The Specification of 2-trees

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Abstract

We derive new functional equations at a species level for certain classes of 2-trees, including a dissymmetry theorem. From these equations we deduce various series expansions for these structures. We obtain formulas for unlabeled 2-trees which are more explicit than previously known results. Moreover, the asymptotic behavior of unlabeled 2-trees is established.

Résumé

Nous présentons de nouvelles équations fonctionnelles pour certaines classes de 2-arbres, incluant un théorème de dissymétrie. Nous en déduisons diverses séries génératrices associés à ces espèces. Nous obtenous ainsi des formules énumératives pour les 2-arbres non-étiquetés qui sont plus explicites que les résultats connus jusquà présent. De plus le comportement asymptotique de ces structures est établi.

1 Introduction

The class $\mathcal{a}$ of 2-trees is defined recursively as the smallest class of simple graphs such that

1. The single edge is in $\mathcal{a}$,
2. If a simple graph G has a vertex x of degree 2 whose neighbors are adjacent and such that $G - x$ is in $\mathcal{a}$, then G is in $\mathcal{a}$.

A *triangle* of a 2-tree is a complete subgraph induced by three vertices. The definition implies that 2-trees are essentially triangles that are glued together along edges in a tree-like fashion. See Figure 1 for an example.

There is extensive literature about 2-trees. These structures and their k-dimensional generalization occur in various situations related, for example, to classification theory and graph colorings. See for instance [9], [18]. We mention here some of the previous work in enumerating 2-trees. Beineke and Pippert give formulas in [6] for counting vertex labeled 2-trees. Palmer gives formulas in [14] for 2-trees which are edge labeled or triangle labeled. In [10] and [11], Harary and Palmer enumerate unlabeled 2-trees as well. Palmer and Read enumerate plane 2-trees in [15]. See also the papers of Beineke and Moon [5], Rényi [17], and Foata [8].

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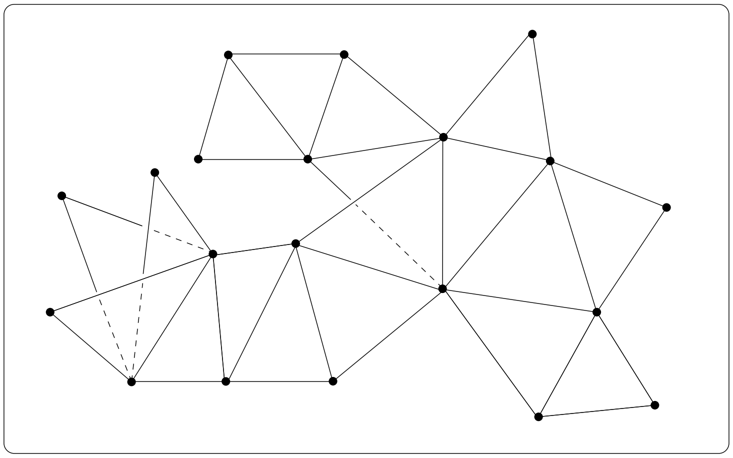

Figure 1: A 2-tree

In this paper we derive new functional equations for certain classes of 2-trees, including a dissymmetry theorem. From these equations we deduce various series expansions for these structures. We obtain enumerative formulas for unlabeled 2-trees which are more explicit than previously known results. In particular, recurrence relations are given for the number of unlabeled rooted 2-trees and (unrooted) 2-trees. Moreover, the asymptotic behavior of unlabeled 2-trees is established. The ultimate goal would be to compute the molecular decomposition of the species of 2-trees, i.e. a classification of these structures according to their automorphism groups. We come short of this but we show how to compute this molecular decomposition for the classes of oriented (edge) rooted and unrooted 2-trees.

The emphasis is put on triangle labeled 2-trees. Thus we consider $\mathfrak{a}$ as a species whose set of structures $\mathfrak{a}[U]$ on a set U is the set of 2-trees where the triangles are (labeled by) the elements of U. This concept should be intuitively clear. See Figure 2 where the set $\mathfrak{a}[U]$ of all 2-trees on U is given, with $U = \{a, b, c\}$. In some cases, for example in defining the edge orientations (see below), it may be necessary to also label the vertices (or the edges) and then go to isomorphism classes, i.e., unlabel the vertices (or edges).

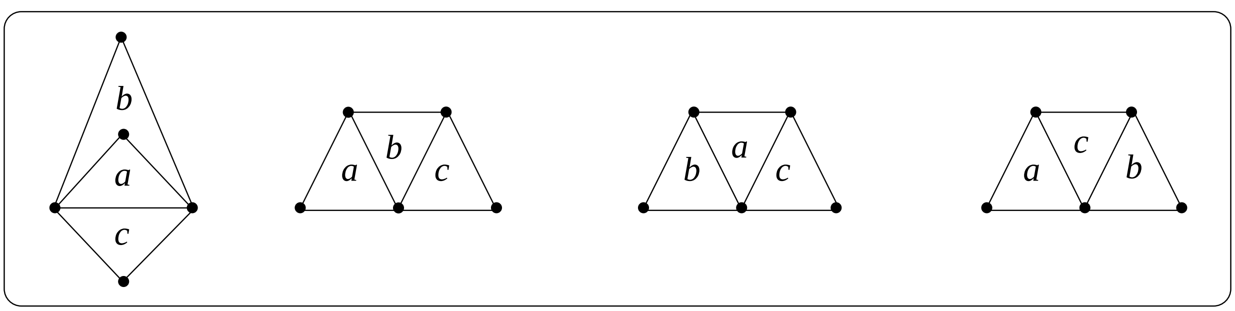

Figure 2: The 2-trees on the set of triangles $\{a, b, c\}$.

The transport of structures $\mathfrak{a}[\sigma] : \mathfrak{a}[U] \to \mathfrak{a}[V]$ along a bijection $\sigma : U \tilde{\to} V$ is simply given by the obvious relabeling procedure. If $\alpha \in \mathfrak{a}[U]$ and $\beta = \mathfrak{a}[\sigma](\alpha) \in \mathfrak{a}[V]$, we say that α and β are *isomorphic* 2-trees. The isomorphism classes of 2-trees then correspond to the usual unlabeled 2-trees.

We usually write $F = G$ to denote the existence of an isomorphism of species $\Phi : F \xrightarrow{\sim} G$, i.e. an invertible natural transformation. This means that there is a bijection $\Phi_U : F[U] \xrightarrow{\sim} G[U]$ for any finite set U, and that these bijections commute with the transport of structures along any bijection $\sigma : U \xrightarrow{\sim} V$, i.e., $G[\sigma] \circ \Phi_U = \Phi_V \circ F[\sigma]$. In that case, all the enumerative series associated to the species F and G will coincide. See [4] for more details on species.

2 Oriented-edge rooted 2-trees

A *coherent orientation* of a 2-tree is an orientation of its edges such that all triangles are circularly oriented. See Figure 3 a). Thus the orientation of a single edge determines a unique coherent orientation of the 2-tree. Let $\mathfrak{a}_O$ denote the species of coherently oriented 2-trees and let $\mathfrak{a}^{\to} = \mathfrak{a}_O^{-}$ denote the species of edge rooted coherently oriented 2-trees. Since the orientation of the root-edge completely determines the orientation of the other edges, it is equivalent to define $\mathfrak{a}^{\to}$ as the species of 2-trees with a distinguished and oriented edge. See Figure 3 b).

