The Sum $16^n \sum_{k=0}^{2n} 4^k \binom{\frac{1}{2}}{k}\binom{-\frac{1}{2}}{k}\binom{-2k}{2n-k}$: A Third Proof of its Closed Form

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Abstract

We present a third proof of an interesting binomial coefficient identity using a generating function approach.

Introduction

In [1] Koepf and Larcombe gave a computer assisted approach to the evaluation of the particular binomial coefficient sum $16^n \sum_{k=1}^{2n} 4^k \binom{\frac{1}{2}}{k}\binom{-\frac{1}{2}}{k}\binom{-2k}{2n-k}$. We state the result as follows:

Theorem For integer $n \geq 1$,

$$16^n \sum_{k=1}^{2n} 4^k \binom{\frac{1}{2}}{k}\binom{-\frac{1}{2}}{k}\binom{-2k}{2n-k} = 2^{4n+1} n \, {}_3F_2\left(\left.\begin{matrix} 1-2n, 1+2n, \frac{1}{2} \\ 2, 2 \end{matrix}\right| 1\right) = (4n+1)\binom{2n}{n}^2.$$

Noting that the evaluation had, prior to this, been dealt with through the application of some results concerning Dixon type sums and formulas [2], in this short paper we detail a new proof using a generating function approach which is rather succinct.

The Proof

Consider the polynomial

$$G(x) = \sum_{m=0}^{\infty} m \; {}_3F_2\left(\begin{matrix} 1-m, 1+m, \frac{1}{2} \\ 2, 2 \end{matrix} \,\middle|\, 1 \right) x^m, \tag{P1}$$

which is an ordinary generating function for the sequence (see Remark 1 later)

$$\left\{ m \; {}_3F_2\left(\begin{matrix} 1-m, 1+m, \frac{1}{2} \\ 2, 2 \end{matrix} \,\middle|\, 1 \right) \right\}_{m=0}^{\infty} = \left\{ 0, 1, \frac{5}{4}, \frac{5}{4}, \frac{81}{64}, \frac{81}{64}, \ldots \right\}. \tag{P2}$$

Then, with $(u)_k$ denoting the usual rising factorial function $(u)_k = u(u+1)(u+2)(u+3)\cdots(u+k-1)$,

$$\begin{aligned} G(x) &= \sum_{m=0}^{\infty} m \left[\sum_{k=0}^{\infty} \frac{(1-m)_k (1+m)_k (\frac{1}{2})_k}{(2)_k^2 k!} \right] x^m \\ &= \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \frac{m(1-m)_k (1+m)_k (\frac{1}{2})_k}{(2)_k^2 k!} x^m, \end{aligned} \tag{P3}$$

since $(1-m)_k = 0$ for $m < k+1$. Continuing,

$$\begin{aligned} G(x) &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(m+k+1)(-m-k)_k (m+k+2)_k (\frac{1}{2})_k}{(2)_k^2 k!} x^{m+k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!}{k!} \frac{(\frac{1}{2})_k}{(2)_k^2} x^{k+1} \sum_{m=0}^{\infty} \binom{m+2k+1}{m} x^m, \end{aligned} \tag{P4}$$

after a little algebra. We now note that the sum in m is merely the series form of $(1-x)^{-2(k+1)}$, and since $(2k+1)!/k! = 4^k (\frac{3}{2})_k$ $(k \geq 0)$, we can further write

$$\begin{aligned} G(x) &= \frac{x}{(1-x)^2} \sum_{k=0}^{\infty} \frac{(\frac{3}{2})_k (\frac{1}{2})_k}{(2)_k^2} \left[\frac{-4x}{(1-x)^2} \right]^k \\ &= -\frac{1}{4} \sum_{k=1}^{\infty} \frac{(\frac{3}{2})_{k-1} (\frac{1}{2})_{k-1}}{(2)_{k-1}^2} \left[\frac{-4x}{(1-x)^2} \right]^k \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{(\frac{1}{2})_k(-\frac{1}{2})_k}{(1)_k} \frac{1}{k!} \left[\frac{-4x}{(1-x)^2}\right]^k \\
&= {}_2F_1\left(\begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \middle| \frac{-4x}{(1-x)^2} \right) - 1 \\
&= \frac{1}{1-x}\, {}_2F_1\left(\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \middle| x^2 \right) - 1, \qquad \text{(P5)}
\end{aligned}
$$

on applying one of Kummer's quadratic transformations [3, (3.1.11), p.191][1]

