FINDING IDENTITIES WITH THE WZ METHOD

Ira M. Gessel*
Department of Mathematics
Brandeis University
Waltham, MA 02254-9110

Abstract. Extending the work of Wilf and Zeilberger on WZ-pairs, we describe how new terminating hypergeometric series identities can be derived by duality from known identities. A large number of such identities are obtained by a Maple program that applies this method systematically.

1. Introduction. Wilf and Zeilberger [10] introduced a new and powerful method, based on Gosper's indefinite summation algorithm [5], for proving identities for hypergeometric series. (See also Petkovšek, Wilf, and Zeilberger [9].) They also showed how their approach led to new identities through duality.

In this paper we extend Wilf and Zeilberger's duality approach, and with the help of Maple, apply it systematically to obtain a large number of identities, many of which are new.

2. The WZ Method. Suppose we want to prove an identity of the form

$$\sum_{k=0}^{\infty} A(n,k) = B(n).$$

If $B(n) \neq 0$, then we may set $F(n,k) = A(n,k)/B(n)$ and rewrite the identity to be proved as

$$\sum_{k=0}^{\infty} F(n,k) = 1.$$

Now suppose that we can find a function $G(n,k)$ such that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{2.1}$$

Then summing (2.1) on k from 0 to ∞, we obtain

$$\sum_{k=0}^{\infty} F(n+1,k) - \sum_{k=0}^{\infty} F(n,k) = \lim_{k\to\infty} G(n,k+1) - G(n,0).$$

If $G(n,0)$ and $\lim_{k\to\infty} G(n,k)$ are 0 then we have

$$\sum_{k=0}^{\infty} F(n+1,k) = \sum_{k=0}^{\infty} F(n,k), \tag{2.2}$$

and thus $\sum_{k=0}^{\infty} F(n,k)$ is independent of n, so if we can evaluate the sum for one value of n (which is usually $n = 0$), we can evaluate it for all n. In most of our applications $F(n,k)$ and $G(n,k)$ will be 0 for k sufficiently large, so we will be working with finite sums.

In order to use this idea, we need to be able to find the function $G(n,k)$, which we call the *mate* of F. The only cases that we will be concerned with are those in which F and G can be expressed explicitly in terms of gamma functions, and in these cases $G(n,k)/F(n,k)$ is a rational function of n and k, which we denote by $Q(n,k)$. Note that if we are given $Q(n,k)$, it is easy to verify that (2.1) holds with $G(n,k) = F(n,k)Q(n,k)$, and thus Wilf and Zeilberger describe $Q(n,k)$ as a "proof certificate" since the verification is purely mechanical. (Actually, Wilf and Zeilberger use the ratio $R(n,k) = G(n,k)/F(n,k-1)$ as their proof certificate rather than our $Q(n,k)$.) We can find $Q(n,k)$, as described in [10], by R. W. Gosper's indefinite summation algorithm [5, 6].

*partially supported by NSF grant DMS-9306297

3. WZ-functions. It turns out to be very helpful to consider functions F and G satisfying (2.1) in which n and k may be complex numbers and F and G have a "closed form." This enables us to verify (2.1) completely formally, without having to worry about the domain of definition of F and G. We may then take a limit, when necessary, to deal with cases in which the formulas for F and G may be ambiguous (and these cases often arise in interesting identities).

Let us call a function $f(n,k)$ of complex variables n and k a *gamma quotient* if

$$f(n,k)=\alpha^n\beta^k\frac{\prod_{i=1}^{r}\Gamma(a_in+b_ik+\gamma_i)}{\prod_{j=1}^{s}\Gamma(c_jn+d_jk+\delta_j)}, \tag{3.1}$$

where α, β, γ_i, and δ_j are complex numbers, and a_i, b_i, c_j, and d_j are integers. If n and k are complex numbers such that (3.1) contains a gamma function evaluated at a nonpositive integer, then we interpret $f(n,k)$ as $\lim_{n'\to n}\bigl(\lim_{k'\to k} f(n',k')\bigr)$ if this limit exists; otherwise $f(n,k)$ is undefined. Note that we are taking the limit in k first. This definition is the one most useful for our applications, even though it is not symmetric in n and k. (For example, if $f(n,k)=(n+2k)/(n+k)$ and $g(n,k)=f(k,n)=(k+2n)/(k+n)$ then $f(0,0)=\lim_{n\to 0} n/n=1$ and $g(0,0)=\lim_{n\to 0}2n/n=2$.)