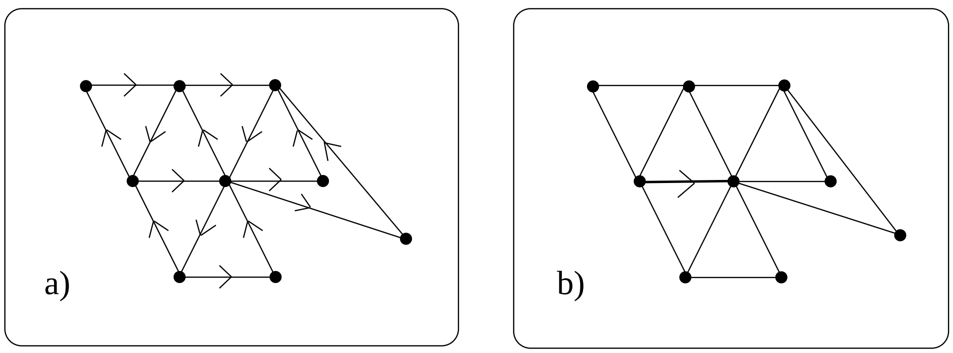

Figure 3: a) An $\mathfrak{a}_O$-structure b) An $\mathfrak{a}^{\to}$-structure

Theorem 1 *The species $B = \mathfrak{a}^{\to}$ satisfies the functional equation*

$$B = E(XB^2), \tag{1}$$

where E denotes the species of sets and X is the species of 2-trees consisting of a singleton triangle.

Proof: Let g be a 2-tree with a distinguished and oriented edge e. Then g can be seen as a set of so-called *pages*, that is, maximal sub-2-trees of g containing only one triangle adjacent to the root edge e. These pages are linked together by the root edge e but there is no specific order or structure between them. Thus we have the species equation

$$B = E(P), \tag{2}$$

where E denotes the species of sets and P denotes the species of pages, that is, of 2-trees with an oriented rooted *end*-edge. Included in (2) is the case of a single (oriented) edge with an empty set of pages. Consider now any page h of g. The triangle of h adjacent to e is called the *base* triangle. The orientation of e induces a coherent orientation of the base triangle. Define e_{in} and e_{out} to be the incoming and outgoing edges (resp.) of the base triangle, with respect to e, and, similarly define by h_{in} and h_{out} the maximal oriented-edge rooted 2-trees attached to the base triangle by the edges $e_{\rm in}$ and $e_{\rm out}$ respectively. These are rooted at the oriented edges $e_{\rm in}$ and $e_{\rm out}$ respectively. See Figure 4. Thus we have

$$P = XB^2,$$

where X represents the base triangle and B^2, the ordered pair $(h_{\rm in}, h_{\rm out})$. □

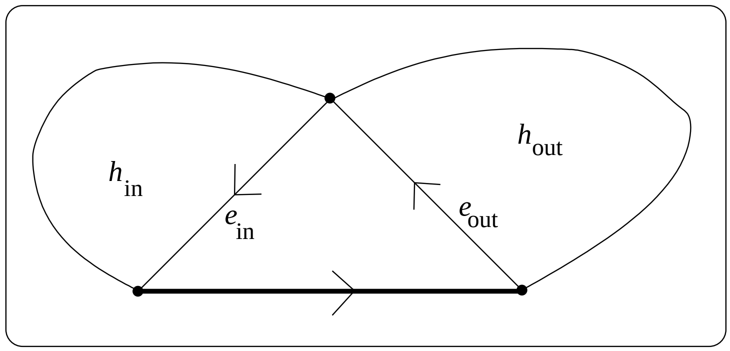

Figure 4: Decomposing a page of an $a^{\rightarrow}$-structure

The functional equation (1) is very strong. In particular, it implies a simple relationship between B and the species A of (vertex labeled) rooted trees. Recall that A is characterized by the functional equation

$$A = XE(A), \tag{3}$$

where X represents here the species of singleton vertices (the root).

Corollary 2 *The species $B = a^{\rightarrow}$ of oriented-edge rooted 2-trees satisfies*

$$B = \sqrt{\frac{A(2X)}{2X}}, \tag{4}$$

where A is the species of rooted trees.

Proof: From (1), we deduce

$$B^2 = E^2(XB^2) = E(2XB^2)$$

and

$$2XB^2 = 2XE(2XB^2).$$

From (3) we have the functional equation $A(2X) = 2XE(A(2X))$ whose solution is unique. Hence we find

$$2XB^2 = A(2X) \tag{5}$$

which is equivalent to (4). □

Let us introduce the generating series

$$B(x) = \sum_{n\geq 0} a_n^{\rightarrow} \frac{x^n}{n!}, \tag{6}$$

$$\widetilde{B}(x) = \sum_{n\geq 0} \widetilde{a}_n^{\rightarrow} x^n, \tag{7}$$

$$Z_B(x_1, x_2, \ldots) = \sum_{n_1, n_2, \ldots} a_{n_1,n_2,\ldots}^{\rightarrow} \frac{x_1^{n_1} x_2^{n_2} \cdots}{1^{n_1} n_1! 2^{n_2} n_2! \cdots}, \tag{8}$$

where $a_n^{\rightarrow}, \widetilde{a}_n^{\rightarrow}$ and $a_{n_1,n_2,\ldots}^{\rightarrow}$ are the numbers of oriented-edge rooted 2-trees which have n labeled triangles, n unlabeled triangles, or which are triangle labeled and left fixed under a given permutation of cycle type $1^{n_1} 2^{n_2} \cdots$, respectively.

Corollary 3 *For the species $B = a^{\rightarrow}$ of oriented edge rooted 2-trees, we have*

$$a_n^{\rightarrow} = (2n+1)^{n-1} \tag{9}$$

and

$$a_{n_1,n_2,\ldots}^{\rightarrow} = \prod_{i=1}^{\infty} (1 + 2\sum_{d|i} dn_d)^{n_i - 1} (1 + 2\sum_{\substack{d|i \\ d<i}} dn_d) \tag{10}$$

Moreover the numbers $b_n = \widetilde{a}_n^{\rightarrow}$ satisfy the recurrence

$$b_n = \frac{1}{n} \sum_{\substack{1\leq k\leq n \\ i+j+1|k}} (i+j+1) b_i b_j b_{n-k}, \qquad b_0 = 1. \tag{11}$$

Proof: Formulas (9) and (10) are proved by applying the composite Lagrange inversion formula for series and for cycle index series, respectively, (See [4], (3.1.10b) and (3.2.55), resp.). When applied to the functional equation $A(x) = x\exp(A(x))$ which follows from (3) with $F(x) = x^m$, Lagrange inversion yields

$$\left(\frac{A(x)}{x}\right)^m = \sum_{n\geq 0} m(m+n)^{n-1} \frac{x^n}{n!}. \tag{12}$$

Since each coefficient is a polynomial in m, we can replace m by $1/2$ and x by $2x$ to get, using (4),

$$a^{\rightarrow}(x) = \left(\frac{A(2x)}{2x}\right)^{\frac{1}{2}} = \sum_{n\geq 0} (2n+1)^{n-1} \frac{x^n}{n!} \tag{13}$$

which gives (9). Note that $m = 2n+1$ is the number of edges of a 2-tree with n triangles. Formula (9) can also be proved directly by giving a Prüfer type encoding for triangle-labeled oriented-edge rooted 2-trees.

Taking $F(X) = X^m$, where m is a positive integer, Lagrange inversion for cycle index series gives

$$[x_1^{n_1} x_2^{n_2} \ldots] Z_{(A(X))^m} = [t_1^{n_1} t_2^{n_2} \ldots]\ t_1^m \prod_{i=1}^{\infty} (1 - t_i) e^{n_i (t_i + \frac{1}{2} t_{2i} + \ldots)}. \tag{14}$$

Algebraic manipulations yield

$$[x_1^{n_1}x_2^{n_2}\ldots]Z_{(A(X)/X)^m} = [t_1^{n_1}t_2^{n_2}\ldots]\prod_{i=1}^{\infty}(1-t_i)e^{(m+\sum_{d|i}dn_d)\frac{t_i}{i}}. \tag{15}$$

Replacing X by X/m, we get

$$Z_{\left(\frac{A(X/m)}{X/m}\right)^m} = \sum_{n_1,n_2,\ldots}\frac{x_1^{n_1}x_2^{n_2}\ldots}{1^{n_1}n_1!2^{n_2}n_2!\ldots}\prod_{i=1}^{\infty}(1+\frac{1}{m}\sum_{d|i}dn_d)^{n_i-1}(1+\frac{1}{m}\sum_{d|i,d<i}dn_d). \tag{16}$$

Since each coefficient in the latter series is a polynomial in $\frac{1}{m}$, we can replace m by any real number. In particular, we can make the substitution $m=\frac{1}{2}$ to obtain (10). Note that formula (9) corresponds to the special case $n_1=n, 0=n_2=n_3=\ldots$.