$$
{}_2F_1\left(\begin{matrix} a, b \\ 2b \end{matrix} \middle| \frac{4x}{(1+x)^2} \right) = (1+x)^{2a}\, {}_2F_1\left(\begin{matrix} a, a+\frac{1}{2}-b \\ b+\frac{1}{2} \end{matrix} \middle| x^2 \right) \qquad \text{(P6)}
$$

with $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $x \to -x$. Appealing next to Euler's well known transformation (see, *e.g.*, [4, (1.3.15), p.10])

$$
{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = (1-z)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix} \middle| z \right), \qquad \text{(P7)}
$$

in which we set $a = b = -\frac{1}{2}$, $c = 1$, $z = x^2$, we arrive at a final form for $G(x)$, namely,

$$
G(x) = (1-x^2)(1+x)g(x) - 1, \qquad \text{(P8)}
$$

where

$$
g(x) = {}_2F_1\left(\begin{matrix} \frac{3}{2}, \frac{3}{2} \\ 1 \end{matrix} \middle| x^2 \right). \qquad \text{(P9)}
$$

The proof now concludes quickly, for equating coefficients of x^{2n} across (P8) yields, for $n \geq 1$,

$$
\begin{aligned}
2n\, {}_3F_2\left(\begin{matrix} 1-2n, 1+2n, \frac{1}{2} \\ 2, 2 \end{matrix} \middle| 1 \right) &= [x^{2n}]\{g(x)\} - [x^{2n-2}]\{g(x)\} \\
&= \left(\frac{(\frac{3}{2})_n}{n!} \right)^2 - \left(\frac{(\frac{3}{2})_{n-1}}{(n-1)!} \right)^2 \\
&= \frac{1}{2^{4n}}(4n+1) \binom{2n}{n}^2 \qquad \text{(P10)}
\end{aligned}
$$

as required, after some simplification.□

Remark 1 We remark, for completeness, on the repetition in terms exhibited

[1] The lower parameter in the l.h.s. of (P6) is printed incorrectly as $2a$ in [3].

by the sequence (P2). Clearly, since the argument of the hypergeometric series $g(x)$ is x^2,

$$\begin{aligned} f(x) &= (1-x^2)g(x) \\ &= \sum_{n=0}^{\infty} a_n x^{2n}, \end{aligned} \qquad (1)$$

say, where $a_0 = 1$. Then (P8) gives

$$\begin{aligned} G(x) &= (1+x)f(x) - 1 \\ &= \sum_{n=0}^{\infty} a_n(x^{2n} + x^{2n+1}) - 1 \\ &= \{a_0(1+x) + a_1(x^2+x^3) + a_2(x^4+x^5) + a_3(x^6+x^7) + \cdots\} - 1 \\ &= a_0 x + a_1(x^2+x^3) + a_2(x^4+x^5) + a_3(x^6+x^7) + \cdots, \end{aligned} \qquad (2)$$

which generates a sequence of form $\{0, a_0, a_1, a_1, a_2, a_2, a_3, a_3, \ldots\}$. Evidently, from (1),(P10) $a_n = [x^{2n}]\{f(x)\} = [x^{2n}]\{g(x)\} - [x^{2n-2}]\{g(x)\} = 2^{-4n}(4n+1)\binom{2n}{n}^2$, so that $a_0 = 1$, $a_1 = \frac{5}{4}$, $a_2 = \frac{81}{64}$, $a_3 = \frac{325}{256}$, *etc.*

Remark 2 Two generalised versions of the Theorem are given in [1], which may themselves succumb to the technique seen here. However, due to their complex nature, this would be a non-trivial exercise and lies beyond the remit of this paper.

References

[1] Koepf, W.A. and Larcombe, P.J. (2009). The sum $16^n \sum_{k=0}^{2n} 4^k \binom{\frac{1}{2}}{k} \binom{-\frac{1}{2}}{k} \binom{-2k}{2n-k}$: a computer assisted proof of its closed form, and some generalised results, *Util. Math.*, **79**, pp.xx-xx.

[2] Larcombe, P.J. and Larsen, M.E. (2009). The sum $16^n \sum_{k=0}^{2n} 4^k \binom{\frac{1}{2}}{k} \binom{-\frac{1}{2}}{k} \binom{-2k}{2n-k}$: a proof of its closed form, *Util. Math.*, **79**, pp.3-7.

[3] Andrews, G.E., Askey, R. and Roy, R. (1999). Special functions (Encyclopaedia of mathematics and its applications, No. 71), Cambridge University Press, Cambridge, U.K.

[4] Slater, L.J. (1966). Generalized hypergeometric functions, Cambridge University Press, London, U.K.