Counting Three-Line Latin Rectangles

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A $k \times n$ Latin rectangle is a $k \times n$ array of numbers such that (i) each row is a permutation of $[n] = \{1, 2, \ldots, n\}$ and (ii) each column contains distinct entries. If the first row is $12 \cdots n$, the Latin rectangle is said to be *reduced*. Since the number $k \times n$ Latin rectangles is clearly $n!$ times the number of reduced $k \times n$ Latin rectangles, we shall henceforth consider only reduced Latin rectangles. It is known [7, exercise 4.5.10, p. 288; solution, p. 507] that the number of (reduced) $3 \times n$ Latin rectangles is the coefficient of $x^n/n!$ in

$$e^{2x} \sum_{n=0}^{\infty} n! \frac{x^n}{(1+x)^{3n+3}}. \tag{1}$$

For other work on the enumeration of $3 \times n$ Latin rectangles, see [1], [2], [8]–[12], and [13, pp. 204-210].

A $3 \times n$ Latin rectangle may be identified with the pair (π, σ) of permutations of $1, \ldots, n$ which are its second and third rows. A pair (π, σ) of permutations corresponds to a $3 \times n$ Latin rectangle if and only if for each i in $[n]$, $\pi(i) \neq i$, $\sigma(i) \neq i$, and $\pi(i) \neq \sigma(i)$; in other words, π, σ, and $\pi\sigma^{-1}$ are derangements.

We generalize (1) to count π and σ by their numbers of cycles, and we obtain the following result:

Theorem 1. *The number of pairs (π, σ) of permutations of $[n]$ such that π, σ, and $\pi\sigma^{-1}$ are derangements, π has j cycles, and σ has k cycles, is the coefficient of $\alpha^j \beta^k x^n/n!$ in*

$$e^{2\alpha\beta x} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!} \frac{x^n}{(1+\alpha x)^{n+\beta}(1+\beta x)^{n+\alpha}(1+x)^{n+\alpha\beta}}, \tag{2}$$

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where $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$.

Our approach is similar to that taken by Foata and others [3]–[6] in their combinatorial study of orthogonal polynomials. We work with digraphs corresponding to permutations. We may identify the permutation π of $[n]$ with the digraph on $[n]$ having an edge from i to $\pi(i)$ for each i.

We define a *Latin configuration* to be a digraph on $[n]$ with edges in three colors, yellow, blue, and green, with the following properties:

1) The yellow edges form a derangement π on $[n]$.
2) The blue edges form a derangement σ on $[n]$.
3) If there is a green edge from i to j, then there must be both a yellow and a blue edge from i to j.

The *weight* of a Latin configuration is defined to be $\alpha^{|\pi|}\beta^{|\sigma|}\gamma^{g}$, where $|\pi|$ is the number of cycles of π, $|\sigma|$ is the number of cycles of σ, and g is the number of green edges. Let $L_n(\alpha,\beta,\gamma)$ be the sum of the weights of all Latin configurations on $[n]$ and let $K_n(\alpha,\beta,\gamma)$ be the sum of $\alpha^{|\pi|}\beta^{|\sigma|}\gamma^{|I(\pi,\sigma)|}$ over all pairs (π,σ) of derangements of $[n]$, where $I(\pi,\sigma)$ is the set of values of i for which $\pi(i)=\sigma(i)$.

Lemma 1.

$$L_n(\alpha,\beta,\gamma) = K_n(\alpha,\beta,\gamma+1).$$

Proof. Let $D(n)$ be the set of derangements of $[n]$. Then

$$K_n(\alpha,\beta,\gamma+1) = \sum_{\pi,\sigma\in D(n)} \alpha^{|\pi|}\beta^{|\sigma|}(\gamma+1)^{|I(\pi,\sigma)|} = \sum_{\substack{\pi,\sigma\in D(n)\\ G\subseteq I(\pi,\sigma)}} \alpha^{|\pi|}\beta^{|\sigma|}\gamma^{|G|}.$$

But a pair (π,σ) of derangements together with a subset G of $I(\pi,\sigma)$ corresponds to a Latin configuration in which the green edges are those from i to $\pi(i)=\sigma(i)$ for $i\in G$.

It follows that $K_n(\alpha,\beta,\gamma) = L_n(\alpha,\beta,\gamma-1)$. We now determine the generating function

$$L(x) = \sum_{n=0}^{\infty} L_n(\alpha,\beta,\gamma)\frac{x^n}{n!}$$

for Latin configurations.

First we may split the vertices of a Latin configuration into two classes: those in green cycles and all others. A green cycle is coextensive with a blue cycle and a yellow cycle, and therefore can have no edges connecting it with any other vertices.

The set of green cycles of a Latin configuration constitutes a derangement, since the only restriction on them is that there be no fixed points. Since the generating function for derangements is $e^{-x}/(1-x)$, the generating function for green cycles, together with their associated blue and yellow cycles, is $\left(e^{-\gamma x}/(1-\gamma x)\right)^{\alpha\beta}$ because every edge is weighted γ and every cycle is weighted $\alpha\beta$. Thus $L(x) = \left(e^{-\gamma x}/(1-\gamma x)\right)^{\alpha\beta} R(x)$, where $R(x)$ is the generating function for *green-acyclic* Latin configurations, that is, Latin configurations with no green cycles.

