

# JACOBI-STIRLING POLYNOMIALS AND $P$ -PARTITIONS

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ABSTRACT. We investigate the diagonal generating function of the Jacobi-Stirling numbers of the second kind  $\text{JS}(n+k, n; z)$  by generalizing the analogous results for the Stirling and Legendre-Stirling numbers. More precisely, letting  $\text{JS}(n+k, n; z) = p_{k,0}(n) + p_{k,1}(n)z + \cdots + p_{k,k}(n)z^k$ , we show that  $(1-t)^{3k-i+1} \sum_{n \geq 0} p_{k,i}(n)t^n$  is a polynomial in  $t$  with nonnegative integral coefficients and provide combinatorial interpretations of the coefficients by using Stanley's theory of  $P$ -partitions.

## 1. INTRODUCTION

Let  $\ell_{\alpha,\beta}[y](t)$  be the Jacobi differential operator:

$$\ell_{\alpha,\beta}[y](t) = \frac{1}{(1-t)^\alpha(1+t)^\beta} \left( -(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t) \right)'$$

It is well known that the Jacobi polynomial  $y = P_n^{(\alpha,\beta)}(t)$  is an eigenvector for the differential operator  $\ell_{\alpha,\beta}$  corresponding to  $n(n+\alpha+\beta+1)$ , i.e.,

$$\ell_{\alpha,\beta}[y](t) = n(n+\alpha+\beta+1)y(t).$$

For each  $n \in \mathbb{N}$ , the Jacobi-Stirling numbers  $\text{JS}(n, k; z)$  of the second kind appeared originally as the coefficients in the expansion of the  $n$ -th composite power of  $\ell_{\alpha,\beta}$  (see [7]):

$$(1-t)^\alpha(1+t)^\beta \ell_{\alpha,\beta}^n[y](t) = \sum_{k=0}^n (-1)^k \text{JS}(n, k; z) \left( (1-t)^{\alpha+k}(1+t)^{\beta+k} y^{(k)}(t) \right)^{(k)},$$

where  $z = \alpha + \beta + 1$ , and can also be defined as the connection coefficients in

$$x^n = \sum_{k=0}^n \text{JS}(n, k; z) \prod_{i=0}^{k-1} (x - i(z+i)). \quad (1.1)$$

The Jacobi-Stirling numbers  $\text{js}(n, k; z)$  of the first kind are defined by

$$\prod_{i=0}^{n-1} (x - i(z+i)) = \sum_{k=0}^n \text{js}(n, k; z) x^k. \quad (1.2)$$

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When  $z = 1$ , the Jacobi-Stirling numbers become the *Legendre-Stirling numbers* [6] of the first and second kinds:

$$\text{ls}(n, k) = \text{js}(n, k; 1), \quad \text{LS}(n, k) = \text{JS}(n, k; 1). \quad (1.3)$$

Generalizing the work of Andrews and Littlejohn [2] on Legendre-Stirling numbers, Gelineau and Zeng [9] studied the combinatorial interpretations of the Jacobi-Stirling numbers and remarked on the connection with Stirling numbers and central factorial numbers. Further properties of the Jacobi-Stirling numbers have been given by Andrews, Egge, Gawronski, and Littlejohn [1].

The Stirling numbers of the second and first kinds  $S(n, k)$  and  $s(n, k)$  are defined by

$$x^n = \sum_{k=0}^n S(n, k) \prod_{i=0}^{k-1} (x - i), \quad \prod_{i=0}^{n-1} (x - i) = \sum_{k=0}^n s(n, k) x^k. \quad (1.4)$$

The lesser known central factorial numbers [14, p. 213–217]  $T(n, k)$  and  $t(n, k)$  are defined by

$$x^n = \sum_{k=0}^n T(n, k) x \prod_{i=1}^{k-1} \left( x + \frac{k}{2} - i \right), \quad (1.5)$$

and

$$x \prod_{i=1}^{n-1} \left( x + \frac{n}{2} - i \right) = \sum_{k=0}^n t(n, k) x^k. \quad (1.6)$$

Starting from the fact that for fixed  $k$ , the Stirling number  $S(n+k, n)$  can be written as a polynomial in  $n$  of degree  $2k$  and there exist nonnegative integers  $c_{k,j}$ ,  $1 \leq j \leq k$ , such that

$$\sum_{n \geq 0} S(n+k, n) t^n = \frac{\sum_{j=1}^k c_{k,j} t^j}{(1-t)^{2k+1}}, \quad (1.7)$$

Gessel and Stanley [10] gave a combinatorial interpretation for the  $c_{k,j}$  in terms of the descents in *Stirling permutations*. Recently, Egge [5] has given an analogous result for the Legendre-Stirling numbers, and Gelineau [8] has made a preliminary study of the analogous problem for Jacobi-Stirling numbers. In this paper, we will prove some analogous results for the diagonal generating function of Jacobi-Stirling numbers. As noticed in [9], the leading coefficient of the polynomial  $\text{JS}(n, k; z)$  is  $S(n, k)$  and the constant term of  $\text{JS}(n, k; z)$  is the central factorial number of the second kind with even indices  $T(2n, 2k)$ . Similarly, the leading coefficient of the polynomial  $\text{js}(n, k; z)$  is  $s(n, k)$  and the constant term of  $\text{js}(n, k; z)$  is the central factorial number of the first kind with even indices  $t(2n, 2k)$ .

**Definition 1.** The Jacobi-Stirling polynomial of the second kind is defined by

$$f_k(n; z) := \text{JS}(n + k, n; z). \quad (1.8)$$

The coefficient  $p_{k,i}(n)$  of  $z^i$  in  $f_k(n; z)$  is called the Jacobi-Stirling coefficient of the second kind for  $0 \leq i \leq k$ . Thus

$$f_k(n; z) = p_{k,0}(n) + p_{k,1}(n)z + \cdots + p_{k,k}(n)z^k. \quad (1.9)$$

The main goal of this paper is to prove Theorems 1 and 2 below.

**Theorem 1.** For each integer  $k$  and  $i$  such that  $0 \leq i \leq k$ , there is a polynomial  $A_{k,i}(t) = \sum_{j=1}^{2k-i} a_{k,i,j} t^j$  with positive integer coefficients such that

$$\sum_{n \geq 0} p_{k,i}(n) t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}. \quad (1.10)$$

In order to give a combinatorial interpretation for  $a_{k,i,j}$ , we introduce the multiset

$$M_k := \{1, \bar{1}, 2, \bar{2}, \dots, k, \bar{k}\},$$

where the elements are ordered by

$$\bar{1} < 1 < \bar{2} < 2 \dots < \bar{k} < k. \quad (1.11)$$

Let  $[\bar{k}] := \{\bar{1}, \bar{2}, \dots, \bar{k}\}$ . For any subset  $S \subseteq [\bar{k}]$ , we set  $M_{k,S} = M_k \setminus S$ .