To prove the recurrence (11), one can use the functional equation (1) directly to deduce that

$$\widetilde{B}(x) = \exp(\sum_{i\geq 1}\frac{x^i\widetilde{B^2}(x^i)}{i}), \tag{17}$$

and then take the logarithmic derivative. □

See Table 1 for the first 30 terms of the sequence $\widetilde{a}_n^{\rightarrow}$.

Remark 4 Formula (9) is coherent with (and equivalent to) the formula

$$\alpha_p = \binom{p}{2}(2p-3)^{p-4} \tag{18}$$

of Beineke and Pippert [6] for the number of 2-trees with p labeled vertices. Indeed we have $n=p-2$ and $m=(2p-3)=(2n+1)$ and, with obvious notations, $\alpha_p^{\rightarrow}=2m\alpha_p$ and $\alpha_p^{\rightarrow}=p(p-1)a_n^{\rightarrow}$. Hence we have

$$\alpha_p = \frac{1}{m}\binom{p}{2}a_n^{\rightarrow}. \tag{19}$$

□

Note that the molecular decomposition of the species $B=a^{\rightarrow}$ can be deduced from that of A (see [4], App. 2, Table 8), using (4). The first few terms are

$$\begin{aligned} a^{\rightarrow}(X) &= 1+X+E_2(X)+2X^2+E_3(X)+2XE_2(X)+7X^3+E_4(X)\\ &\quad 2XE_3(X)+2E_2(X^2)+10X^2E_2(X)+24X^4+\ldots \end{aligned} \tag{20}$$

A complete table, up to order 8, is found in the appendix. Roughly speaking, the molecular decomposition of a species corresponds to a classification of the unlabeled structures according to their order and their automorphism group (up to conjugacy). For example, the terms $E_3(X)$, $2XE_2(X)$ and $7X^3$ correspond to the unlabeled 2-trees with three triangles. They are illustrated in Figure 5. The automorphism group of the 2-tree corresponding to $E_3(X)$ is the symmetric group S_3 while the 7 different unlabeled 2-trees which correspond to the molecular species X^3 have a trivial automorphism group.

Note that the molecular decomposition of the species A of rooted trees can be recursively computed, using the functional equation (4) and addition formulas for the species E of sets (see [1]).

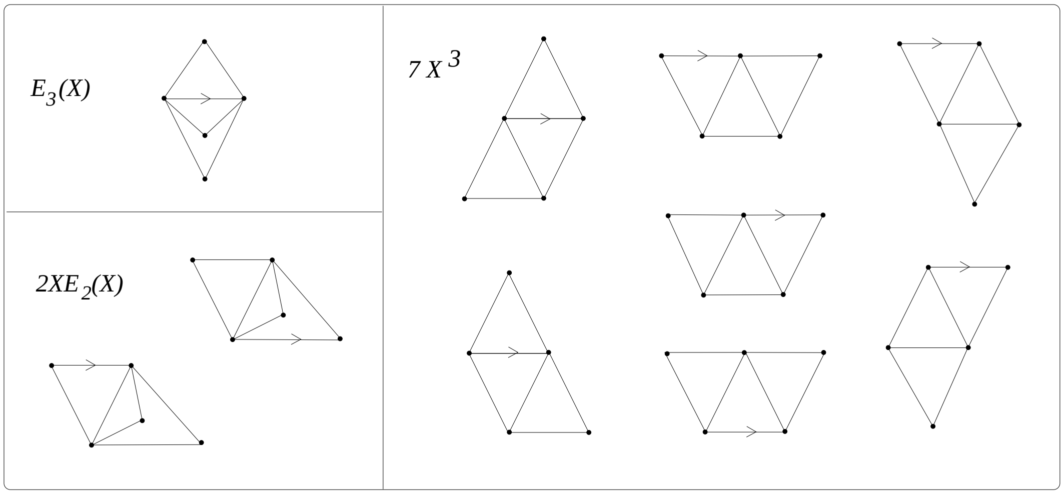

Figure 5: Molecular decomposition of the species of 2-trees over three triangles

3 The dissymmetry theorem for 2-trees

The goal of this section is to express the species $\mathfrak{a}$ of 2-trees in terms of the species $\mathfrak{a}^{\rightarrow}$ of oriented edge rooted 2-trees. The main tool for this is a dissymmetry theorem for 2-trees which is stated below. This theorem is closely related to the Dissimilarity Characteristic Theorem for 2-trees ([10, p. 74]). It yields the same unlabeled enumerative result but carries more information, for example for the cycle or asymmetry index series and the molecular decompositions. We first introduce some concepts which are fundamental in the proof.

A triangle t and edge e in a 2-tree are said to be *adjacent* if e is an edge of t. A *bi-path* in a 2-tree is a finite sequence $[x_1, x_2, \ldots, x_k]$, $k \geq 1$, such that the x_i are alternately edges and triangles, all distinct, and, for $1 \leq i \leq k-1$, x_i and x_{i+1} are adjacent. Any 2-tree, like a tree, has a well defined *center*, namely, the midpoint of a longest bi-path. This center, in analogy with the case of a bi-colored tree all of whose leaves have the same color, will be either an edge or a triangle.

Consider the following auxiliary species of rooted 2-trees:

$\mathfrak{a}^{-}$ = species of edge rooted 2-trees,
$\mathfrak{a}^{\triangle}$ = species of triangle rooted 2-trees,
$\mathfrak{a}^{\underline{\triangle}}$ = species of based-triangle rooted 2-trees.

By a based triangle, we mean a triangle with a distinguished edge. See Figure 6 for a schematic illustration of a based-triangle rooted 2-tree.

Theorem 5 Dissymmetry theorem for 2-trees *There is an isomorphism of species*

$$\mathfrak{a}^{-} + \mathfrak{a}^{\triangle} = \mathfrak{a} + \mathfrak{a}^{\underline{\triangle}}. \tag{21}$$

Proof: Let (g, p) be an $(\mathfrak{a}^{-} + \mathfrak{a}^{\triangle})$-structure, that is, a 2-tree g rooted at p, where p is either an edge or a triangle. If p is the center of g, then we map (g, p) to g. This corresponds to the term $\mathfrak{a}$ in the right-hand side. The rest of the proof is dedicated to the case when p is not the center of g.

If p is a triangle, set $t = p$ and let e be the edge of p which is closest to the center of g. If p is an edge, set $e = p$ and let t be the triangle containing e which is the closest to the center of g. In both cases map (g, p) to the $\mathfrak{a}^{\underline{\triangle}}$-structure $(g, (t, e))$.

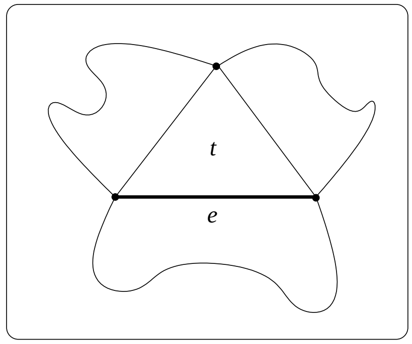

Figure 6: A based-triangle rooted 2-tree

Conversely, given an $\mathfrak{a}$-structure g, we root g at its center. Given an $\mathfrak{a}^{\underline{\triangle}}$-structure $(g,(t,e))$, we look for the center c of g. If t is in the unique bi-path from e to c, we root g at e. If instead e is in the bi-path from t to c, we root g at t.