We call a pair (F,G) a *WZ-pair* if F and G are gamma quotients satisfying (2.1) for all complex n and k for which both sides are defined. Note that to verify (2.1) it is sufficient to do so under the assumption that all gamma functions involved are defined; we may then take a limit, if necessary, for the remaining values. (Our definition of a WZ-pair is slightly different from that of Wilf and Zeilberger [10] in that they do not require F and G to be gamma quotients.) It is easy to show that if (F,G) is a WZ-pair then $G(n,k)/F(n,k)$ must be a rational function of n and k. In fact, (2.1) may be written

$$\frac{G(n,k)}{F(n,k)}=\frac{F(n+1,k)/F(n,k)-1}{G(n,k+1)/G(n,k)-1},$$

so if $F(n+1,k)/F(n,k)$ and $G(n,k+1)/G(n,k)$ are rational then so is G/F.

We call a gamma quotient F a *WZ-function* if there exists a gamma quotient G such that (F,G) is a WZ-pair. If F is a WZ-function then its mate G can be found by Gosper's algorithm.

Once we have a WZ-pair (F,G), we can easily find other WZ-functions. The following theorem is straightforward:

Theorem 1. *Let (F,G) be a WZ-pair.*

(i) *For any complex numbers α and β,*

$$\bigl(F(n+\alpha,k+\beta),G(n+\alpha,k+\beta)\bigr)$$

is a WZ-pair.

(ii) *For any complex number γ, $\bigl(\gamma F(n,k),\gamma G(n,k)\bigr)$ is a WZ-pair.*

(iii) *If $p(n,k)$ is a gamma product such $p(n+1,k)=p(n,k+1)=p(n,k)$ for all complex n and k for which $p(n,k)$ is defined then*

$$\bigl(p(n,k)F(n,k),p(n,k)G(n,k)\bigr)$$

is a WZ-pair.

(iv) $\bigl(F(-n,k),-G(-n-1,k)\bigr)$ *is a WZ-pair.*

(v) $\bigl(F(n,-k),-G(n,-k+1)\bigr)$ *is a WZ-pair.*

(vi) $\bigl(G(k,n),F(k,n)\bigr)$ *is a WZ-pair.*

We shall call the WZ-pairs obtained from (F,G) by any combination of (i)–(v) the *associates* of (F,G), and (extending the terminology of Wilf and Zeilberger) we call (vi) and all its associates the *duals* of (F,G). We shall also use the terms associate and dual to refer to the WZ-functions themselves and to the identities obtained from them.

Although (ii) is a special case of (iii), it is so useful that it deserves separate mention. The most important application of (iii) for nonconstant p arises through the reflection formula for the gamma function, $\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$: Let a and b be integers, let γ be an arbitrary complex number, and let

$$p(n,k)=(-1)^{an+bk}\Gamma(an+bk+\gamma)\Gamma(1-an-bk-\gamma),$$

where $(-1)^x$ may be unambiguously defined as $e^{i\pi x}$. Then by the reflection formula,

$$p(n,k) = (-1)^{an+bk}\pi/\sin\pi(an+bk+\gamma)$$

and thus $p(n,k)$ satisfies the conditions in (iii). So from any WZ-function F we may obtain a new WZ-function by replacing any factor $\Gamma(an+bk+\gamma)$ with $(-1)^{an+bk}/\Gamma(1-an-bk-\gamma)$. (Cf. section 4 of [10].)

We shall use the standard notation for hypergeometric series: The *rising factorial* is

$$(a)_k = a(a+1)\cdots(a+k-1),$$

if k is a nonnegative integer, and more generally,

$$(a)_k = \Gamma(a+k)/\Gamma(a)$$

for all k, with our usual interpretation for gamma quotients. The *hypergeometric series* is

$$ {}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(1)_k(b_1)_k\cdots(b_q)_k} z^k, \tag{3.2}$$

which when not displayed we write as ${}_pF_q\left(a_1,\dots,a_p; b_1,\dots,b_q \mid z\right)$. The a_i are called *numerator parameters* and the b_j are called *denominator parameters*. Our interpretation of gamma quotients implies that a factor of $(a+1)_k/(a)_k$ should be interpreted as $(a+k)/a$, even if a is a negative integer, and this must be interpreted as 1 if $k=0$ and $a=0$.