**Definition 2.** A permutation  $\pi$  of  $M_{k,S}$  is a Jacobi-Stirling permutation if whenever  $u < v < w$  and  $\pi(u) = \pi(w)$ , we have  $\pi(v) > \pi(u)$ . We denote by  $\mathcal{JSP}_{k,S}$  the set of Jacobi-Stirling permutations of  $M_{k,S}$  and

$$\mathcal{JSP}_{k,i} = \bigcup_{\substack{S \subseteq [\bar{k}] \\ |S|=i}} \mathcal{JSP}_{k,S}.$$

For example, the Jacobi-Stirling permutations of  $\mathcal{JSP}_{2,1}$  are:

$$\begin{aligned} & 22\bar{2}11, \bar{2}2211, \bar{2}1221, \bar{2}1122, 221\bar{2}1, 122\bar{2}1, \bar{1}2221, \bar{1}2122, 2211\bar{2}, 1221\bar{2}, \\ & 1122\bar{2}, 11\bar{2}22, 2211\bar{1}, 1221\bar{1}, 1122\bar{1}, 11\bar{1}22, 22\bar{1}11, \bar{1}2211, \bar{1}1221, \bar{1}1122. \end{aligned}$$

Let  $\pi = \pi_1 \pi_2 \dots \pi_m$  be a word on a totally ordered alphabet. We say that  $\pi$  has a descent at  $l$ , where  $1 \leq l \leq m - 1$ , if  $\pi_l > \pi_{l+1}$ . Let  $\text{des } \pi$  be the number of descents of  $\pi$ . The following is our main interpretation for the coefficients  $a_{k,i,j}$ .

**Theorem 2.** For  $k \geq 1$ ,  $0 \leq i \leq k$ , and  $1 \leq j \leq 2k - i$ , the coefficient  $a_{k,i,j}$  is the number of Jacobi-Stirling permutations in  $\mathcal{JSP}_{k,i}$  with  $j - 1$  descents.

The rest of this paper is organized as follows. In Section 2, we investigate some elementary properties of the Jacobi-Stirling polynomials and prove Theorem 1. In Section 3 we apply Stanley's  $P$ -partition theory to derive a first interpretation of the integers  $a_{k,i,j}$  and then reformulate it in terms of descents of Jacobi-Stirling permutations in Section 4.

In Section 5, we construct Legendre-Stirling posets in order to prove a similar result for the Legendre-Stirling numbers, and then to deduce Egge's result for Legendre-Stirling numbers [5] in terms of descents of Legendre-Stirling permutations. A second proof of Egge's result is given by making a link to our result for Jacobi-Stirling permutations, namely Theorem 2. We end this paper with a conjecture on the real-rootedness of the polynomials  $A_{k,i}(t)$ .

## 2. JACOBI-STIRLING POLYNOMIALS

**Proposition 3.** *For  $0 \leq i \leq k$ , the Jacobi-Stirling coefficient  $p_{k,i}(n)$  is a polynomial in  $n$  of degree  $3k - i$ . Moreover, the leading coefficient of  $p_{k,i}(n)$  is*

$$\frac{1}{3^{k-i} 2^i i! (k-i)!} \quad (2.1)$$

for all  $0 \leq i \leq k$ .

*Proof.* We proceed by induction on  $k \geq 0$ . For  $k = 0$ , we have  $p_{0,0}(n) = 1$  since  $f_0(n) = \text{JS}(n, n; z) = 1$ . Let  $k \geq 1$  and suppose that  $p_{k-1,i}$  is a polynomial in  $n$  of degree  $3(k-1) - i$  for  $0 \leq i \leq k-1$ . From (1.1) we deduce the recurrence relation:

$$\begin{cases} \text{JS}(0, 0; z) = 1, & \text{JS}(n, k; z) = 0, \text{ if } k \notin \{1, \dots, n\}, \\ \text{JS}(n, k; z) = \text{JS}(n-1, k-1; z) + k(k+z) \text{JS}(n-1, k; z), & \text{for } n, k \geq 1. \end{cases} \quad (2.2)$$

Substituting in (1.8) yields

$$f_k(n; z) - f_k(n-1; z) = n(n+z)f_{k-1}(n; z). \quad (2.3)$$

It follows from (1.9) that for  $0 \leq i \leq k$ ,

$$p_{k,i}(n) - p_{k,i}(n-1) = n^2 p_{k-1,i}(n) + n p_{k-1,i-1}(n). \quad (2.4)$$

Applying the induction hypothesis, we see that  $p_{k,i}(n) - p_{k,i}(n-1)$  is a polynomial in  $n$  of degree at most

$$\max(3(k-1) - i + 2, 3(k-1) - (i-1) + 1) = 3k - i - 1.$$

Hence  $p_{k,i}(n)$  is a polynomial in  $n$  of degree at most  $3k - i$ . It remains to determine the coefficient of  $n^{3k-i}$ , say  $\beta_{k,i}$ . Extracting the coefficient of  $n^{3k-i-1}$  in (2.4) we have

$$\beta_{k,i} = \frac{1}{3k-i} (\beta_{k-1,i} + \beta_{k-1,i-1}).$$

Now it is fairly easy to see that (2.1) satisfies the above recurrence.  $\square$

**Proposition 4.** *For all  $k \geq 1$  and  $0 \leq i \leq k$ , we have*

$$p_{k,i}(0) = p_{k,i}(-1) = p_{k,i}(-2) = \dots = p_{k,i}(-k) = 0. \quad (2.5)$$

TABLE 1. The first values of  $A_{k,i}(t)$

$k \setminus i$	0	1	2	3
0	1			
1	$t + t^2$	$t$		
2	$t + 14t^2 + 21t^3 + 4t^4$	$2t + 12t^2 + 6t^3$	$t + 2t^2$	
3	$t + 75t^2 + 603t^3 + 1065t^4 + 460t^5 + 36t^6$	$3t + 114t^2 + 501t^3 + 436t^4 + 66t^5$	$3t + 55t^2 + 116t^3 + 36t^4$	$t + 8t^2 + 6t^3$

*Proof.* We proceed by induction on  $k$ . By definition, we have

$$f_1(n; z) = \text{JS}(n + 1, n; z) = p_{1,0}(n) + p_{1,1}(n)z.$$

As noticed in [9, Theorem 1], the leading coefficient of the polynomial  $\text{JS}(n, k; z)$  is  $S(n, k)$  and the constant term is  $T(2n, 2k)$ . We derive from (1.4) and (1.5) that

$$\begin{aligned} p_{1,1}(n) &= S(n + 1, n) = n(n + 1)/2, \\ p_{1,0}(n) &= T(2n + 2, 2n) = n(n + 1)(2n + 1)/6. \end{aligned}$$

