

COUNTING THE NUMBER OF DYCK PATHS ACCORDING TO DIFFERENT TYPES OF PARAMETERS

OLEG SOFRYGIN

ABSTRACT. This paper is intended to count the number of Dyck paths according to several parameters, such as length, number of peaks, number of double rises, and number of returns to the x -axis. Generating function methods are used to count the number of paths as well as bijective methods are used to prove patterns between different parameters.

1. COUNTING DYCK PATHS ACCORDING TO THEIR LENGTH AND THE NUMBER OF PEAKS

Definition. A Dyck path is a path on the square lattice with steps $(1, 1)$ and $(1, -1)$ from $(0, 0)$ to $(2n, 0)$ that never falls below the x -axis.

Definition. We say that a Dyck path is prime if the only vertices at level 0 are the first and the last ones.

To count Dyck paths by peaks and length: assign to a Dyck path with $2n$ steps and j peaks a weight $x^n t^j$. Now the generating function $D(x, t)$ is equal to the sum of the weights of all Dyck paths,

$$(1) \quad D(x, t) = \sum_{k, n=0}^{\infty} f(n, k) x^n t^k,$$

where $f(n, k)$ is the number of Dyck paths of length $2n$ with k peaks. Hence $[x^n t^j] D(x, t)$ is equal to the number of Dyck paths with $2n$ steps and j peaks.

Let $P(x, t)$ be the generating function for prime Dyck paths with weights assigned to each peak and to each step. Then

$$D(x, t) = 1 + P(x, t) + P(x, t)^2 + P(x, t)^3 + \dots = \frac{1}{1 - P(x, t)}.$$

Next we will show that

$$P(x, t) = x(D(x, t) - 1) + tx.$$

Every prime path consists of an up step, an arbitrary Dyck path Q , and a down step. A prime path will have the same number of peaks as the path Q , unless the original prime Dyck path is xy , i.e. it consists of only one up step and one down step. In this case $xD(x, t)$ will not take into account a peak. So that's why

$$P(x, t) = x(D(x, t) - 1) + tx.$$

Therefore the generating function for the number of paths of length $2n$ with j peaks satisfies

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$$(2) \quad D(x, t) = \frac{1}{1 - x(D(x, t) - 1) - tx}.$$

Let's solve this for $D(x, t)$. From now on we write D for $D(x, t)$. We can write equation (2) as:

$$xD^2 + (tx - x - 1)D + 1 = 0.$$

Solving this gives 2 solutions :

$$D = \frac{1 + x - tx + \sqrt{(1 + x - tx)^2 - 4x}}{2x}$$

and

$$D = \frac{1 + x - tx - \sqrt{(1 + x - tx)^2 - 4x}}{2x}.$$

But the first solutions will contain a negative power of x in case of expanding it to the power series and expanding the second solution will contain only positive powers of x . Hence the second solution is the correct one.

Narayana numbers are:

$$(3) \quad N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.$$

One of the generating functions for the Narayana numbers is

$$(4) \quad \sum_{k,j=0}^{\infty} N(k, j) x^k t^j = \frac{1 - x - tx - \sqrt{(1 - x - tx)^2 - 4x^2 t}}{2x}.$$

We will show that our generating function $D(x, t)$ is equal to (4).

$$(5) \quad D = \frac{1 + x - tx - \sqrt{(1 + x - tx)^2 - 4x}}{2x}.$$

The expression under the square root of the generating function (5) is,

$$(1 + x - tx)^2 - 4x = 1 - 2x - 2tx + x^2 - 2tx^2 + t^2 x^2.$$

When the expression under the square root of the generating function (4) is,

$$(1 - x - tx)^2 - 4x^2 t = 1 - 2x - 2tx + x^2 - 2tx^2 + t^2 x^2.$$

Hence both expressions are equal and this shows that $D(x, t)$ and the generating function for the Narayana numbers are actually the same, with the only difference in the constant term. Therefore

$$f(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}, \text{ except that } f(0, 0) = 1$$

2. COUNTING DYCK PATHS ACCORDING TO THEIR LENGTH AND THE NUMBER OF DOUBLE RISES

To count Dyck paths by double rises and length: assign a Dyck path with $2n$ steps and j double rises a weight of $x^n t^j$. Now the generating function for the number of such Dyck paths is

$$K(x, t) = \sum_{k,n=0}^{\infty} l(n, k) x^n t^k$$

Let $p'(x, t)$ be the generating function for prime Dyck paths with weights assigned for each double rise and for each single step. Then

$$K(x, t) = 1 + p'(x, t) + p'(x, t)^2 + p'(x, t)^3 + \dots = \frac{1}{1 - p'(x, t)}.$$