We shall count green-acyclic Latin configurations by constructing them in two steps, with each step translating into a generating function operation. We first insert the green edges, obtaining a set of green paths and a set of isolated vertices. We then "mark" certain vertices with indeterminates. Finally, based on the marks, we insert the yellow and blue edges.

First we make two observations which are the basis for the derivation of our generating function.

1) The green edges constitute a set of disjoint paths, each of which has at least two vertices.
2) If the green edges are contracted (together with their associated yellow and blue edges), what remains is a digraph consisting of a yellow permutation and a blue permutation. Any yellow or blue loop in the contracted graph must be attached to a vertex that was contracted.

Now we do the construction. We start with a set of green paths, each of at least two vertices, and a set of isolated vertices. We mark each isolated vertex with A and B. We mark the head of each green path with either A or α and with B or β.

The generating function for the configurations we have constructed so far is

$$\exp\left(\frac{(\mathrm{A}+\alpha)(\mathrm{B}+\beta)\gamma x^2}{1-\gamma x} + \mathrm{AB}x\right), \tag{3}$$

where each green edge is weighted γ.

Next, we put in the yellow and blue edges as follows:

1. Construct a yellow derangement through all the vertices marked A and a blue derangement through all the vertices marked B.
2. For each vertex v which is the head of a green path, do the following: Let u be the tail of the path. If v is marked A, there is a yellow edge (w, v). Replace this edge with the yellow edge (w, u). If v is marked α, add the yellow edge (v, u).
3. Repeat step 2 with A, α, and "yellow" replaced by B, β, and "blue."
4. Add a yellow edge and blue edge parallel to every green edge.

It is clear that only Latin configurations are obtained, and each is obtained exactly once.

We now describe the operation on the generating function (3) which corresponds to the insertion of yellow and blue edges. Let $D_n(s) = \sum_\pi s^{|\pi|}$, where the sum is over all derangements π of $[n]$. Then the Latin configurations coming from a term $\mathrm{A}^i\mathrm{B}^j\alpha^k\beta^l\gamma^m x^n/n!$ in (3) will have total weight $D_i(\alpha)D_j(\beta)\alpha^k\beta^l\gamma^m x^n/n!$. This is because each vertex marked α yields a yellow cycle, each vertex marked β yields a blue cycle, and the contribution from the vertices marked A and B is $D_i(\alpha)D_j(\beta)$.

It is convenient to adopt the "symbolic" or "umbral" convention

$$\begin{aligned} \mathrm{A}^i &= D_i(\alpha) \\ \mathrm{B}^j &= D_j(\beta), \end{aligned}$$

by which we mean that henceforth after an expression is expanded, any occurrence of A^i is to be replaced by $D_i(\alpha)$ and any occurrence of B^j is to be replaced by $D_j(\beta)$. As shown by Rota [14], this means formally that we apply to all our formulas a linear operator Φ defined by

$$\Phi\left(\mathrm{A}^i\mathrm{B}^j\alpha^k\beta^l\gamma^m\frac{x^n}{n!}\right) = D_i(\alpha)D_j(\beta)\alpha^k\beta^l\gamma^m\frac{x^n}{n!}.$$

Thus we need only evaluate (3) with this interpretation. First we study some properties of these "umbral variables." We note that

$$\sum_{i=0}^{\infty} D_i(\alpha)\frac{x^i}{i!} = \left(\frac{e^{-x}}{1-x}\right)^{\alpha},$$

and thus

$$e^{\mathrm{A}x} = \left(\frac{e^{-x}}{1-x}\right)^{\alpha}.$$

Now we introduce umbral variables a and b defined by $\mathrm{a} = \alpha + \mathrm{A}$ and $\mathrm{b} = \beta + \mathrm{B}$, that is,

$$\mathrm{a}^n = (\alpha + \mathrm{A})^n$$

and

$$\mathrm{b}^n = (\beta + \mathrm{B})^n$$

for all n. Then we have

$$\sum_{n=0}^{\infty} \mathrm{a}^n \frac{x^n}{n!} = e^{\mathrm{a}x} = e^{\alpha x} e^{\mathrm{A}x} = (1-x)^{-\alpha} = \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!}.$$

Thus $\mathrm{a}^n = (\alpha)_n$, and similarly, $\mathrm{b}^n = (\beta)_n$.