Hence (2.5) is true for  $k = 1$ . Assume that (2.5) is true for some  $k \geq 1$ . By (2.4) we have

$$p_{k,i}(n) - p_{k,i}(n - 1) = n^2 p_{k-1,i}(n) + n p_{k-1,i-1}(n).$$

Since  $\text{JS}(0, k; z) = 0$  if  $k \geq 2$ , we have  $p_{k,i}(0) = 0$ . The above equation and the induction hypothesis imply successively that

$$p_{k,i}(-1) = 0, \quad p_{k,i}(-2) = 0, \quad \dots, \quad p_{k,i}(-k + 1) = 0, \quad p_{k,i}(-k) = 0.$$

The proof is thus complete.  $\square$

**Lemma 5.** *For each integer  $k$  and  $i$  such that  $0 \leq i \leq k$ , there is a polynomial  $A_{k,i}(t) = \sum_{j=1}^{2k-i} a_{k,i,j} t^j$  with integer coefficients such that*

$$\sum_{n \geq 0} p_{k,i}(n) t^n = \frac{A_{k,i}(t)}{(1 - t)^{3k-i+1}}. \quad (2.6)$$

*Proof.* By Proposition 3 and standard results concerning rational generating functions (cf. [16, Corollary 4.3.1]), for each integer  $k$  and  $i$  such that  $0 \leq i \leq k$ , there is a polynomial  $A_{k,i}(t) = a_{k,i,0} + a_{k,i,1}t + \dots + a_{k,i,3k-i}t^{3k-i}$  satisfying (2.6). Now, by [16, Proposition 4.2.3], we have

$$\sum_{n \geq 1} p_{k,i}(-n) t^n = -\frac{A_{k,i}(1/t)}{(1 - 1/t)^{2k-i+1}}. \quad (2.7)$$

Applying (2.5) we see that  $a_{k,i,2k-i+1} = \dots = a_{k,i,3k-i} = 0$ .  $\square$

The first values of  $A_{k,i}(t)$  are given in Table 1. The following result gives a recurrence for the coefficients  $a_{k,i,j}$ .

**Proposition 6.** *Let  $a_{0,0,0} = 1$ . For  $k, i, j \geq 0$ , we have the following recurrence for the integers  $a_{k,i,j}$ :*

$$\begin{aligned} a_{k,i,j} = & j^2 a_{k-1,i,j} + [2(j-1)(3k-i-j-1) + (3k-i-2)] a_{k-1,i,j-1} \\ & + (3k-i-j)^2 a_{k-1,i,j-2} + j a_{k-1,i-1,j} + (3k-i-j) a_{k-1,i-1,j-1}, \end{aligned} \quad (2.8)$$

where  $a_{k,i,j} = 0$  if any of the indices  $k, i, j$  is negative or if  $j \notin \{1, \dots, 2k-i\}$ .

*Proof.* For  $0 \leq i \leq k$ , let

$$F_{k,i}(t) = \sum_{n \geq 0} p_{k,i}(n) t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}. \quad (2.9)$$

The recurrence relation (2.4) is equivalent to

$$F_{k,i}(t) = (1-t)^{-1} [t^2 F''_{k-1,i}(t) + t F'_{k-1,i}(t) + t F'_{k-1,i-1}(t)] \quad (2.10)$$

with  $F_{0,0} = (1-t)^{-1}$ . Substituting (2.9) into (2.10) we obtain

$$\begin{aligned} A_{k,i}(t) = & (1-t)^{3k-i} [t^2 (A_{k-1,i}(t)(1-t)^{-(3k-i-2)})'' \\ & + t (A_{k-1,i}(t)(1-t)^{-(3k-i-2)})' + t (A_{k-1,i-1}(t)(1-t)^{-(3k-i-1)})'] \\ = & [t^2 A''_{k-1,i}(t)(1-t)^2 + 2(3k-i-2)t^2 A'_{k-1,i}(t)(1-t) \\ & + (3k-i-2)(3k-i-1)t^2 A_{k-1,i}(t)] \\ & + [t A'_{k-1,i}(t)(1-t)^2 + (3k-i-2)t A_{k-1,i}(t)(1-t)] \\ & + [t A'_{k-1,i-1}(t)(1-t) + (3k-i-1)t A_{k-1,i-1}(t)]. \end{aligned}$$

Taking the coefficient of  $t^j$  in both sides of the above equation, we have

$$\begin{aligned} a_{k,i,j} = & j(j-1)a_{k-1,i,j} - 2(j-1)(j-2)a_{k-1,i,j-1} + (j-2)(j-3)a_{k-1,i,j-2} \\ & + 2(3k-i-2)(j-1)a_{k-1,i,j-1} - 2(3k-i-2)(j-2)a_{k-1,i,j-2} \\ & + (3k-i-2)(3k-i-1)a_{k-1,i,j-2} + j a_{k-1,i,j} - 2(j-1)a_{k-1,i,j-1} \\ & + (j-2)a_{k-1,i,j-2} + (3k-i-2)a_{k-1,i,j-1} - (3k-i-2)a_{k-1,i,j-2} \\ & + j a_{k-1,i-1,j} - (j-1)a_{k-1,i-1,j-1} + (3k-i-1)a_{k-1,i-1,j-1}, \end{aligned}$$

which gives (2.8) after simplification.  $\square$

**Corollary 7.** *For  $k \geq 0$  and  $0 \leq i \leq k$ , the coefficients  $a_{k,i,j}$  are positive integers for  $1 \leq j \leq 2k-i$ .*

*Proof.* This follows from (2.8) by induction on  $k$ . Clearly, this is true for  $k=0$  and  $k=1$ . Suppose that this is true for some  $k \geq 1$ . As each term in the right-hand side of (2.8) is nonnegative, we only need to show that at least one term on the right-hand side of (2.8) is strictly positive. Indeed, for  $k \geq 2$ , the induction hypothesis and (2.8) imply that

- if  $j = 1$ , then  $a_{k,i,1} \geq a_{k-1,i-1,1} > 0$ ;
- if  $2 \leq j \leq 2k-i$ , then  $a_{k,i,j} \geq (3k-i-j)a_{k-1,i-1,j-1} \geq k a_{k-1,i-1,j-1} > 0$ .

These two cases cover all possibilities. □

Theorem 1 follows then from Lemma 5, Proposition 6 and Corollary 7.