Every prime path consists of an up step, an arbitrary Dyck path Q , and a down step. Adding one up step (and a down step) to an arbitrary Dyck path Q adds one more double rise, unless the original Dyck path Q is empty. Then the modified path will only consist of an up and a down step, hence it will have no double rises. This leads to the following generating function

$$p'(x, t) = xt(K(x, t) - 1) + x.$$

Hence

$$(6) \quad K(x, t) = \frac{1}{1 - xtK(x, t) + xt - x}.$$

Let's solve (6) for $K(x, t)$. From now on we write K for $K(x, t)$.

$$xtK^2 + (x - xt - 1)K + 1 = 0$$

We get the following generating function for K :

$$(7) \quad K = \frac{1 + xt - x - \sqrt{(1 + xt - x)^2 - 4xt}}{2xt}.$$

Next we show that (7) is the generating function for the Narayana numbers. In $K(x, t)$ we substitute t for $\frac{y}{x}$, making $K(x, \frac{y}{x})$

$$K = \frac{1 + y - x - \sqrt{(1 + y - x)^2 - 4y}}{2y}.$$

Now we change the variables x to y and y to x $K(x, \frac{y}{x})$ to $K(y, \frac{x}{y})$

$$K = \frac{1 + x - y - \sqrt{(1 + x - y)^2 - 4x}}{2x}.$$

On the next step we substitute y for xt , $K(xt, \frac{x}{xt}) = K(xt, \frac{1}{t})$

$$(8) \quad K = \frac{1 + x - xt - \sqrt{(1 + x - xt)^2 - 4x}}{2x}.$$

Equation (8) is equal to the generating function of $D(x, t)$, which is the generating function for the Narayana numbers. Hence we get that the number of Dyck paths of length $2n$ with k double rises is $l(n, k) = f(n, n - k)$.

3. COUNTING DYCK PATHS ACCORDING TO THEIR LENGTH AND THE NUMBER OF PEAKS OF HEIGHT AT LEAST 2

Generating functions method.

Assign a Dyck path with $2n$ steps and j peaks of height at least 2 a weight of $x^n t^j$. The generating function for the number of such paths is:

$$R(x, t) = \sum_{k, n=0}^{\infty} m(n, k) x^n t^k.$$

Let $p''(x, t)$ be the generating function for prime Dyck paths with weights assigned for each peak of height at least 2 and for each up step. Then

$$R(x, t) = 1 + p''(x, t) + p''(x, t)^2 + p''(x, t)^3 + \dots = \frac{1}{1 - p''(x, t)}.$$

But using results of the first section we get that

$$p''(x, t) = xD(x, t),$$

where $D(x, t)$ is the generating function for an arbitrary Dyck path with assigned weights for length and peaks at any height:

$$D(x, t) = \frac{1 + x - tx - \sqrt{(1 + x - tx)^2 - 4x}}{2x}.$$

Hence we get that the generating function $R(x, t)$ is expressed through the generating function for the number of Dyck paths of length $2n$ with k peaks that can be at any height,

$$(9) \quad R(x, t) = \frac{1}{1 - xD(x, t)}.$$

Next we show that the generating function $R(x, t)$ is the same as the generating function (7) from the previous section. To show that compute the value of $\frac{K(x, t)}{R(x, t)}$:

$$K(x, t)(1 - xD(x, t)),$$

which is the same as

$$\frac{1}{4xt}((1 + xt - x) - \sqrt{(1 + xt - x)^2 - 4xt})((1 + xt - x) + \sqrt{(1 + xt - x)^2 - 4xt})$$

this simplifies into $\frac{1}{4xt}(4xt) = 1$. This shows that the number of Dyck paths with length $2n$ and with k peaks of height at least 2 is actually the same as the number of Dyck paths with the length $2n$ and with k double rises. Hence $m(n, k) = f(n, n - k)$.

Bijjective proof.