Lemma 2. *For all k,*

$$e^{\mathrm{a}x}\mathrm{a}^k = (1-x)^{-\alpha} \left(\frac{\mathrm{a}}{1-x}\right)^k.$$

Proof. We have

$$\begin{aligned} e^{\mathrm{a}x}\mathrm{a}^k &= \sum_{n=0}^{\infty} \mathrm{a}^{n+k} \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha)_{n+k} \frac{x^n}{n!} \\ &= (\alpha)_k \sum_{n=0}^{\infty} (\alpha + k)_n \frac{x^n}{n!} = \frac{(\alpha)_k}{(1-x)^{\alpha+k}} = (1-x)^{-\alpha} \left(\frac{\mathrm{a}}{1-x}\right)^k. \end{aligned}$$

It follows by linearity that for any series f for which $f(\mathrm{a})$ makes sense,

$$e^{\mathrm{a}x} f(\mathrm{a}) = (1-x)^{-\alpha} f\left(\frac{\mathrm{a}}{1-x}\right).$$

Similarly, we have

$$e^{\mathrm{b}x} f(\mathrm{b}) = (1-x)^{-\beta} f\left(\frac{\mathrm{b}}{1-x}\right).$$

Putting these together, we have

Lemma 3.

$$e^{\mathrm{a}u+\mathrm{b}v}f(\mathrm{a},\mathrm{b}) = (1-u)^{-\alpha}(1-v)^{-\beta} f\left(\frac{\mathrm{a}}{1-u}, \frac{\mathrm{b}}{1-v}\right).$$

Now we write (3) as

$$\exp\left(\frac{\mathrm{ab}\gamma x^2}{1-\gamma x} + \mathrm{ab}x - \mathrm{a}\beta x - \alpha \mathrm{b}x + \alpha\beta x\right) = e^{\alpha\beta x} e^{-\mathrm{a}\beta x - \alpha \mathrm{b}x} \exp\left(\frac{\mathrm{ab}x}{1-\gamma x}\right).$$

It then follows from Lemma 3 that this is

$$e^{\alpha\beta x}(1+\beta x)^{-\alpha}(1+\alpha x)^{-\beta}\exp\left(\frac{\mathrm{ab}x}{(1+\beta x)(1+\alpha x)(1-\gamma x)}\right) = e^{\alpha\beta x}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!}\frac{x^n}{(1+\alpha x)^{n+\beta}(1+\beta x)^{n+\alpha}(1-\gamma x)^n},$$

so multiplying by the generating function for green cycles, we get the generating function for Latin configurations,

$$\sum_{n=0}^{\infty} L_n(\alpha,\beta,\gamma)\frac{x^n}{n!} = e^{\alpha\beta(1-\gamma)x}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!}\frac{x^n}{(1+\alpha x)^{n+\beta}(1+\beta x)^{n+\alpha}(1-\gamma x)^{n+\alpha\beta}}.$$

Then changing γ to $\gamma-1$, we have

$$\sum_{n=0}^{\infty} K_n(\alpha,\beta,\gamma)\frac{x^n}{n!} = e^{\alpha\beta(2-\gamma)x}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!}\frac{x^n}{(1+\alpha x)^{n+\beta}(1+\beta x)^{n+\alpha}(1-(\gamma-1)x)^{n+\alpha\beta}}$$

as the generating function for pairs (π,σ) of derangements by cycles of π, cycles of σ, and equal values of π and σ. Setting $\gamma=0$ yields Theorem 1.

Note in particular that setting $\gamma=1$ yields

$$\sum_{n=0}^{\infty} D_n(\alpha)D_n(\beta)\frac{x^n}{n!} = e^{\alpha\beta x}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!}\frac{x^n}{(1+\alpha x)^{n+\beta}(1+\beta x)^{n+\alpha}}, \qquad (4)$$

and setting $\alpha=\beta=1$ in (4) yields

$$\sum_{n=0}^{\infty} D_n^2\frac{x^n}{n!} = e^x\sum_{n=0}^{\infty} n!\frac{x^n}{(1+x)^{2n+2}},$$

where $D_n = D_n(1)$ is the derangement number.

The methods used in this paper can also be used to count pairs (π,σ) of permutations by the number of cycles of π and σ and the number of fixed points of π, σ, and $\pi\sigma^{-1}$. If we omit from this generating function the number of fixed points of $\pi\sigma^{-1}$ then we obtain a formula equivalent to the bilinear generating function for Charlier polynomials (of which (4) is a specialization).

The first six values of the polynomials $K_n(\alpha, \beta, \gamma)$ are as follows:

$$K_0 = 1$$

$$K_1 = 0$$

$$K_2 = \alpha\beta\gamma^2$$

$$K_3 = 2\alpha\beta(1 + \gamma^3)$$

$$K_4 = 3\alpha\beta(2 + 2\alpha + 2\beta + 2\alpha\beta + 8\gamma + 4\alpha\gamma^2 + 4\beta\gamma^2 + 2\gamma^4 + \alpha\beta\gamma^4)$$

$$K_5 = 4\alpha\beta(48 + 30\alpha + 30\beta + 30\alpha\beta + 30\gamma + 60\alpha\gamma + 60\beta\gamma + 30\alpha\beta\gamma + 60\gamma^2 + 35\alpha\beta\gamma^2 + 30\alpha\gamma^3 + 30\beta\gamma^3 + 6\gamma^5 + 5\alpha\beta\gamma^5)$$

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