These two constructions are clearly inverses of each other and do not depend on the particular triangle labelings. Therefore they give the desired isomorphism of species. $\square$

The next step is to express the species $\mathfrak{a}^{-}$, $\mathfrak{a}^{\triangle}$ and $\mathfrak{a}^{\underline{\triangle}}$ in terms of $\mathfrak{a}^{\rightarrow}$. Consider first the species $\mathfrak{a}^{-}$ of edge-rooted 2-trees. An $\mathfrak{a}^{-}$-structure g is similar to an $\mathfrak{a}^{\rightarrow}$-structure except that the rooted edge is not oriented. See Figure 7 a). Nevertheless, we can look at the various pages which are attached to the root edge. A crucial observation is that the non-root edges of the base triangle of each page can be naturally oriented, away from the root edge; see Figure 7 b). We then introduce the *skeleton* of g by restricting it to the root edge and the base triangle of each page. We keep the orientation of the non-root edges and we also label these edges in order to be able to reconstruct g. Note that the root edge is not labeled.

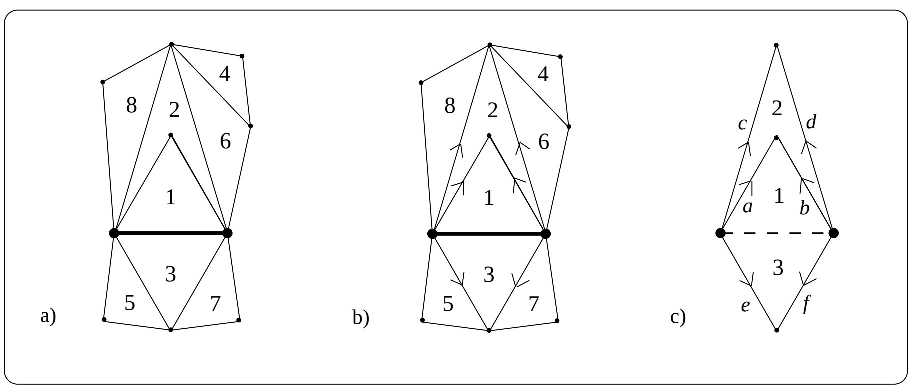

Figure 7: a) An $\mathfrak{a}^{-}$-structure b) The natural orientation of the base triangles c) The skeleton

These skeleton graphs actually define a two-sort species $Q(X,Y)$, where X represents the sort of triangles (the set $\{1,2,3\}$ in Figure 7 c)) and Y the sort of oriented edges (the set $\{a,b,\ldots,f\}$ in Figure 7 c)). To reconstruct g from its skeleton, we replace each oriented edge by the corresponding $\mathfrak{a}^{\rightarrow}$-structure. For example, the oriented edge d is replaced by the induced $\mathfrak{a}^{\rightarrow}$-structure containing the triangles 4 and 6 in Figure 7 b). This two sort species $Q(X,Y)$ will be analyzed further in the next section. However the above construction can be readily expressed in terms of species substitution (partitional composition). In the following, we write $\mathfrak{a}=\mathfrak{a}(X)$, $\mathfrak{a}^{-}=\mathfrak{a}^{-}(X)$, etc, to emphasize the fact these are triangle-labeled structures; note that X represents the species of singleton triangles.

Proposition 6 *The species $a^- = a^-(X)$ of edge rooted 2-trees can be expressed as*

$$a^-(X) = Q(X, a^{\rightarrow}(X)). \tag{22}$$

□

Similar decompositions can be carried out for the species $a^{\triangle}$ and $a^{\underline{\triangle}}$.

Proposition 7 *There exists skeleton species $S(X,Y)$ and $U(X,Y)$ such that*

$$\begin{aligned} a^{\triangle}(X) &= X \cdot S(X, a^{\rightarrow}(X)); & (23)\\ a^{\underline{\triangle}}(X) &= X \cdot U(X, a^{\rightarrow}(X)). & (24)\end{aligned}$$

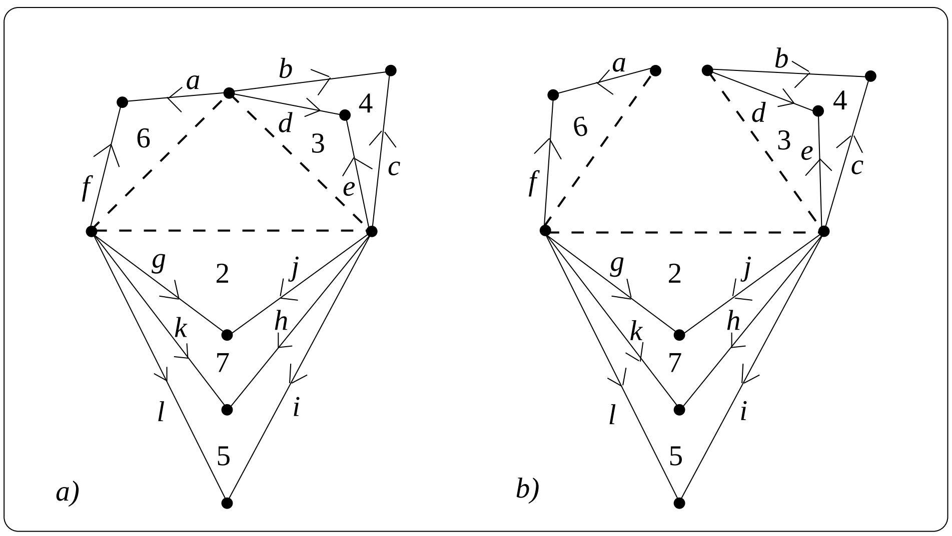

Figure 8: a) $S(X,Y)$-structure b) $U(X,Y)$-structure

Proof: In the first case, given a triangle rooted 2-tree, the edges of the root triangle form a 3-cycle (unoriented) of base edges to which various pages are attached. The skeleton is obtained by removing the root triangle (this explains the factor X in (23)) and keeping only the base triangle of each page. Again, the non-base edges are canonically oriented away from the base and labeled. This defines a species $S(X,Y)$, illustrated in Figure 8 a).

The case of an $a^{\underline{\triangle}}$-structure is similar. The main difference here is that the root triangle has a distinguished edge. It is then natural to separate the 3-cycle of this triangle at the vertex opposite to the distinguished edge. This yields a 3-chain (unoriented) of base edges to which various pages are attached. The resulting skeleton species $U(X,Y)$ is represented in Figure 8 b). □

4 Coherently oriented 2-trees

We would like to mention that there is also a dissymmetry theorem for coherently oriented 2-trees, with almost the same proof. Indeed, with the obvious notation, we have

$$a_O^- + a_O^{\triangle} = a_O + a_O^{\triangleq}. \tag{25}$$

Observe that $a^{\rightarrow} = a_O^-$ and that $a_O^{\triangle}$ and $a_O^{\triangleq}$ can be easily expressed as

$$a_O^{\triangle} = X \cdot C_3(a^{\rightarrow}), \qquad\qquad a_O^{\triangleq} = X \cdot (a^{\rightarrow})^3, \tag{26}$$

where C_3 denotes the species of circular permutations of order 3. Hence we have the following theorem.

Theorem 8 Dissymmetry theorem for coherently oriented 2-trees. *There is an isomorphism of species*

$$a^{\rightarrow} + X \cdot C_3(a^{\rightarrow}) = a_O + X \cdot (a^{\rightarrow})^3. \tag{27}$$

□

From Theorem 8, it is immediate to derive the next proposition, using the known formulas for $C_3(X)$ and Z_{C_3} (see (42)). Moreover, the molecular decomposition for a_O can be deduced from the dissymetry theorem for coherently oriented 2-trees, using (27) and addition formulas for C_3 (see [1]).