Theorem 1. *Any prime Dyck path P , having peaks of height at least 2, can be decomposed into N prime Dyck paths, each having height $H \geq 2$ and peaks only at height H , where N is the number of different heights at which path P has peaks. Moreover, the sum of all peaks of height at least 2 of the decomposed prime paths is equal to the number of peaks of the path P ; the sum of double rises of the decomposed paths is equal to the number of double rises of P , and the sum of the lengths of the decomposed paths is equal to the length of P .*

Bijjective Proof (Theorem 1): Let P be a prime Dyck path, having $K \geq 2$ peaks of height at least 2, R the number of double rises, and N the number of different heights at which path P has peaks. Let S be the height of the lowest peak in P and T be the number of peaks at height S . Apply the following bijective mapping to the path P :

Remove $S - 1$ up and down steps from the bottom of P . M is the number of prime Dyck paths of height at least 2 left after removing $S - 1$ up and down steps from P . If $M > 1$ then do the the following bijective mapping to put these prime Dyck paths into one prime Dyck path: put M prime Dyck paths of height at least 2 next to each other, remove the last down step from the first path, the first and the last up and down steps from the next paths, and only the first up step from the last path in the sequence of M paths. Attach these paths to one another, eliminating

the gaps from up and down steps removal and add $M - 1$ up and down steps to the bottom of the resultant path. Figure 1 gives an example of such a bijection.

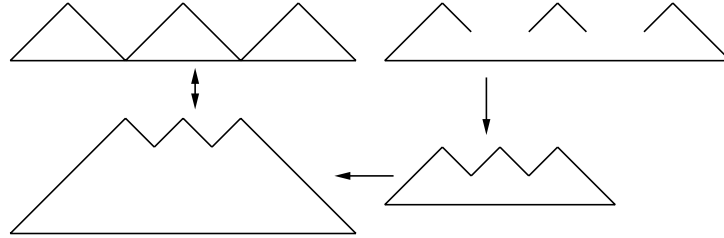


FIGURE 1. Making a prime Dyck path out of 3 prime Dyck paths of height 2

Let \hat{P} be this newly created path.

Concatenate T prime Dyck paths of length 2 together in one path and add $S - 1$ up and down steps to the bottom of it. We removed $S - 1$ double rises and T peaks from the original prime Dyck path P , creating a new path with $S - 1$ double rises and T peaks at height S . Hence the number of double rises and the number of peaks in the decomposition is the same as in the prime path P . Now apply the same mapping to the prime path \hat{P} $N - 1$ times.

Each iteration will create one more prime path of height at least 2, with peaks only at one level. After N iterations all paths in the decomposition will have peaks only at one level. Figure 2 gives an example of decomposition of a prime Dyck path with peaks at N different heights, into N prime paths having peaks only at one level.

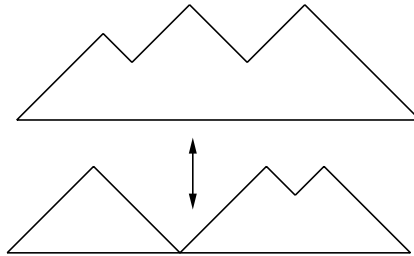


FIGURE 2. Making a prime Dyck path out of 3 prime Dyck paths of height 2

This completes the bijective proof for the **Theorem 1**.

Now we find a bijection from the Dyck path Q' of length $2n$ with k double rises to the Dyck path Q of length $2n$ with k peaks of height at least 2. This will prove that the number of Dyck paths of length $2n$ with k peaks of height at least 2 is the same as the number of Dyck paths of length $2n$ with k double rises. Or that $m(n, k) = f(n, n - k)$.

Mapping from Q to Q'

1) Decompose every prime path P of Q , having peaks of height at least 2 into N prime paths each of height at least 2, where N is the number of heights at which path P has peaks. This gives a new path \hat{Q} .

2) Every prime path in \hat{Q} has now peaks only at one height. For each prime path \hat{P} with peaks of height at least 2, let K be its height minus 2. For each path \hat{P} add K prime Dyck paths of length 2 to the left of \hat{P} and remove K up and down steps from the bottom of the path \hat{P} . Figure 3 shows an example of such mapping.



FIGURE 3. Removing extra double rises from the Dyck path

3) Now all the prime paths from \hat{Q} have height 2 or 1 and the sum of the number of peaks of height at least 2 the same as the number of peaks in the original prime path Q . Each modified prime path \hat{P} has only one double rise. For every modified path \hat{P} having S peaks remove $S - 1$ peaks (with up and down steps), adding $S - 1$ up and down steps to the bottom of the path.



FIGURE 4. Making a path with 2 double rises out of a path with 2 peaks

Now every path \hat{P} that originally had S peaks of height at least 2, has only one peak and S double rises.