Now, define the *Jacobi-Stirling polynomial of the first kind*  $g_k(n; z)$  by

$$g_k(n; z) = \text{js}(n, n - k; z). \tag{2.11}$$

**Proposition 8.** *For  $k \geq 1$ , we have*

$$g_k(n; z) = f_k(-n; -z). \tag{2.12}$$

If we write  $g_k(n; z) = q_{k,0}(n) + q_{k,1}(n)z + \cdots + q_{k,k}(n)z^k$ , then

$$\sum_{n \geq 1} q_{k,i}(n)t^n = (-1)^k \frac{\sum_{j=1}^{2k-i} a_{k,i,3k-i+1-j} t^j}{(1-t)^{3k-i+1}}. \tag{2.13}$$

*Proof.* From (1.2) we deduce

$$\begin{cases} \text{js}(0, 0; z) = 1, & \text{js}(n, k; z) = 0, & \text{if } k \notin \{1, \dots, n\}, \\ \text{js}(n, k; z) = \text{js}(n-1, k-1; z) - (n-1)(n-1+z) \text{js}(n-1, k; z), & n, k \geq 1. \end{cases} \tag{2.14}$$

It follows from the above recurrence and (2.11) that

$$g_k(n; z) - g_k(n-1; z) = -(n-1)(n-1+z)g_{k-1}(n-1; z).$$

Comparing with (2.3) we get (2.12), which implies that  $q_{k,i}(n) = (-1)^i p_{k,i}(-n)$ . Finally (2.13) follows from (2.7). □

### 3. JACOBI-STIRLING POSETS

We first recall some basic facts about Stanley's theory of  $P$ -partitions (see [15] and [16, §4.5]). Let  $P$  be a poset, and let  $\omega$  be a labeling of  $P$ , i.e., an injection from  $P$  to a totally ordered set (usually a set of integers). A  $(P, \omega)$ -*partition* (or  $P$ -partition if  $\omega$  is understood) is a function  $f$  from  $P$  to the positive integers satisfying

- (1) if  $x <_P y$  then  $f(x) \leq f(y)$
- (2) if  $x <_P y$  and  $\omega(x) > \omega(y)$  then  $f(x) < f(y)$ .

A *linear extension* of a poset  $P$  is an extension of  $P$  to a total order. We will identify a linear extension of  $P$  labeled by  $\omega$  with the permutation obtained by taking the labels of  $P$  in increasing order with respect to the linear extension. For example, the linear extensions of the poset shown in Figure 1 are 213 and 231. We write  $\mathcal{L}(P)$  for the set of linear extensions of  $P$  (which also depend on the labeling  $\omega$ ).

The *order polynomial*  $\Omega_P(n)$  of  $P$  is the number of  $(P, \omega)$ -partitions with parts in  $[n] = \{1, 2, \dots, n\}$ . It is known that  $\Omega_P(n)$  is a polynomial in  $n$  whose degree is the number

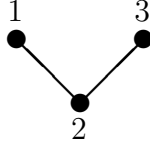


FIGURE 1. A poset

of elements of  $P$ . The following is a fundamental result in the  $P$ -partition theory [16, Theorem 4.5.14]:

$$\sum_{n \geq 1} \Omega_P(n) t^n = \frac{\sum_{\pi \in \mathcal{L}(P)} t^{\text{des } \pi + 1}}{(1-t)^{k+1}}, \quad (3.1)$$

where  $k$  is the number of elements of  $P$  and  $\text{des } \pi$  is computed according to the natural order of the integers.

For example, the two linear extensions of the poset shown in Figure 1 each have one descent, and the order polynomial for this poset is  $2 \binom{n+1}{3}$ . So equation (3.1) reads

$$\sum_{n \geq 1} 2 \binom{n+1}{3} t^n = \frac{2t^2}{(1-t)^4}.$$

By (2.2) the Jacobi-Stirling numbers have the generating function

$$\sum_{n \geq 0} \text{JS}(n, k; z) t^n = \frac{t^k}{(1-(z+1)t)(1-2(z+2)t) \cdots (1-k(z+k)t)}, \quad (3.2)$$

As  $f_k(n; z) = \text{JS}(n+k, n; z)$ , switching  $n$  and  $k$  in the last equation yields

$$\sum_{k \geq 0} f_k(n; z) t^k = \frac{1}{(1-(z+1)t)(1-2(z+2)t) \cdots (1-n(z+n)t)}.$$

Identifying the coefficients of  $t^k$  gives

$$f_k(n; z) = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} j_1(z+j_1) \cdot j_2(z+j_2) \cdots j_k(z+j_k). \quad (3.3)$$

For any subset  $S$  of  $[k]$ , we define  $\gamma_{S,m}(j)$  by

$$\gamma_{S,m}(j) = \begin{cases} j & \text{if } m \in S, \\ j^2 & \text{if } m \notin S, \end{cases}$$

and define  $p_{k,S}(n)$  by

$$p_{k,S}(n) = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} \gamma_{S,1}(j_1) \gamma_{S,2}(j_2) \cdots \gamma_{S,k}(j_k). \quad (3.4)$$



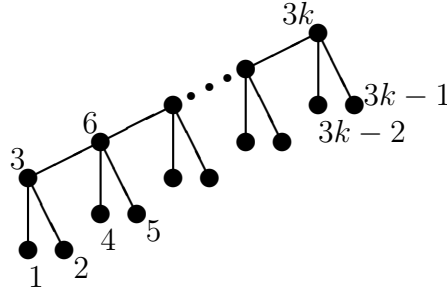


FIGURE 2. The labeled poset  $R_k$ .

For example, if  $k = 2$  and  $S = \{1\}$  then

$$p_{k,S}(n) = \sum_{1 \leq j_1 \leq j_2 \leq n} j_1 j_2^2 = n(n+1)(n+2)(12n^2 + 9n - 1)/120.$$

**Definition 3.** Let  $R_k$  be the labeled poset in Figure 2. Let  $S$  be a subset of  $[k]$ . The poset  $R_{k,S}$  obtained from  $R_k$  by removing the points  $3m - 2$  for  $m \in S$  is called a Jacobi-Stirling poset.

For example, the posets  $R_{2,\{1\}}$  and  $R_{2,\{2\}}$  are shown in Figure 3.

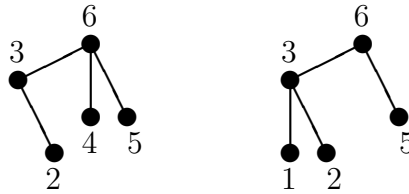


FIGURE 3. The labeled posets  $R_{2,\{1\}}$  and  $R_{2,\{2\}}$ .