Proposition 9 *We have the following expressions for series associated with the species a_O of coherently oriented 2-trees in terms of the species $a^{\rightarrow}$ of oriented-edge rooted 2-trees.*

$$\begin{aligned} a_O(x) &= a^{\rightarrow}(x) - \frac{2}{3}x(a^{\rightarrow}(x))^3, & (28)\\ \widetilde{a}_O(x) &= \widetilde{a}^{\rightarrow}(x) + \frac{2}{3}x(\widetilde{a}^{\rightarrow}(x^3) - (\widetilde{a}^{\rightarrow}(x))^3), & (29)\\ Z_{a_O} &= Z_{a^{\rightarrow}} + \frac{2}{3}x_1((Z_{a^{\rightarrow}})_3 - (Z_{a^{\rightarrow}})^3). & (30) \end{aligned}$$

□

Corollary 10 *We have the following expressions for the number of labeled and unlabeled (respectively) coherently oriented 2-trees with n triangles:*

$$\begin{aligned} a_{O,n} &= (2n+1)^{n-2}, & (31)\\ \widetilde{a}_{O,n} &= b_n + \frac{2}{3}b_{\frac{n-1}{3}} - \frac{2}{3}\sum_{i+j+k=n-1} b_i b_j b_k, & (32) \end{aligned}$$

where $b_n = \widetilde{a}_n^{\rightarrow}$ is given by (11), and $b_x^{\rightarrow} = 0$ when x is non-integral.

Proof: Taking $m = \frac{3}{2}$ in (12), we have

$$(a^{\rightarrow}(x))^3 = \left(\frac{A(2x)}{2x}\right)^{\frac{3}{2}} = 3\sum_{n\geq 0}(3+2n)^{n-1}\frac{x^n}{n!} \tag{33}$$

from which (31) follows by using (28). It can also be seen directly that $a_n^{\rightarrow} = (2n+1)a_{O,n}$. Formula (32) immediately follows from (29). □

Remark 11 The following (known) identity follows directly from (28). It is used by Beineke and Moon in [5] as the basis of an inductive proof for the enumeration of labeled 2-trees.

$$(2n+1)^{n-2} = \frac{1}{3} \sum_{i+j+k=n-1} \binom{n-1}{i,j,k} (2i+1)^{i-1}(2j+1)^{j-1}(2k+1)^{k-1}. \tag{34}$$

5 Quotient species

In order to better understand the skeleton species Q, S and U, and to compute their associated series, we express them as quotient species. Information about quotient species can be found in [4] and [7]. We present here briefly an application of this concept to two-sort species. A group H is said to *act* on a $2-$sort species of structures F if for any ordered pair (U,V) of finite sets, there is a group action of H on the set $F[U,V]$ of F-structures on the pair (U,V),

$$\rho_{(U,V)} : H \times F[U,V] \to F[U,V]$$

such that for any α, β, the following diagram commutes.

$$\begin{array}{ccc} H \times F[U,V] & \stackrel{\rho_{(U,V)}}{\longrightarrow} & F[U,V] \\ \Big\downarrow 1_H \times F[\alpha,\beta] & & \Big\downarrow F[\alpha,\beta] \\ H \times F[U',V'] & \stackrel{\rho_{(U',V')}}{\longrightarrow} & F[U',V'] \end{array} \tag{35}$$

Here, $\alpha : U \to U'$ and $\beta : V \to V'$ are bijections, and $F[\alpha,\beta]$ is the transportation of structures along (α,β). See [4], section 2.4 for more information about multi-sort species.

The quotient species of F by H, denoted F/H, is defined by setting

$$(F/H)[U,V] = F[U,V]/H$$

for each ordered pair of finite sets (U,V), and by defining an appropriate transportation of structures using the commutative diagram (35). Here, $F[U,V]/H$ is the set of orbits of H in $F[U,V]$.

As a first example, we show that the species $Q(X,Y)$ can be described as the quotient of another species by the cyclic group of order 2, $\mathbb{Z}_2 = <\tau>$. Indeed, we can orient the base edge of a $Q(X,Y)$-structure so that there is now a left side and a right side. This defines an associated species $P(X,Y)$ which in fact is given by

$$P(X,Y) = E(XY^2). \tag{36}$$

We now define the action of $\mathbb{Z}_2$ on $P(X,Y)$ by reversing the orientation. More precisely, if $s = \{(1,(a_1,a_2)), (2,(a_3,a_4), \ldots, (k,(a_{2k-1},a_{2k}))\}$ is an $E(XY^2)$-structure, then we set $\tau \cdot s = \{(1,(a_2,a_1), (2,(a_4,a_3)), \ldots, (k,(a_{2k},a_{2k-1}))\}$. An orbit under this group action corresponds clearly to the original unoriented $Q(X,Y)$-structure. Hence we have:

Proposition 12 *There is an isomorphism of species*

$$Q(X,Y) = E(XY^2)/\mathbb{Z}_2. \tag{37}$$

□

Similarly, for the species $S(X,Y)$, we can orient the 3-cycles to obtain a species $R(X,Y) = C_3(E(XY^2))$ and have the generator τ of $\mathbb{Z}_2$ act by reversing the orientation. More precisely, if (s_1, s_2, s_3) is an oriented 3-cycle of $E(XY^2)$-structures, we set $\tau \cdot (s_1, s_2, s_3) = (\tau \cdot s_1, \tau \cdot s_3, \tau \cdot s_2)$. Finally, for $U(X,Y)$, the 3-chains can be oriented, yielding the species $T(X,Y) = E^3(XY^2)$ and the group action will reverse this orientation. More precisely, if $[s_1, s_2, s_3]$ is an ordered triple of $E(XY^2)$-structures, we set $\tau \cdot [s_1, s_2, s_3] = [\tau \cdot s_3, \tau \cdot s_2, \tau \cdot s_1]$. In conclusion, we have:

Proposition 13 *There are isomorphisms of species*

$$S(X,Y) = C_3(E(XY^2))/\mathbb{Z}_2, \tag{38}$$
$$U(X,Y) = E^3(XY^2)/\mathbb{Z}_2. \tag{39}$$

□

In order to compute the cycle index series of a quotient species, we need a lemma which is stated below for the case of one-sort species. Its extension to multi-sort species is straightforward. Let $F = F(X)$ be a species and H be a group acting on F. Let $s_1, s_2, \ldots$ be an infinite set of variables and let $x_k = p_k(s_1, s_2, \ldots) = \sum_{i \geq 1} s_i^k$ denote the power sum symmetric functions for $k = 1, 2, \ldots$ A *colored* F-structure on a set U is an F-structure f on U together with a coloring function $c : U \to \mathbb{N}^+ = \{1, 2, \ldots\}$. The weight $w(f, c)$ of a colored F-structure is the product of the weight of f (which is 1 if F is not a weighted species) times the monomial $\prod_i s_i^{|c^{-1}(i)|}$. In other words, the variables s_i act as a counter for the occurrences of the color i. Now the symmetric group S_n acts on the colored structures (f, c) on $[n]$, by the formula $\sigma \cdot (f, c) = (\sigma \cdot f, c \cdot \sigma^{-1})$. Let us denote by $F(1_{\mathbf{s}})$ the set of all orbits of colored F-structures, $n \geq 0$, or in other words, the set of all unlabeled colored F-structures. The weight function w extends to unlabeled colored F-structures and the total weight $|F(1_{\mathbf{s}})|_w := \sum_{\alpha \in F(1_{\mathbf{s}})} w(\alpha)$ is a symmetric function of the variables s_i. It is well known (see [4], §4.3) that when expressed in terms of the power sums x_i, this symmetric function yields the cycle index series of F:

$$Z_F(x_1, x_2, \ldots) = |F(1_{\mathbf{s}})|_w. \tag{40}$$

Example 1 For the species $F = E$, of sets, and $F = C_3$ of circular permutations of order 3, we have

$$\begin{aligned} Z_E(x_1, x_2, \ldots) &= h(s_1, s_2, \ldots) \\ &= \exp\Big(\sum_{k \geq 1} \frac{x_k}{k}\Big), \end{aligned} \tag{41}$$

where h denotes the complete homogeneous symmetric function, and

$$\begin{aligned} Z_{C_3}(x_1, x_2, x_3, \ldots) &= 2 \sum_{i<j<k} s_i s_j s_k + \sum_{i \neq j} s_i s_j^2 + \sum_{i \geq 1} s_i^3 \\ &= \frac{1}{3}(x_1^3 + 2x_3). \end{aligned} \tag{42}$$

Also recall the following basic formula for composite species:

$$Z_{F\circ G} = Z_F \circ Z_G, \tag{43}$$

where the $\circ$ on the right hand side denotes the plethystic composition of symmetric functions. (See [4], Definition 2.3.9)

Now the action of the group H on F extends to an action on the set $F(1_s)$ of unlabeled colored F-structures. The following lemma is a simple consequence of the Cauchy-Frobenius Theorem (alias Burnside's Lemma).