Therefore we got a Dyck path Q' , which has k double rises and length $2n$.

Mapping from Q' to Q

1) Decompose every prime path P' of Q' , having peaks of height at least 2 into N prime paths each of height at least 2, where N is the number of heights at which path P' has peaks. This gives a new path \hat{Q}' .

2) Every prime path in \hat{Q}' has now peaks only at one height. For each prime path \hat{P}' with peaks of height at least 2, let K be the number of its peaks. Remove $K - 1$ peaks from each path \hat{P}' (with up and down steps) and add $K - 1$ prime



FIGURE 5. Removing extra peaks from a Dyck path

Dyck paths to the left of the path \hat{P}' . Example transformation is shown at the Figure 5

3) Now all the prime paths from \hat{Q}' have only 1 peak and the sum of the number of double rises each prime path has is the same as the number of double rises the original prime path Q' has. For every modified path \hat{P}' having height $S \geq 2$ add $S - 2$ peaks at height S and remove $S - 2$ up and down steps from the bottom of the path \hat{P}' .

Assuming the original path had $S - 1$ double rises, this mapping gives a path having $S - 1$ peaks of height 2. Example mapping is shown at Figure 6.



FIGURE 6. Making a path with 2 peaks out of a path with 2 double rises

Therefore we got a Dyck path Q , which has k peaks of height at least 2 and length $2n$.

The bijection between Q and Q' completes the proof of $m(n, k) = f(n, n - k)$.

$r \backslash t$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	5	4	3	2	1
2	0	4	9	8	10	0
3	0	3	8	6	0	0
4	0	2	10	0	0	0
5	0	1	0	0	0	0

TABLE 1. Joint distribution for the number of paths of length 12 with r double rises and t peaks of height at least 2

4. JOINT DISTRIBUTION OF DOUBLE RISES AND PEAKS OF HEIGHT AT LEAST 2
IN PATHS OF LENGTH $2n$.

Let $J(r, t)$ be the number of paths of length $2n$ with r double rises and t peaks of height at least 2. The sample values for the path of length 12 can be found in the Table 1. This data suggests that the joint distribution of the peaks of height at least 2 and the number of double rises is symmetric or, in other words, that the number of paths of length $2n$ with r double rises and t peaks of height at least 2 is equal to the number of paths of length $2n$ with t double rises and r peaks of height at least 2. We prove this by finding a bijection between the paths of length $2n$ with r double rises and t peaks of height at least 2, where $t > 1$ and paths of length $2n$ with $r + 1$ double rises and $t - 1$ peaks of height at least 2, hence proving that $J(r, t) = J(r + 1, t - 1)$. The more general result $J(r, t) = J(t, r)$ easily follows from this.

4.1. Bijection between the Dyck paths of length $2n$ with r double rises and t peaks of height at least 2, where $t > 1$ and Dyck paths of length $2n$, $r + 1$ double rises and $t - 1$ peaks of height at least 2. Let Q be the path of length $2n$, r double rises, t peaks of height at least 2 and Q' the path of length $2n$, $r + 1$ double rises, $t - 1$ peaks of height at least 2.

Mapping from Q to Q'

1) Decompose every prime path P in Q into N prime paths, each having peaks only at one height, where N is the number of different heights at which path P has peaks. (this is possible due to **Theorem 1** from Section 3).

Notice: Both, the sum of the peaks of height at least 2 of N prime Dyck paths and the sum of the double rises of N prime paths are equal to the number of peaks of height at least 2 and the number of double rises in the path P respectively.

The decomposition of all prime paths of Q gives us a new path \hat{Q} .

2) 2 cases are possible:

a) Consider a case where there are no prime paths in \hat{Q} with more than one peak of height at least 2. Then there must be at least 2 prime Dyck paths having height more or equal to 2. Take any prime path \hat{P} from \hat{Q} , with height $K \geq 2$. Remove K up and down steps from the bottom of the path \hat{P} , add K up and down steps to the bottom of any other prime path from \hat{Q} , having height more or equal to 2. This mapping removes one peak of height at least 2 from \hat{Q} , but adds one more double rise to the number of double rises of \hat{Q} .