A modification to the definition of the hypergeometric series is necessary for some identities. Suppose that after replacing each $(a+1)_k/(a)_k$ with $(a+k)/a$, there remains a numerator parameter which is a nonpositive integer. If $-m$ is the one of least absolute value, then we interpret the sum in (3.2) as stopping with the term $k=m$. For example, if n is a nonnegative integer, we interpret the hypergeometric series ${}_2F_1\left(a,-n;-2n \mid z\right)$ as

$$\sum_{k=0}^{n} \frac{(a)_k(-n)_k}{(1)_k(-2n)_k} z^k \tag{3.3}$$

even though according to our definitions $(-n)_k/(-2n)_k$ is nonzero for $k>2n$. There is one further complication: if $n=1$, we want ${}_2F_1\left(a,-n;-2n \mid z\right)$ to be interpreted as (3.3), even though the terminating numerator parameter -1 is one more than the denominator parameter -2. For the general rule used in this paper, suppose that $f(n)$ is a hypergeometric series in which every numerator and denominator parameter is a linear function of n. Then if a numerator parameter $rn+s$ is a nonpositive integer for some value of n, we take it as a terminating parameter unless there is a denominator parameter given by the formula $rn+s-1$.

We shall also use the notation

$$\Gamma\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) = \frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)}.$$

4. An example: Gauss's second theorem. We now give an example of the approach we shall take. We start with a hypergeometric series identity which we write in the form

$$\sum_{k=0}^{\infty} f(k,a,b,\dots) = 1,$$

where $f(k,a,b,\dots)$ is a gamma quotient involving k and the parameters $a,b,\dots$, but not n. (It does not matter if the identity holds only when it terminates.) We construct a potential WZ-function $F(n,k)$ by making a substitution in f that introduces n, and we apply Gosper's algorithm to determine whether a mate can be found for $F(n,k)$. If so, we try to find an associate of $F(n,k)$ that is a terminating hypergeometric series, and if its mate $G(n,k)$ satisfies $G(n,0)=0$ and $\lim_{k\to\infty} G(n,k)=0$, then we will have a WZ proof of a terminating form of our original identity or of a closely related identity. Next we look for terminating hypergeometric series among the duals of F, and again check that their mates satisfy the right conditions.

As our first example, we take one of the standard hypergeometric summation formulas, which is sometimes called Gauss's second summation theorem (Bailey [1, p. 11, equation (2)]):

$$ {}_2F_1\left(\begin{matrix} A, B \\ \frac{1}{2}(A+B+1) \end{matrix} \,\Big|\, \frac{1}{2}\right) = \Gamma\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}(A+B+1) \\ \frac{1}{2}(A+1), \frac{1}{2}(B+1) \end{matrix}\right). \tag{4.1} $$

We first express this identity in such a way that the coefficients of the parameters on both sides are integers. Thus we set $A = 2a$ and $B = 2b$, in (4.1), obtaining

$$ {}_2F_1\left(\begin{matrix} 2a, 2b \\ a+b+\frac{1}{2} \end{matrix} \,\Big|\, \frac{1}{2}\right) = \Gamma\left(\begin{matrix} \frac{1}{2}, a+b+\frac{1}{2} \\ a+\frac{1}{2}, b+\frac{1}{2} \end{matrix}\right), $$

which we rewrite as

$$ \sum_{k=0}^{\infty} \Gamma\left(\begin{matrix} 2a+k, 2b+k, a+\frac{1}{2}, b+\frac{1}{2} \\ 1+k, a+b+\frac{1}{2}+k, 2a, 2b, \frac{1}{2} \end{matrix}\right) \left(\frac{1}{2}\right)^k = 1. \tag{4.2} $$

To get a potential WZ-function from (4.2), we make a substitution of the form $a := a + pn$, $b := b + qn$, $k := k + rn$, where p, q, and r are integers. There are many ways to do this that give WZ-functions, but we shall consider only two of them. First let us replace a with $a - n$. Applying Gosper's algorithm, we find that the function

$$ F(n,k) = \Gamma\left(\begin{matrix} 2a-2n+k, 2b+k, a-n+\frac{1}{2}, b+\frac{1}{2} \\ 1+k, a-n+b+\frac{1}{2}+k, 2a-2n, 2b, \frac{1}{2} \end{matrix}\right) \left(\frac{1}{2}\right)^k \tag{4.3} $$

is a WZ-function with mate

$$ G(n,k) = F(n,k) \frac{2k\,(a-n+b+k-\frac{1}{2})}{(2a-2n+k-1)(2a-2n+k-2)} \tag{4.4} $$

Note that we could have reduced the computation required in Gosper's algorithm by computing (4.4) first in the case $a = 0$, and then replacing n by $n - a$, as allowed by Theorem 1(i).

