The Sum $16^n \sum_{k=0}^{2n} 4^k \left( \frac{1}{k} \right) \left( \frac{-1}{k} \right) \left( \frac{-2k}{2n-k} \right)$: 
A Third Proof of its Closed Form

Ira M. Gessel and Peter J. Larcombe

Department of Mathematics  
Brandeis University, Waltham, MA 02454-9110, U.S.A.  
{gessel@brandeis.edu}

†Derbyshire Business School  
University of Derby  
Kedleston Road, Derby DE22 1GB, U.K.  
{P.J.Larcombe@derby.ac.uk}

Abstract

We present a third proof of an interesting binomial coefficient identity using a generating function approach.

Introduction

In [1] Koepf and Larcombe gave a computer assisted approach to the evaluation of the particular binomial coefficient sum $16^n \sum_{k=1}^{2n} 4^k \left( \frac{1}{k} \right) \left( \frac{-1}{k} \right) \left( \frac{-2k}{2n-k} \right)$. We state the result as follows:

Theorem For integer $n \geq 1$,

$$16^n \sum_{k=1}^{2n} 4^k \left( \frac{1}{k} \right) \left( \frac{-1}{k} \right) \left( \frac{-2k}{2n-k} \right) = 2^{4n+1} \sum_{k=2}^{2n} 3F_2 \left( 1 - 2n, 1 + 2n; \frac{1}{2}, \frac{1}{2}; 1 \right) = (4n+1) \binom{2n}{n}^2 .$$
Noting that the evaluation had, prior to this, been dealt with through the application of some results concerning Dixon type sums and formulas [2], in this short paper we detail a new proof using a generating function approach which is rather succinct.

The Proof

Consider the polynomial

\[ G(x) = \sum_{m=0}^{\infty} m \binom{3}{2} F_2 \left( \begin{array}{c} 1-m, 1+m, \frac{1}{2} \\ 2, 2 \end{array} \right)_{m} x^m, \]  

(P1)

which is an ordinary generating function for the sequence (see Remark 1 later)

\[ \left\{ m \binom{3}{2} F_2 \left( \begin{array}{c} 1-m, 1+m, \frac{1}{2} \\ 2, 2 \end{array} \right)_{m} \right\}_{m=0}^{\infty} = \{ 0, 1, 5, 5, 81, 81, 64, \ldots \}. \]  

(P2)

Then, with \((u)_k\) denoting the usual rising factorial function \((u)_k = u(u + 1)(u + 2)(u + 3) \cdots (u + k - 1),\)

\[ G(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(1-m)_k (1+m)_k (\frac{1}{2})_k}{(2)_k k!} x^m \]

\[ = \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \frac{m(1-m)_k (1+m)_k (\frac{1}{2})_k}{(2)_k k!} x^m, \]  

(P3)

since \((1-m)_k = 0\) for \(m < k + 1\). Continuing,

\[ G(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(m+k+1)(-m-k)_k (m+k+2)_k (\frac{1}{2})_k}{(2)_k k!} x^{m+k+1} \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)! (\frac{1}{2})_k}{(2)_k k!} x^{k+1} \sum_{m=0}^{\infty} \binom{m+2k+1}{m} x^m, \]  

(P4)

after a little algebra. We now note that the sum in \(m\) is merely the series form of \((1-x)^{-2(4k+1)}\), and since \((2k+1)!/k! = 4^k (\frac{1}{2})_k\) \((k \geq 0),\) we can further write

\[ G(x) = \frac{1}{(1-x)^2} \sum_{k=0}^{\infty} \binom{\frac{3}{2}}{k} \binom{\frac{1}{2}}{k} \left[ \frac{-4x}{(1-x)^2} \right]^{k} \]