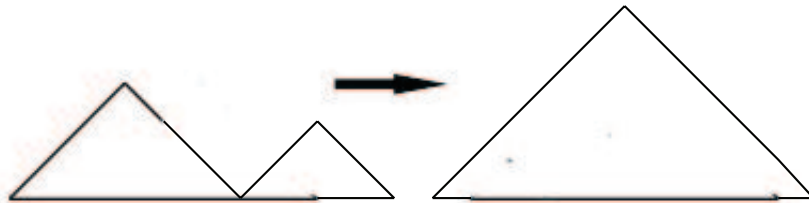


FIGURE 7. Removing one peak of height at least 2 and adding one more double rise to the path

b) Now consider the situation where there is at least one prime Dyck path with at least 2 peaks. Add one up step and one down step to the bottom of any prime Dyck path \hat{P} having more than 1 peak. Remove one peak from \hat{P} (both the up and down steps of the peak) moving the two pieces of the path together to close the gap created by the removed peak. This makes \hat{Q} with one less peak but one more double rise. The example of such mapping is in the Figure 8

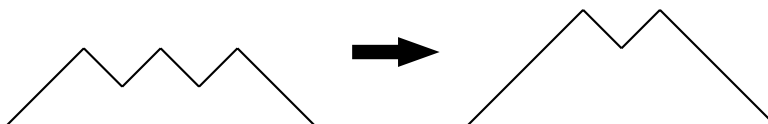


FIGURE 8. Removing one peak of height at least 2 and adding one more double rise to the path

In both of the cases we get a mapping from Q to Q' .

Mapping from Q' to Q

1) Decompose every path P' of Q' into N prime paths, each having peaks only at one height, where N is the number of different heights at which path P' has peaks. (this is possible due to **Theorem 1** from Section 3). This decomposition of all prime paths of Q' gives us a new path \hat{Q}' .

2) As in the above mapping of Q to Q' there are two cases to consider:

a) The maximum height of the prime Dyck path in \hat{Q}' is 2. Then there are at least 2 prime Dyck paths of height 2, since $r + 1 \geq 2$. Do the mapping, as below, adding one more peak and removing one double rise from \hat{Q}' .



FIGURE 9. Removing one double rise and adding one more peak to the path

b) Secondly, suppose there is a prime Dyck path \hat{P}' of height more than 2. Remove one up and down step from the bottom of \hat{P}' and add one more peak to \hat{P}' to the same height where all the other peaks of \hat{P}' are located:

In both cases we get a mapping of Q' to Q .

5. COUNTING DYCK PATHS ACCORDING THEIR LENGTH AND THE NUMBER OF RETURNS TO THE X-AXIS

Definition. The ballot numbers are defined by the following formula

$$(10) \quad B(m, n) = \frac{m-n}{m+n} \binom{m+n}{n} \text{ for } (m, n) \neq (0, 0).$$



FIGURE 10. Removing one double rise and adding one more peak to the path

The number of Dyck paths having length $2n + 2m$ and m returns to the x -axis is counted by the ballot numbers. This is proved by finding a bijection between such paths and paths having length $2n + m$, ending at height m and never touching the x -axis, these paths are known to be counted by the ballot numbers.

The number of returns to the x -axis in an arbitrary Dyck path Q is equal to the number of prime paths into which the arbitrary Dyck path Q can be factored. Each of the ballot numbers $B(n, m)$ counts the number of paths that start at $(0, 0)$ and end at (m, n) . Hence the ballot number $B(n + m, n)$ counts the number of paths that start at $(0, 0)$, never touch the x -axis and end at $(2n + m, m)$, with steps $(1, 1)$ and $(1, -1)$.

Now we show a bijection between paths that end at height 1 and prime Dyck paths: if path U ends at height 1 and never touches the x -axis, adding one down step to the end of the path U makes it a prime Dyck path.

The next step is to describe a bijection between a path S that ends at $(2n + m, m)$ and a Dyck path D having length $2n + 2m$ and consisting of m prime paths. Any path ending at height m can be factored uniquely into m paths ending at height 1, each of them having a bijection to a prime Dyck path. So we get bijection to m prime paths from any path ending at height m and never touching the x -axis. These m prime paths can be concatenated together at zero level to get the resulting Dyck path consisting of m prime paths and having length $2n + 2m$.

Hence the number of paths that end at height m with the length $2n + m$ and never touch the x -axis, with steps $(1, 1)$ $(1, -1)$ is equal to the number of Dyck paths of length $2n + 2m$ that can be decomposed into m prime paths. Hence the ballot number $B(n + m, n)$ counts the number of Dyck paths of length $2n + 2m$ and with m returns to x -axis.