**Lemma 9.** For any subset  $S \subseteq [k]$ , let  $A_{k,S}(t)$  be the descent polynomial of  $\mathcal{L}(R_{k,S})$ , i.e., the coefficient of  $t^j$  in  $A_{k,S}(t)$  is the number of linear extensions of  $R_{k,S}$  with  $j - 1$  descents, then

$$\sum_{n \geq 0} p_{k,S}(n) t^n = \frac{A_{k,S}(t)}{(1-t)^{3k-|S|+1}}. \tag{3.5}$$

*Proof.* It is easy to see that  $\Omega_{R_{k,S}}(n) = p_{k,S}(n)$  and the result follows from (3.1). □

For  $0 \leq i \leq k$ ,  $R_{k,i}$  is defined as the set of  $\binom{k}{i}$  posets

$$R_{k,i} = \{ R_{k,S} \mid S \subseteq [k] \text{ with cardinality } i \}.$$

The posets in  $R_{2,1}$  are shown in Figure 3. We define  $\mathcal{L}(R_{k,i})$  to be the (disjoint) union of  $\mathcal{L}(P)$ , over all  $P \in R_{k,i}$ ; i.e.,

$$\mathcal{L}(R_{k,i}) = \bigcup_{\substack{S \subseteq [k] \\ |S|=i}} \mathcal{L}(R_{k,S}). \quad (3.6)$$

Now we are ready to give the first interpretation of the coefficients  $a_{k,i,j}$  in the polynomial  $A_{k,i}(t)$  defined in (2.6).

**Theorem 10.** *We have*

$$A_{k,i}(t) = \sum_{\substack{S \subseteq [k] \\ |S|=i}} A_{k,S}(t). \quad (3.7)$$

*In other words, the integer  $a_{k,i,j}$  is the number of elements of  $\mathcal{L}(R_{k,i})$  with  $j-1$  descents.*

*Proof.* Extracting the coefficient of  $z^i$  in both sides of (3.3), then applying (1.9) and (3.4), we obtain

$$p_{k,i}(n) = \sum_{\substack{S \subseteq [k] \\ |S|=i}} p_{k,S}(n),$$

so that

$$\sum_{n \geq 0} p_{k,i}(n)t^n = \sum_{n \geq 0} \sum_S p_{k,S}(n)t^n = \sum_S \sum_{n \geq 0} p_{k,S}(n)t^n,$$

where the summations on  $S$  are over all subsets of  $[k]$  with cardinality  $i$ . The result follows then by comparing (2.6) and (3.5).  $\square$

It is easy to compute  $A_{k,S}(1)$  which is equal to  $|\mathcal{L}(R_{k,S})|$  and is also  $(3k-i)!$  times the leading coefficient of  $p_{k,S}(n)$ .

**Proposition 11.** *Let  $S \subseteq [k]$ ,  $|S| = i$  and let  $l_j(S) = |\{s \in S \mid s \leq j\}|$  for  $1 \leq j \leq k$ . We have*

$$A_{k,S}(1) = \frac{(3k-i)!}{\prod_{j=1}^k (3j - l_j(S))}. \quad (3.8)$$

*Proof.* We construct a permutation in  $\mathcal{L}(R_{k,S})$  by reading the elements of  $R_{k,S}$  in increasing order of their labels and inserting each one into the permutation already constructed from the earlier elements. Each element of  $R_{k,S}$  will have two natural numbers associated to it: the reading number and the insertion-position number. It is clear that the insertion-position number of  $3j$  must be equal to its reading number, which is  $3j - l_j(S)$ , since it must be inserted to the right of all the previously inserted elements (those with labels less than  $3j$ ). On the other hand, an element not divisible by 3 may be inserted anywhere, so its number of possible insertion positions is equal to its reading number. So the number of possible linear extensions of  $R_{k,S}$  is equal to the product of the reading numbers of all elements with labels not divisible by 3. Since the product of all the reading

numbers is  $(3j - i)!$ , we obtain the result by dividing this number by the product of the reading numbers of the elements with labels  $3, 6, \dots, 3k$ .  $\square$

From (3.8) we can derive the formula for  $A_{k,i}(1)$ , which is equivalent to Proposition 3.

**Proposition 12.** *We have*

$$|\mathcal{L}(R_{k,i})| = A_{k,i}(1) = \frac{(3k - i)!}{3^{k-i} 2^i i! (k - i)!}.$$

*Proof.* By Proposition 11 it is sufficient to prove the identity

$$\sum_{1 \leq s_1 < \dots < s_i \leq k} \frac{(3k - i)!}{\prod_{j=1}^k (3j - l_j(S))} = \frac{(3k - i)!}{3^{k-i} 2^i i! (k - i)!}, \quad (3.9)$$

where  $S = \{s_1, \dots, s_i\}$  and  $l_j(S) = |\{s \in S : s \leq j\}|$ .

The identity is obvious if  $S = \emptyset$ , i.e.,  $i = 0$ . When  $i = 1$ , it is easy to see that (3.9) is equivalent to the  $a = 2/3$  case of the indefinite summation

$$\sum_{s=0}^{k-1} \frac{(a)_s}{s!} = \frac{(a+1)_{k-1}}{(k-1)!}, \quad (3.10)$$

where  $(a)_n = a(a+1) \cdots (a+n-1)$  and  $(a)_0 = 1$ . Since the left-hand side of (3.9) can be written as

$$\sum_{s_i=i}^k \frac{(3k - i)!}{\prod_{j=s_i}^k (3j - i)} \sum_{1 \leq s_1 < \dots < s_{i-1} \leq s_i - 1} \frac{1}{\prod_{j=1}^{s_i-1} (3j - l_j(S))}, \quad (3.11)$$

we derive (3.9) from the induction hypothesis and (3.10).  $\square$

**Remark 1.** *Alternatively, we may prove the formula for  $A_{k,i}(1)$  as follows:*

$$\begin{aligned} A_{k,i}(1) &= \sum_{\substack{S \subseteq [k] \\ |S|=i}} A_{k,S}(1) \\ &= \sum_{\substack{S \subseteq [k] \\ |S|=i, k \in S}} A_{k,S}(1) + \sum_{\substack{S \subseteq [k] \\ |S|=i, k \notin S}} A_{k,S}(1) \\ &= (3k - i - 1)A_{k-1,i-1}(1) + (3k - i - 1)(3k - i - 2)A_{k-1,i}(1), \end{aligned}$$

from which we easily deduce that  $A_{k,i}(1) = (3k - i)!/3^{k-i} 2^i i! (k - i)!$ .