\[ = -\frac{1}{4} \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k-1} \frac{(-4x)^k}{(1-x)^2} \]
\[
= \sum_{k=1}^{\infty} \frac{(\frac{1}{2})_k(-\frac{1}{2})_k}{(1)_k} \frac{1}{k!} \left[ \frac{-4x}{(1-x)^2} \right]^k
\]
\[= \; 2F_1 \left( \frac{1}{2}, -\frac{1}{2} \middle| \frac{-4x}{(1-x)^2} \right) - 1 \]
\[= \; \frac{1}{1-x} \; 2F_1 \left( -\frac{1}{2}, -\frac{1}{2} \middle| x^2 \right) - 1, \quad \text{(P5)}
\]
on applying one of Kummer’s quadratic transformations [3, (3.1.11), p.191]\(^1\)
\[2F_1 \left( \frac{a}{2b}, b \middle| \frac{4x}{(1+x)^2} \right) = (1+x)^{2a} \; 2F_1 \left( \frac{a+a+b}{b+b+\frac{1}{2}} \middle| x^2 \right) \quad \text{(P6)}
\]
with \(a = -\frac{1}{2}, \; b = \frac{1}{2}, \; x \to -x\). Appealing next to Euler’s well known transformation (see, e.g., [4, (1.3.1.5), p.10])
\[2F_1 \left( \frac{a}{c}, b \middle| z \right) = (1-z)^{-a-b} \; 2F_1 \left( \frac{c-a}{c-b} \middle| z \right), \quad \text{(P7)}
\]
in which we set \(a = b = -\frac{1}{2}, \; c = 1, \; z = x^2\), we arrive at a final form for \(G(x)\), namely,
\[G(x) = (1-x^2)(1+x)g(x) - 1, \quad \text{(P8)}
\]
where
\[g(x) = 2F_1 \left( \frac{3}{2}, \frac{3}{2} \middle| x^2 \right). \quad \text{(P9)}
\]
The proof now concludes quickly, for equating coefficients of \(x^{2n}\) across (P8) yields, for \(n \geq 1,
\[2n \; 3F_2 \left( \frac{1-2n, 1+2n, \frac{1}{2}}{2, 2, 1} \middle| 1 \right) = \; [x^{2n}]\{g(x)\} - [x^{2n-2}]\{g(x)\}\]
\[= \; \left( \frac{(\frac{3}{2})_n}{n!} \right)^2 - \left( \frac{(\frac{5}{2})_{n-1}}{(n-1)!} \right)^2\]
\[= \; \frac{1}{24n}(4n+1) \left( \frac{2n}{n} \right)^2 \quad \text{(P10)}
\]
as required, after some simplification. □

Remark 1 We remark, for completeness, on the repetition in terms exhibited

\(^1\)The lower parameter in the l.h.s. of (P6) is printed incorrectly as \(2n\) in [3].
by the sequence (P2). Clearly, since the argument of the hypergeometric series \( g(x) \) is \( x^2 \),

\[
f(x) = (1 - x^2)g(x)
\]

\[
= \sum_{n=0}^{\infty} a_n x^{2n},
\]

say, where \( a_0 = 1 \). Then (P8) gives

\[
G(x) = (1 + x)f(x) - 1
\]

\[
= \sum_{n=0}^{\infty} a_n (x^{2n} + x^{2n+1}) - 1
\]

\[
= \{a_0 (1 + x) + a_1 (x^2 + x^3) + a_2 (x^4 + x^5) + a_3 (x^6 + x^7) + \cdots \} - 1
\]

\[
= a_0 x + a_1 (x^2 + x^3) + a_2 (x^4 + x^5) + a_3 (x^6 + x^7) + \cdots,
\]

which generates a sequence of form \( \{0, a_0, a_1, a_2, a_3, \ldots \} \). Evidently, from (1), (P10) \( a_n = [x^{2n} \{ f(x) \}] = [x^{2n} \{ g(x) \}] - [x^{2n-2} \{ g(x) \}] = 2^{-4n}(4n + 1) \binom{2n}{n}^2 \), so that \( a_0 = 1, a_1 = \frac{5}{1}, a_2 = \frac{51}{4}, a_3 = \frac{325}{24}, \) etc.

Remark 2 Two generalised versions of the Theorem are given in [1], which may themselves succumb to the technique seen here. However, due to their complex nature, this would be a non-trivial exercise and lies beyond the remit of this paper.
References

[1] Koepf, W.A. and Larcombe, P.J. (2009). The sum $16^n \sum_{k=0}^{2n} 4^k \binom{n-k}{k}$ $\left(\frac{-1}{2} \right) \binom{-2k}{n-k}$: a computer assisted proof of its closed form, and some generalised results, *Util. Math.*, **79**, pp.xx-xx.

[2] Larcombe, P.J. and Larsen, M.E. (2009). The sum $16^n \sum_{k=0}^{2n} 4^k \binom{n-k}{k}$ $\left(\frac{-1}{2} \right) \binom{-2k}{n-k}$: a proof of its closed form, *Util. Math.*, **79**, pp.3-7.