Lemma 14 *Let H be a finite group acting on a species F. Then the cycle index series of the quotient species F/H is given by*

$$Z_{F/H}(x_1, x_2, \ldots) = \frac{1}{|H|}\sum_{h\in H} |\mathrm{Fix}_{F(1_\mathbf{s})}(h)|_w, \tag{44}$$

where $\mathrm{Fix}_{F(1_\mathbf{s})}(h) = \{\alpha \in F(1_\mathbf{s}) | h\cdot\alpha = \alpha\}$.

□

For two-sort species $F(X,Y)$, the cycle index series involves two sorts of variables x_k and y_k, $k \geq 1$, which represent power sum symmetric functions

$$x_k = p_k(s_1, s_2, \ldots), \qquad y_k = p_k(t_1, t_2, \ldots),$$

in two infinities of variables s_i and t_i. For example, we have, for $P(X,Y) = E(XY^2)$, $R(X,Y) = C_3(E(XY^2))$ and $T(X,Y) = E^3(XY^2)$,

$$\begin{aligned} Z_P(x_1, x_2, \ldots : y_1, y_2, \ldots) &= \exp\sum_{k\geq 1}\frac{x_k y_k^2}{k} \\ &= h\circ(x_1 y_1^2), \end{aligned} \tag{45}$$

where $\circ$ denotes the plethystic composition,

$$Z_R = Z_{C_3}\circ Z_P \qquad \text{and} \qquad Z_T = Z_P^3. \tag{46}$$

The following result is obtained by applying the obvious extension of Lemma 14 to two-sort species.

Proposition 15 *The cycle index series $Z_F(x_1, x_2, \ldots; y_1, y_2, \ldots)$ for the quotient species $Q(X,Y) = E(XY^2)/\mathbb{Z}_2$, $S(X,Y) = C_3(XY^2)/\mathbb{Z}_2$ and $U(X,Y) = E^3(XY^2)/\mathbb{Z}_2$ are given by:*

$$Z_Q = \frac{1}{2}(Z_P + q), \tag{47}$$

$$Z_S = \frac{1}{2}(Z_{C_3}\circ Z_P + q\cdot(p_2\circ Z_P)), \tag{48}$$

$$Z_U = \frac{1}{2}((Z_P)^3 + q\cdot(p_2\circ Z_P)), \tag{49}$$

where $q = h\circ(x_1 y_2 + p_2\circ(x_1\frac{y_1^2 - y_2}{2}))$.

Proof: To prove the stated formula for Z_Q, it suffices by Lemma 14 to show that $|\mathrm{Fix}_{P(1_s,1_t)}(\tau)| = h \circ (x_1 y_2 + p_2 \circ (x_1(y_1^2 - y_2)/2))$, where τ is the generator of the group $\mathbb{Z}_2$. We color triangles with $s_1, s_2, \ldots$ and edges with $t_1, t_2, \ldots$ and enumerate the oriented but non-labeled structures which remain fixed under τ. Notice first that the action of τ leaves triangles fixed, so the coloring of the triangles can be arbitrary. This accounts for the x_1 in the term $x_1 y_2$ and the x_1 in the term $x_1(y_1^2 - y_2)/2$.

An ordered pair of the form (t_i, t_i) for any $i \in \{1, 2, \ldots\}$ from the species Y^2 is fixed under the action of τ, and this explains the y_2 in the term $x_1 y_2$. An ordered pair of the form $(t_i, t_j) \in s \in \mathrm{Fix}_{P(1_s,1_t)}(\tau)$, with $i < j$, will imply that there is another ordered pair of the form $(t_j, t_i) \in s$. This explains the term $p_2 \circ (x_1(y_1^2 - y_2)/2)$, since $(y_1^2 - y_2)/2 = e_2(t_1, t_2, \ldots) = \sum_{i<j} t_i t_j$ is the elementary symmetric function of degree 2, and the proof of (47) is now complete. The proofs of (48) and (49) are similar and left to the reader. □

6 Enumeration of 2-trees

It is now possible to put together the equations (21)–(24) to express the species $\mathfrak{a}$ in terms of $\mathfrak{a}^{\rightarrow}$. This gives

$$\begin{aligned} \mathfrak{a} &= \mathfrak{a}^{-} + \mathfrak{a}^{\triangle} - \mathfrak{a}^{\triangleq} \\ &= (Q(X,Y) + XS(X,Y) - XU(X,Y))|_{Y:=\mathfrak{a}^{\rightarrow}(X)}. \end{aligned} \tag{50}$$

Moreover, formulas (47)–(49) can be used to compute the generating series $Z_{\mathfrak{a}}(x_1, x_2, \ldots)$ and deduce the series $\mathfrak{a}(x)$ and $\widetilde{\mathfrak{a}}(x)$.

Proposition 16 *We have the following expressions for the three main series associated with the species $\mathfrak{a}$ of 2-trees in terms of the species $\mathfrak{a}^{\rightarrow}$ of oriented-edge rooted 2-trees.*

$$\mathfrak{a}(x) = \frac{1}{2}(\mathfrak{a}^{\rightarrow}(x) + e^x) + \frac{x}{3}(1 - (\mathfrak{a}^{\rightarrow}(x))^3), \tag{51}$$

$$\begin{aligned} \widetilde{\mathfrak{a}}(x) &= \frac{1}{2}(\widetilde{\mathfrak{a}}^{\rightarrow}(x) + \exp(\sum_{i\geq 1} \frac{1}{2i}(2x^i \widetilde{\mathfrak{a}}^{\rightarrow}(x^{2i}) + x^{2i}(\widetilde{\mathfrak{a}}^{\rightarrow}(x^{2i}))^2 - x^{2i}\widetilde{\mathfrak{a}}^{\rightarrow}(x^{4i})))) \\ &\quad + \frac{x}{3}(\widetilde{\mathfrak{a}}^{\rightarrow}(x^3) - (\widetilde{\mathfrak{a}}^{\rightarrow}(x))^3), \end{aligned} \tag{52}$$

$$\begin{aligned} Z_{\mathfrak{a}} &= \frac{1}{2}(Z_{\mathfrak{a}^{\rightarrow}} + \exp(\sum_{i\geq 1} \frac{1}{2i}(2x_i(Z_{\mathfrak{a}^{\rightarrow}})_{2i} + x_{2i}((Z_{\mathfrak{a}^{\rightarrow}})_{2i}^2) - x_{2i}(Z_{\mathfrak{a}^{\rightarrow}})_{4i})) \\ &\quad + \frac{x_1}{3}((Z_{\mathfrak{a}^{\rightarrow}})_3 - (Z_{\mathfrak{a}^{\rightarrow}})^3). \end{aligned} \tag{53}$$

□

Note that we have used the plethystic notation $(Z_F)_k = Z_F(x_k, x_{2k}, x_{3k}, \ldots)$ in (53).