Since both of the above proofs of Proposition 12 use mathematical induction, it is desirable to have a more conceptual proof. Here we give such a proof based on the fact that Proposition 12 is equivalent to

$$|\mathcal{L}(R_{k,i})| = 2^{k-i} \cdot \frac{(3k - i)!}{2^i i! 3^{k-i} (k - i)!}. \quad (3.12)$$

*A combinatorial proof of Proposition 12.* We show that  $|\mathcal{L}(R_{k,i})|$  is equal to  $2^{k-i}$  times the number of partitions of  $[3k-i]$  with  $k-i$  blocks of size 3 and  $i$  blocks of size 2.

Let  $S$  be an  $i$ -element subset of  $[k]$  and let  $\pi$  be an element  $\mathcal{L}(R_{k,S})$ , viewed as a bijection from  $[3k-i]$  to  $R_{k,S}$ . Let  $\sigma = \pi^{-1}$ . Then  $\sigma$  is a natural labeling of  $R_{k,S}$ , i.e., an order-preserving bijection from the poset  $R_{k,S}$  to  $[3k-i]$ , and conversely, every natural labeling of  $R_{k,S}$  is the inverse of an element of  $\mathcal{L}(R_{k,S})$ .

We will describe a map from the set of natural labelings of elements of  $R_{k,i}$  to the set of partitions of  $[3k-i]$  with  $k-i$  blocks of size 3 and  $i$  blocks of size 2, for which each such partition is the image of  $2^{k-i}$  natural labelings. Given a natural labeling  $\sigma$  of  $R_{k,S}$ , the blocks of the corresponding partition are the sets  $\{\sigma(3m-2), \sigma(3m-1), \sigma(3m)\}$  for  $m \notin S$  and the sets  $\{\sigma(3m-1), \sigma(3m)\}$  for  $m \in S$ . We note that since  $\sigma$  is a natural labeling,  $\sigma(3m)$  is always the largest element of its block and  $\sigma(3) < \sigma(6) < \dots < \sigma(3m)$ .

Now let  $P$  be a partition of  $[3k-i]$  with  $k-i$  blocks of size 3 and  $i$  blocks of size 2. We shall describe all natural labelings  $\sigma$  of posets  $R_{k,S}$  that correspond to  $P$  under the map just defined. First, we list the blocks of  $P$  as  $B_1, B_2, \dots, B_k$  in increasing order of their largest elements. Then  $\sigma(3m)$  must be the largest element of  $B_m$ . If  $B_m$  has two elements, then the smaller element must be  $\sigma(3m-1)$ , and  $m$  must be an element of  $S$ . If  $B_m$  has three elements then  $m \notin S$ , and  $\sigma(3m-2)$  and  $\sigma(3m-1)$  are the two smaller elements of  $B_m$ , but in either order. Thus  $S$  is uniquely determined by  $P$ , and there are exactly  $2^{k-i}$  natural labelings of  $R_{k,S}$  in the preimage of  $P$ . So  $|\mathcal{L}(R_{k,i})|$  is  $2^i$  times the number of partitions of  $[3k-i]$  with  $k-i$  blocks of size 3 and  $i$  blocks of size 2, and is therefore equal to the right-hand side of (3.12).  $\square$

#### 4. TWO PROOFS OF THEOREM 2

We shall give two proofs of Theorem 2. We first derive Theorem 2 from Theorem 10 by constructing a bijection from the linear extensions of Jacobi-Stirling posets to permutations. The second proof consists of verifying that the cardinality of Jacobi-Stirling permutations in  $\mathcal{JSP}_{k,i}$  with  $j-1$  descents satisfies the recurrence relation (2.8). Given a word  $w = w_1 w_2 \dots w_m$  of  $m$  letters, we define the  $j$ th slot of  $w$  by the pair  $(w_j, w_{j+1})$  for  $j = 0, \dots, m$ . By convention  $w_0 = w_{m+1} = 0$ . A slot  $(w_j, w_{j+1})$  is called a descent (resp. non-descent) slot if  $w_j > w_{j+1}$  (resp.  $w_j \leq w_{j+1}$ ).

**4.1. First proof of Theorem 2.** For any subset  $S = \{s_1, \dots, s_i\}$  of  $[k]$  we define  $\bar{S} = \{\bar{s}_1, \dots, \bar{s}_i\}$ , which is a subset of  $[\bar{k}]$ . Recall that  $\mathcal{JSP}_{k,\bar{S}}$  is the set of Jacobi-Stirling permutations of  $M_{k,\bar{S}}$ . We construct a bijection  $\phi : \mathcal{L}(R_{k,S}) \rightarrow \mathcal{JSP}_{k,\bar{S}}$  such that  $\text{des } \phi(\pi) = \text{des } \pi$  for any  $\pi \in \mathcal{L}(R_{k,S})$ .

If  $k = 1$ , then  $\mathcal{L}(R_{1,0}) = \{123, 213\}$  and  $\mathcal{L}(R_{1,1}) = \{23\}$ . We define  $\phi$  by

$$\phi(123) = \bar{1}11, \quad \phi(213) = 11\bar{1}, \quad \phi(23) = 11.$$

Suppose that  $k \geq 2$  and  $\phi : \mathcal{L}(R_{k-1,S}) \rightarrow \mathcal{JSP}_{k-1,\bar{S}}$  is defined for any  $S \subseteq [k-1]$ . If  $\pi \in \mathcal{L}(R_{k,S})$  with  $S \subseteq [k]$ , we consider the following two cases:

- (i)  $k \notin S$ , denote by  $\pi'$  the word obtained by deleting  $3k$  and  $3k - 1$  from  $\pi$ , and  $\pi''$  the word obtained by further deleting  $3k - 2$  from  $\pi'$ . As  $\pi'' \in \mathcal{L}(R_{k-1,S})$ , by induction hypothesis, the permutation  $\phi(\pi'') \in \mathcal{JSP}_{k-1,\bar{S}}$  is well defined. Now,
- a) if  $3k - 2$  is in the  $r$ th descent (or nondescent) slot of  $\pi''$ , then we insert  $\bar{k}$  in the  $r$ th descent (or nondescent) slot of  $\phi(\pi'')$  and obtain a word  $\phi_1(\pi'')$ ;
  - b) if  $3k - 1$  is in the  $s$ th descent (or nondescent) slot of  $\pi'$ , we define  $\phi(\pi)$  by inserting  $kk$  in the  $s$ th descent (or nondescent) slot of  $\phi_1(\pi'')$ .
- (ii)  $k \in S$ , denote by  $\pi'$  the word obtained from  $\pi$  by deleting  $3k$  and  $3k - 1$ . As  $\pi' \in \mathcal{L}(R_{k-1,i-1})$ , the permutation  $\phi(\pi') \in \mathcal{JSP}_{k-1,\bar{S}}$  is well defined. If  $3k - 1$  is in the  $r$ th descent (or nondescent) slot of  $\pi'$ , we define  $\phi(\pi)$  by inserting  $kk$  in the  $r$ th descent (or nondescent) slot of  $\phi(\pi')$ .

