the "tail" we can deduce a generating function for the directed-convex polyominoes.

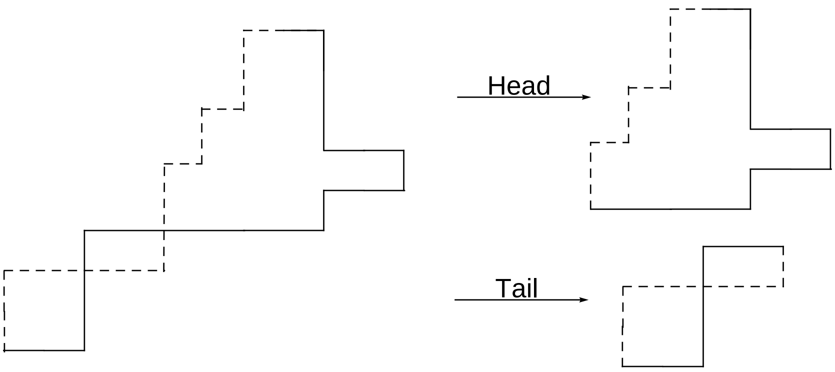

Figure 3. Decomposition of a tail-dc-polyomino

4. The tail part and its generating function

The tail of a tail-dc-polyomino, if it exists, could be defined as a pair of paths that start at $(0, 0)$, one with a north step and the other with an east step, and ending at some point (m, n) in the $x - y$ plane.

In a previous paper we were able to establish the generating function for ordered pairs of paths that started at $(0, 0)$ and ended at some point (m, n) of the $x-y$ plane. The generating function for these pairs of paths is

$$G(x, y) = \frac{1}{\sqrt{(1 - x - y)^2 - 4xy}}.$$

So the generating function for the tail is:

$$T_2(x,y) = \frac{G(x,y) - xG(x,y) - yG(x,y) - 1}{2} + 1.$$

Since order doesn't matter the first steps for the two paths are forced and the tail could be a pair of empty paths.

5. Generating function for the directed-convex polyominoes

We now have computed all the generating function needed in order to deduce the generating function for the directed-convex polyominoes which is:

$$T(x,y) = \frac{T_1(x,y)}{T_2(x,y)}.$$

After simplifying with Mathematica we obtain,

$$T(x,y) = \frac{2}{(1-x-y)(x^2+(1-y)^2-2x(1+y))} + \frac{2}{((-2+x)x+(1-y)^2))\sqrt{(1-x-y)^2-4xy}}$$

where the coefficient $x^m y^n$ is the number of directed-convex polyominoes whose minimal bounding rectangle has its upper right corner at coordinate (m,n). But we are interested in counting the directed-convex polyominoes by their perimeter. Thus the generating function of interest is

$$\begin{aligned} T_0(x) &= T(x,x) \\ &= \frac{2}{1-6x+8x^2+(1+2(-2+x)x)\sqrt{1-4x}}. \end{aligned}$$

After computing the first terms of the power series associated with $T_0(x)$ we noticed that they were the same as the coefficients of

$$C(x) = \sum_{n=2}^{\infty} \binom{2n}{n-2} x^n.$$

which we know has as an explicit formule:

$$C(x) = \frac{(1-\sqrt{1-4x})^4}{16x^4\sqrt{1-4x}}.$$

By subtracting $T_0(x)$ from $C(x)$ we can see using Mathematica that the difference is zero. The two generating function are equal. Therefore the number of directed-convex polyominoes that are not parallelogram polyominoes with a perimeter of $2n+4$ is $\binom{2n}{n-2}$.

6. Counting all the directed-convex polyominoes according to their perimeter

In 1969 George Polya established that the number of parallelogram polyominoes with perimeter $2n+4$ is the Catalan number

$$C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+2)(n+1)!(n+1)!}.$$

Let's compute the following:

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-2} &= \frac{(2n)!}{n!^2} - \frac{(2n)!}{(n-2)!(n+2)!} \\ &= \frac{(2n)!}{n!(n+2)!}\left((n+1)(n+2) - n(n-1)\right) = \frac{(2n)!}{n!(n+2)!}(4n+2) \\ &= \frac{(2n)!}{n!(n+2)!}\frac{2(2n+1)(2n+2)}{2(n+1)} \\ &= \frac{(2n+2)!}{(n+1)!(n+2)!} \\ &= C_{n+1}.\end{aligned}$$

So the number of directed-convex polyominoes with a perimeter of $2n+4$ is $\binom{2n}{n-2} + \left[\binom{2n}{n} - \binom{2n}{n-2}\right] = \binom{2n}{n}$.