Remark 17 It follows from (51) and (9) that the number $\mathfrak{a}_n$ of 2-trees with n labeled triangles is given by

$$\mathfrak{a}_n = \frac{1 + (2n+1)^{n-2}}{2}. \tag{54}$$

This formula was first proved by E. Palmer in [14]. It can be shown directly by observing that

$$(2n+1)a_n = a_n^- + n \qquad \text{and} \qquad 2a_n^- = a_n^{\rightarrow} + 1. \tag{55}$$

The first half of (55) can be derived by observing that with one exception (the 2−tree in which every triangle shares a common edge), rooting at an edge introduces a dissymmetry. The second half of (55) follows similarly, and this, together with (9), yields (54).

Also, equation (52) can be used to find a recurrence formula for the coefficients $\widetilde{a}_n$ of $\widetilde{a}(x)$ involving the $b_n = \widetilde{a}_n^{\rightarrow}$, by taking the logarithmic derivative. That recurrence is:

Corollary 18 *For* $n \geq 1$,

$$\widetilde{a}_n = \frac{1}{2n}\sum_{j=1}^{n}(\sum_{k|j} kd_k)(\widetilde{a}_{n-j} - c_{n-j}) + c_n, \tag{56}$$

where, for $i \geq 1$,

$$d_i = 2b_{\frac{i-1}{2}} + \sum_{k+j=\frac{i-2}{2}} b_j b_k - b_{\frac{i-2}{4}}, \tag{57}$$

and, for $i \geq 0$,

$$c_i = \frac{1}{2}b_i + \frac{1}{3}b_{\frac{i-1}{3}} - \frac{1}{3}\sum_{j+k+l=i-1} b_j b_k b_l \tag{58}$$

Here, $b_k = \widetilde{a}_k^{\rightarrow}$ *and* $\widetilde{a}_h^{\rightarrow}$ *is defined to be zero for* h *fractional or negative.*

□

7 Asymptotics

We now determine the asymptotic behavior of the numbers $\widetilde{a}_n$ and $b_n = \widetilde{a}_n^{\rightarrow}$ of unlabeled 2-trees and oriented-edge rooted 2-trees respectively. We adapt the approach of Pólya and Otter for ordinary trees (see [2], [3], [13], [16]) as presented in [11], §9.5.

The method relies on the equation (17), for b_n and on the relation (52), for $\widetilde{a}_n$. Consider the power series

$$y = b(x) = \sum_{n\geq 0} b_n x^n. \tag{59}$$

Because of (17), it satisfies the functional equation $f(x, y) = 0$, where

$$f(x,y) = y - e^{xy^2}\omega(x), \tag{60}$$

with

$$\omega(x) = \exp\left(\frac{x^2}{2}b^2(x^2) + \frac{x^3}{3}b^2(x^3) + \dots\right). \tag{61}$$

We have

$$f_y(x,y) := \frac{\partial}{\partial y} f(x,y) = 1 - 2xye^{xy^2}\omega(x). \tag{62}$$

Let ξ denote the radius of convergence of the series $b(x)$. The asymptotic behavior of $b_n = \widetilde{a}_n^{\rightarrow}$ depends closely on ξ and the first task is to determine ξ. It is clear that $\xi < 1$ and it can be shown that $\xi \geq \frac{4}{27}$. Indeed, it follows from the recurrence (11) that $b_n \leq t_n$ where t_n is defined by $t_0 = 1$ and, for $n \geq 1$,

$$t_n = \sum_{i+j+h=n-1} t_i t_j t_h. \tag{63}$$

The power series $t = t(x) = \sum_{n\geq 0} t_n x^n$ then satisfies the functional equation

$$t = 1 + xt^3, \tag{64}$$

which yields $t_n = \frac{1}{3n+1}\binom{3n+1}{n}$ by Lagrange inversion. Note that t_n is the number of unlabeled ternary rooted trees. Using Stirling's formula, we find that the radius of convergence ρ of $t(x)$ is $\rho = 1/\lim_{n\to\infty} t_n^{1/n} = \frac{4}{27}$.

Hence the series $y = b(x)$ defines an analytic function around the origin, for which the point $x = \xi$ is a singularity. Note that the series $\omega(x)$ defined by (61) has a radius of convergence equal to $\sqrt{\xi} > \xi$ and defines an analytic function in a larger region than $y = b(x)$. This shows that the function $f(x,y)$ is an analytic function of two variables around the origin. It also implies using the equation

$$b(x)e^{-xb^2(x)} = \omega(x)$$

that $b(x)$ is a bounded function of x in the interval $(0,\xi)$. Since $b(x)$ is an increasing function, the limit

$$\tau = \lim_{x\to\xi^-} b(x)$$

exists and $\tau = b(\xi)$.

It follows from the implicit function theorem that any singularity of the function $y = y(x)$ satisfying $f(x,y) = 0$ must come from a point $(x,y) = (\xi,\tau)$ for which $f(x,y) = 0$ and $f_y(x,y) = 0$. Hence we must have

$$\tau - e^{\xi\tau^2}\omega(\xi) = 0 \qquad \text{and} \qquad 1 - 2\xi\tau e^{\xi\tau^2}\omega(\xi) = 0$$

or, equivalently, $2\xi\tau^2 = 1$ or $\tau = \frac{1}{\sqrt{2\xi}}$ and $1 - 2\xi e\omega^2(\xi) = 0$. This shows that ξ is the least positive root of the equation

$$1 - 2xe\omega^2(x) = 0. \tag{65}$$

Since $f_{yy}(\xi,\tau) \neq 0$, ξ is an algebraic singularity of degree 2 and, for x near ξ, we have an expression of the form

$$b(x) = \tau_0 + \tau_1\left(1 - \frac{x}{\xi}\right)^{\frac{1}{2}} + \tau_2\left(1 - \frac{x}{\xi}\right) + \tau_3\left(1 - \frac{x}{\xi}\right)^{\frac{3}{2}} + \ldots \tag{66}$$

with $\tau_0 = \tau = b(\xi)$. Computations give

$$\tau_1 = -\sqrt{2}\sqrt{\frac{f_x(\xi,\tau)\xi}{f_{yy}(\xi,\tau)}} \quad = \quad -\frac{1}{2}\left(\frac{1}{\xi} + \frac{2\omega'(\xi)}{\omega(\xi)}\right)^{\frac{1}{2}} \tag{67}$$

$$\tau_2 = \frac{f_{xy}(\xi,\tau)\xi - \frac{1}{6}f_{yyy}(\xi,\tau)\tau_1^2}{f_{yy}(\xi,\tau)} \quad = \quad \frac{1}{12\sqrt{2\xi}}\left(7 + \xi\frac{2\omega'(\xi)}{\omega(\xi)}\right). \tag{68}$$

We then have the following result.

Theorem 19 *Let $\omega(x)$ be defined by (61) and ξ be the least positive root x of the equation (65). The asymptotic behavior for the number $b_n = \widetilde{a}_n^{\rightarrow}$ of unlabeled oriented-edge rooted 2-trees and the number $\widetilde{a}_n$ of unlabeled 2-trees is given by*

$$b_n \sim \alpha_1 \beta^n n^{-\frac{3}{2}}, \tag{69}$$
$$\widetilde{a}_n \sim \alpha_2 \beta^n n^{-\frac{5}{2}}, \tag{70}$$

where

$$\beta = \frac{1}{\xi}, \quad \alpha_1 = \frac{1}{4\sqrt{\pi}}\left(\frac{1}{\xi} + \frac{2\omega'(\xi)}{\omega(\xi)}\right)^{\frac{1}{2}}, \quad \alpha_2 = \frac{\xi}{16\sqrt{\pi}}\left(\frac{1}{\xi} + \frac{2\omega'(\xi)}{\omega(\xi)}\right)^{\frac{3}{2}}. \tag{71}$$

Proof: In the expansion (66) of $b(x)$, the dominating term for the asymptotic estimation of b_n is $\tau_1\left(1-\frac{x}{\xi}\right)^{\frac{1}{2}}$. The asymptotic equivalence (69), with β and α_1 given by (71) then follows using Stirling's Formula.