Clearly this mapping is a bijection and preserves the number of descents. For example, if  $k = 3$  and  $S = \{2\}$ , then  $\phi(25137869) = 112\bar{3}233\bar{1}$ . This can be seen by applying the mapping  $\phi$  as follows:

$$\begin{aligned} 213 &\rightarrow 25136 \rightarrow 251376 \rightarrow 25137869, \\ 11\bar{1} &\rightarrow 1122\bar{1} \rightarrow 112\bar{3}2\bar{1} \rightarrow 112\bar{3}233\bar{1}. \end{aligned}$$

Clearly we have  $\text{des}(25137869) = 2$  and  $\text{des}(112\bar{3}233\bar{1}) = 2$ .

**4.2. Second proof of Theorem 2.** Let  $\mathcal{JSP}_{k,i,j}$  be the set of Jacobi-Stirling permutations in  $\mathcal{JSP}_{k,i}$  with  $j - 1$  descents. Let  $a'_{0,0,0} = 1$  and  $a'_{k,i,j}$  be the cardinality of  $\mathcal{JSP}_{k,i,j}$  for  $k, i, j \geq 0$ . By definition,  $a'_{k,i,j} = 0$  if any of the indices  $k, i, j < 0$  or  $j \notin \{1, \dots, 2k - i\}$ . We show that  $a'_{k,i,j}$ 's satisfy the same recurrence (2.8) and initial conditions as  $a_{k,i,j}$ 's.

Any Jacobi-Stirling permutation of  $\mathcal{JSP}_{k,i,j}$  can be obtained from one of the following five cases:

- (i) Choose a Jacobi-Stirling permutation in  $\mathcal{JSP}_{k-1,i,j}$ , insert  $\bar{k}$  and then  $kk$  in one of the descent slots (an extra descent at the end of the permutation). Clearly, there are  $a'_{k-1,i,j}$  ways to choose the initial permutation,  $j$  ways to insert  $\bar{k}$ , and  $j$  ways to insert  $kk$ .
- (ii) Choose a Jacobi-Stirling permutation of  $\mathcal{JSP}_{k-1,i,j-1}$ ,
  - 1) insert  $\bar{k}$  in a descent slot and then  $kk$  in a non-descent slot. In this case, there are  $a'_{k-1,i,j-1}$  ways to choose the initial permutation,  $j - 1$  ways to insert  $\bar{k}$ , and  $3k - i - j - 1$  ways to insert  $kk$ .
  - 2) insert  $\bar{k}$  in a non-descent slot and then  $kk$  in a descent slot. In this case, there are  $a'_{k-1,i,j-1}$  ways to choose the initial permutation,  $3k - i - j - 1$  ways to insert  $\bar{k}$ , and  $j$  ways to insert  $kk$ .
- (iii) Choose a Jacobi-Stirling permutation in  $\mathcal{JSP}_{k-1,i,j-2}$ , insert  $\bar{k}$  and then  $kk$  in one of the non-descent slots. In this case, there are  $a'_{k-1,i,j-2}$  ways to choose the initial permutation,  $3k - i - j$  ways to insert  $\bar{k}$ , and  $3k - i - j$  ways to insert  $kk$ .



**Theorem 13.** *Let  $b_{k,j}$  be the number of linear extensions of Legendre-Stirling posets  $P_k$  with exactly  $j$  descents. Then*

$$\sum_{n \geq 0} \text{LS}(n+k, n) t^n = \frac{\sum_{j=1}^{2k-1} b_{k,j} t^j}{(1-t)^{3k+1}}. \quad (5.2)$$

We now apply the above theorem to deduce a result of Egge [5, Theorem 4.6].

**Definition 4.** *A Legendre-Stirling permutation of  $M_k$  is a Jacobi-Stirling permutation of  $M_k$  with respect to the order:  $\bar{1} = 1 < \bar{2} = 2 < \dots < \bar{k} = k$ .*

Here  $\bar{1} = 1$  means that neither  $1\bar{1}$  nor  $\bar{1}1$  counts as a descent. Thus, the Legendre-Stirling permutation  $122\bar{2}1\bar{1}$  has one descent at position 4, while as a Jacobi-Stirling permutation, it has three descents, at positions 3, 4 and 5.

**Theorem 14** (Egge). *The coefficient  $b_{k,j}$  equals the number of Legendre-Stirling permutations of  $M_k$  with exactly  $j - 1$  descents.*

*First proof.* Let  $\mathcal{LSP}_k$  be the set of Legendre-Stirling permutations of  $M_k$ . By Theorem 13, it suffices to construct a bijection  $\psi : \mathcal{LSP}_k \rightarrow \mathcal{L}(P_k)$  such that  $\text{des } \psi(\pi) - 1 = \text{des } \pi$  for any  $\pi \in \mathcal{LSP}_k$ . If  $k = 1$ , then  $\mathcal{LSP}_1 = \{11\bar{1}, \bar{1}11\}$  and  $\mathcal{L}(P_1) = \{132, 312\}$ . We define  $\psi$  by

$$\psi(11\bar{1}) = 132, \quad \psi(\bar{1}11) = 312.$$

Clearly  $\text{des } 132 - 1 = \text{des } 11\bar{1} = 0$  and  $\text{des } 312 - 1 = \text{des } \bar{1}11 = 0$ . Suppose that the bijection  $\psi : \mathcal{LSP}_{k-1} \rightarrow \mathcal{L}(P_{k-1})$  is constructed for some  $k \geq 2$ . Given  $\pi \in \mathcal{LSP}_k$ , we denote by  $\pi'$  the word obtained by deleting  $\bar{k}$  from  $\pi$ , and by  $\pi''$  the word obtained by further deleting  $kk$  from  $\pi'$ . We put  $3k - 1$  at the end of  $\psi(\pi'')$  and obtain a word  $\psi_1(\pi'')$ . In the following two steps, the slot after  $3k - 1$  is excluded, because we cannot insert  $3k$  and  $3k - 2$  to the right of  $3k - 1$ .

- a) if  $\bar{k}$  is in the  $r$ th descent (or nondescent) slot of  $\pi''$ , then we insert  $3k$  in the  $r$ th descent (or nondescent) slot of  $\psi_1(\pi'')$  and obtain a word  $\psi_2(\pi'')$ ;
- b) if  $kk$  is in the  $s$ th descent slot or in the non-descent slot before  $\bar{k}$  (in the  $j$ th non-descent slot other than the non-descent slot before  $\bar{k}$ ) of  $\pi'$ , we define  $\psi(\pi)$  by inserting  $3k - 2$  in the  $s$ th descent slot or in the non-descent slot before  $3k$  (in the  $j$ th non-descent slot other than the non-descent slot before  $3k$ ) of  $\psi_2(\pi'')$ .

For example, we can compute  $\psi(\bar{2}12233\bar{3}1\bar{1}) = 614793258$  by the following procedure:

$$\begin{aligned} 11\bar{1} &\rightarrow \bar{2}11\bar{1} \rightarrow \bar{2}1221\bar{1} \rightarrow \bar{2}122\bar{3}1\bar{1} \rightarrow \bar{2}12233\bar{3}1\bar{1} \\ 132 &\rightarrow 61325 \rightarrow 614325 \rightarrow 61493258 \rightarrow 614793258. \end{aligned}$$

This construction can be easily reversed and the number of descents is preserved. □

*Second proof.* By (1.8), (1.9), and (1.10), we have

$$\sum_{n=0}^{\infty} \text{JS}(n+k, n; z) t^n = \sum_{i=0}^k z^i \frac{\sum_{j=1}^{2k-i} a_{k,i,j} t^j}{(1-t)^{3k-i+1}}.$$

Setting  $z = 1$  and using (1.3) gives

$$\sum_{n=0}^{\infty} \text{LS}(n+k, n) t^n = \sum_{i=0}^k (1-t)^i \frac{\sum_{j=1}^{2k-i} a_{k,i,j} t^j}{(1-t)^{3k+1}}.$$

Multiplying both sides by  $(1-t)^{3k+1}$  and applying (5.2) gives

$$\sum_{j=1}^{2k-1} b_{k,j} t^j = \sum_{i=0}^k (1-t)^i \sum_{j=1}^{2k-i} a_{k,i,j} t^j,$$

so

$$\sum_{i=0}^k \sum_{l=0}^i (-1)^l \binom{i}{l} a_{k,i,j-l} = b_{k,j}. \quad (5.3)$$

For any  $S \subseteq [\bar{k}]$ , let  $\mathcal{JSP}_{k,S,j}$  be the set of all Jacobi-Stirling permutations of  $M_{k,S}$  with  $j-1$  descents. Let  $B_{k,j} = \bigcup_{S \subseteq [\bar{k}]} \mathcal{JSP}_{k,S,j}$  be the set of Jacobi-Stirling permutations with  $j-1$  descents. We show that the left-hand side of (5.3) is the number  $N_0$  of permutations in  $B_{k,j}$  with no pattern  $u\bar{u}$ .

For any  $T \subseteq [\bar{k}]$ , let  $B_{k,j}(T, \geq)$  be the set of permutations in  $B_{k,j}$  containing all the patterns  $u\bar{u}$  for  $\bar{u} \in T$ . By the principle of inclusion-exclusion [16, Chapter 2],

$$N_0 = \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} |B_{k,j}(T, \geq)|. \quad (5.4)$$

Now, for any subsets  $T, S \subseteq [\bar{k}]$  such that  $T \subseteq [\bar{k}] \setminus S$ , define the mapping

$$\varphi : \mathcal{JSP}_{k,S,j} \cap B_{k,j}(T, \geq) \rightarrow \mathcal{JSP}_{k,S \cup T, j-|T|}$$

by deleting the  $\bar{u}$  in every pattern  $u\bar{u}$  of  $\pi \in \mathcal{JSP}_{k,S,j} \cap B_{k,j}(T, \geq)$ . Clearly, this is a bijection. Hence, we can rewrite (5.4) as

$$\begin{aligned} N_0 &= \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} \sum_{\substack{S, T \subseteq [\bar{k}] \\ T \cap S = \emptyset}} |\mathcal{JSP}_{k,S \cup T, j-|T|}| \\ &= \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} \sum_{\substack{S \subseteq [\bar{k}] \\ T \subseteq S}} |\mathcal{JSP}_{k,S, j-|T|}|. \end{aligned}$$

For any subset  $S$  of  $[\bar{k}]$  with  $|S| = i$ , and any  $l$  with  $0 \leq l \leq i$ , there are  $\binom{i}{l}$  subsets  $T$  of  $S$  such that  $|T| = l$ , and, by definition,

$$\sum_{\substack{S \subseteq [\bar{k}] \\ |S|=i}} |\mathcal{JSP}_{k,S, j-|T|}| = a_{k,i,j-l}.$$

This proves that  $N_0$  is equal to the left-hand side of (5.3).

Let  $\mathcal{LSP}_{k,j}$  be the set of all Legendre-Stirling permutations of  $M_k$  with  $j-1$  descents. It is easy to identify a permutation  $\pi \in B_{k,j}$  with no pattern  $u\bar{u}$  with a Legendre-Stirling



permutation  $\pi' \in \mathcal{LSP}_{k,j}$  by inserting each missing  $\bar{u}$  just to the right of the second  $u$ . This completes the proof.  $\square$

Finally, the numerical experiments suggest the following conjecture, which has been verified for  $0 \leq i \leq k \leq 9$ .

**Conjecture 15.** *For  $0 \leq i \leq k$ , the polynomial  $A_{k,i}(t)$  has only real roots.*

Note that by a classical result [4, p. 141], the above conjecture would imply that the sequence  $a_{k,i,1}, \dots, a_{k,i,2k-i}$  is unimodal. Let  $G_k$  be the multiset  $\{1^{m_1}, 2^{m_2}, \dots, k^{m_k}\}$  with  $m_i \in \mathbb{N}$ . A permutations  $\pi$  of  $G_k$  is a *generalized Stirling permutation* (see [3, 12]) if whenever  $u < v < w$  and  $\pi(u) = \pi(w)$ , we have  $\pi(v) > \pi(u)$ . For any  $S \subseteq [k]$ , the set of generalized Stirling permutations of  $M_k \setminus S$  is equal to  $\mathcal{JSP}_{k,S}$ . By Lemma 9 and Theorem 2, the descent polynomial of  $\mathcal{JSP}_{k,S}$  is  $A_{k,S}(t)$ . It follows from a result of Brenti [3, Theorem 6.6.3] that  $A_{k,S}(t)$  has only real roots. By (3.7), this implies, in particular, that the above conjecture is true for  $i = 0$  and  $i = k$ .

One can also use the methods of Haglund and Visontai [11] to show that  $A_{k,S}(t)$  has only real roots, though it is not apparent how to use these methods to show that  $A_{k,i}(t)$  has only real roots.

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