For the asymptotic estimation of the coefficients $\widetilde{a}_n = [x^n]\widetilde{a}(x)$, we can replace $\widetilde{a}(x)$ by

$$\frac{1}{2}b(x) - \frac{x}{3}b^3(x), \tag{72}$$

since this last expression differs from $\widetilde{a}(x)$ by an analytic function around 0 having radius of convergence $\sqrt{\xi} > \xi$, in virtue of (52).

Taking into account that $2\xi\tau_0^2 = 1$, computation shows that

$$\frac{1}{2}b(x) - \frac{x}{3}b^3(x) = \widehat{\tau}_0 + \widehat{\tau}_1\left(1-\frac{x}{\xi}\right)^{\frac{1}{2}} + \widehat{\tau}_2\left(1-\frac{x}{\xi}\right) + \widehat{\tau}_3\left(1-\frac{x}{\xi}\right)^{\frac{3}{2}} + \ldots \tag{73}$$

where

$$\widehat{\tau}_0 = \frac{1}{3}\tau_0, \quad \widehat{\tau}_1 = 0, \quad \widehat{\tau}_2 = \frac{\tau_0^2 - 3\tau_1^2}{6\tau_0}, \quad \widehat{\tau}_3 = \frac{\tau_1(3\tau_0^2 - \tau_1^2 - 6\tau_0\tau_2)}{6\tau_0^2}. \tag{74}$$

For the asymptotic estimation of $\widetilde{a}_n$, the dominating term is now $\widehat{\tau}_3\left(1-\frac{x}{\xi}\right)^{\frac{3}{2}}$, since $\widehat{\tau}_1 = 0$. It turns out that $\widehat{\tau}_3 = -\frac{\tau_1^3}{3\tau_0^2}$. The asymptotic equivalence (70), with β and α_2 given by (71), follows. □

Numerically, rounded to 25 digits after the decimal point, we have

$$\xi = 0.1770995223032856176934613, \tag{75}$$
$$\beta = 5.6465426162329497128927135, \tag{76}$$
$$\alpha_1 = 0.3492613817423114439729529, \tag{77}$$
$$\alpha_2 = 0.0948154165230993989741843. \tag{78}$$

These numerical estimations were computed using Maple software with `Digits:=100` and `Order:=150`. The value of ξ is found by solving numerically the equation (65). The series $\omega(x)$ is approximated using exact values of b_n (i.e., not truncated at 100 digits) to calculate all coefficients of x^i in the series expansion of $\omega(x)$ exactly for $i \leq 150$. It turns out that the term corresponding to x^{149} in $\omega'(x)$ is less than 10^{-49} for $x = 0.2 > \xi$. This strongly suggests that the 25 digits after the decimal point in (75)-(78) are indeed correct.

Remark 20 It follows from general principles that there are asymptotic expansions of the form

$$x_n := b_n/(\alpha_1\beta^n n^{-\frac{3}{2}}) \quad \sim \quad 1+\frac{u_1}{n}+\frac{u_2}{n^2}+\frac{u_3}{n^3}+\ldots \tag{79}$$

$$y_n := \widetilde{a}_n/(\alpha_2\beta^n n^{-\frac{5}{2}}) \quad \sim \quad 1+\frac{v_1}{n}+\frac{v_2}{n^2}+\frac{v_3}{n^3}+\ldots \tag{80}$$

where the u_i's and the v_i's are suitable constants.

Numerical computations show that for β, α_1 and α_2 given by (76)-(78),

$$x_{150} \approx 0.9958115736, \qquad y_{150} \approx 0.9944793144, \tag{81}$$

where the difference with 1 is smaller than 1/150. Making use of standard numerical acceleration techniques in (79) and (80), successively eliminating (u_1, v_1), (u_2, v_2), etc., we obtain the numbers

$$x'_{149} \approx 0.9999773738 \quad , \quad y'_{149} \approx 0.9999252317, \tag{82}$$

$$x''_{148} \approx 1.0000000251 \quad , \quad y''_{148} \approx 1.0000007122, \tag{83}$$

$$x'''_{147} \approx 0.9999999974 \quad , \quad y'''_{147} \approx 0.9999999683, \tag{84}$$

where, for $z = x$ or y,

$$z'_n = (n+1)z_{n+1} - nz_n, \tag{85}$$

$$z''_n = ((n+2)^2 z_{n+2} - 2(n+1)^2 z_{n+1} + n^2 z_n)/2, \tag{86}$$

$$z'''_n = ((n+3)^3 z_{n+3} - 3(n+2)^3 z_{n+2} + 3(n+1)^3 z_{n+1} - n^3 z_n)/3!, \tag{87}$$

as further confirmation of the numerical values given by (75)-(78).

The values of $b_n = \widetilde{a}^{\rightarrow}_n$, $\widetilde{a}_n$ and $\widetilde{a}_{O,n}$ are given in Table 1 for $n \leq 30$. It can be observed empirically that $\widetilde{a}_{O,n} \sim 2\widetilde{a}_n$. This can also be established rigorously, using the following identity which is a consequence of (29) and (52) or which can be established directly:

$$\widetilde{a}_O(x) = 2\widetilde{a}(x) - f(x), \tag{88}$$

where the correcting term is given by

$$f(x) = \exp\left(\sum_{i\geq 1} \frac{1}{2i}(2x^i\widetilde{a}^{\rightarrow}(x^{2i}) + x^{2i}(\widetilde{a}^{\rightarrow}(x^{2i}))^2 - x^{2i}\widetilde{a}^{\rightarrow}(x^{4i}))\right).$$

Similarly to $\omega(x)$, the series $f(x)$ has a radius of convergence equal to $\sqrt{\xi} > \xi$ and its contribution to the asymptotic estimate of $\widetilde{a}_{O,n}$ is negligible.

8 Conclusion

As we have seen, the class $B = a^{\rightarrow}$ of oriented-edge rooted 2-trees is completely specified by the functional equation $B = E(XB^2)$. All the interesting enumerative series expansions of $a^{\rightarrow}$ can be deduced from this equation. We have exhibited the molecular decomposition of $a^{\rightarrow}$ and the three main series $a^{\rightarrow}(x)$, $\widetilde{a}^{\rightarrow}(x)$ and $Z_{a^{\rightarrow}}(x_1, x_2, \ldots)$. We could also have computed the asymmetric series $\overline{a}^{\rightarrow}(x)$ which enumerates the asymmetric structures, i.e. those structures having a trivial automorphism group.

n	$\widetilde{a}_n^{\rightarrow}$	$\widetilde{a}_n$	$\widetilde{a}_{O,n}$
0	1	1	1
1	1	1	1
2	3	1	1
3	10	2	2
4	39	5	7
5	160	12	18
6	702	39	68
7	3177	136	251
8	14830	529	1020
9	70678	2171	4258
10	342860	9368	18580
11	1686486	41534	82716
12	8393681	188942	377207
13	42187148	874906	1748250
14	213828802	4115060	8227066
15	1091711076	19602156	39197164
16	5609297942	94419351	188824506
17	28982708389	459183768	918333933
18	150496728594	2252217207	4504366940
19	784952565145	11130545494	22260929867
20	4110491658233	55382155396	110763984273
21	21602884608167	277255622646	554510459987
22	113907912618599	1395731021610	2791460440109
23	602414753753310	7061871805974	14123739733754
24	3194684310627727	35896206800034	71792405634223
25	16984594260224529	183241761631584	366483503897357
26	90509181437849422	939081790240231	1878163540497162
27	483353806062219857	4830116366008952	9660232634407657
28	2586459035232330374	24927175920361855	49854351638299036
29	13866086598360333093	129047003236769110	258094005977582875
30	74465649185934253879	670024248072778235	1340048495113095791

Table 1: Values of $\widetilde{a}_n^{\rightarrow}$, $\widetilde{a}_n$, and $\widetilde{a}_{O,n}$ for $n \leq 30$