We would now like to specialize the parameters in (4.2) so that $\sum_k F(n,k)$ is a terminating hypergeometric series when n is a nonnegative integer. We may write $F(n,k)$ as

$$ \frac{(2a-2n)_k (2b)_k}{(1)_k (a+b-n+\frac{1}{2})_k} \left(\frac{1}{2}\right)^k U_1(n) \tag{4.5} $$

for some function $U_1(n)$. Summing on k will yield a terminating hypergeometric series if $2a - 2n$ is a nonpositive integer, and without loss of generality we may take $a = 0$ or $a = -1/2$.

In the case $a = 0$ we have

$$ F(n,k) = \frac{(-2n)_k (2b)_k}{(1)_k (b-n+\frac{1}{2})_k} \left(\frac{1}{2}\right)^k \frac{(-b+\frac{1}{2})_n}{(\frac{1}{2})_n} $$

and

$$ G(n,k) = F(n,k) \frac{2k(-n+b+k-\frac{1}{2})}{(-2n+k-1)(-2n+k-2)} $$

for all nonnegative integers n. We have $G(n,0) = 0$ and $G(n,k) = 0$ for all integers $k \geq 2n+2$, so we may conclude that

$$ {}_2F_1\left(\begin{matrix} -2n, 2b \\ b-n+\frac{1}{2} \end{matrix} \,\Big|\, \frac{1}{2}\right) = \frac{(\frac{1}{2})_n}{(-b+\frac{1}{2})_n}, \tag{4.6} $$

when n is a nonnegative integer, which is, of course, just the case $a = -n$ of (4.1).

Similarly, the case $a = -1/2$ gives

$$ F(n,k) = \frac{(-2n-1)_k (2b)_k}{(1)_k (b-n)_k} \left(\frac{1}{2}\right)^k \frac{(-b+1)_n}{(1)_n} \Gamma\left(\begin{matrix} \frac{1}{2}, 1-b \\ \frac{1}{2}-b \end{matrix}\right). \tag{4.7} $$

and we find that

$$ {}_2F_1\left(\begin{matrix}-2n-1, 2b\\ b-n\end{matrix}\;\middle|\;\frac{1}{2}\right)\frac{(-b+1)_n}{(1)_n} $$

is a constant for n a nonnegative integer. Checking the case $n=0$, we find that the constant is 0, so

$$ {}_2F_1\left(\begin{matrix}-2n-1, 2b\\ b-n\end{matrix}\;\middle|\;\frac{1}{2}\right)=0, $$

which is the other terminating form of (4.1).

We now look at the associates of F. Let us first replace k with $k+c$ in (4.3). (There is no need for a similar substitution with n, since the parameter a already shifts n arbitrarily.) We obtain

$$ \frac{(2a-2n+c)_k(2b+c)_k}{(1+c)_k(a+b-n+\frac{1}{2}+c)_k}\left(\frac{1}{2}\right)^k U_2(n) \tag{4.8} $$

for some function $U_2(n)$. To get a terminating hypergeometric series from (4.8), we attempt to choose the parameters so that when n is a nonnegative integer, one of the numerator parameters $2a-2n+c$ and $2b+c$ is a nonpositive integer, and one of the denominator parameters $1+c$ and $a+b-n+\frac{1}{2}+c$ is 1 (independent of n).[1] It is clear that the denominator condition requires $c=0$, and the numerator condition implies that in addition, either $a=0$, giving the case we have already considered, or b is a negative integer or half-integer, say $b=-m/2$. This second case leads to the identity that for any nonnegative integer m,

$$ {}_2F_1\left(\begin{matrix}2a-2n, -m\\ \frac{1}{2}(1-m)+a-n\end{matrix}\;\middle|\;\frac{1}{2}\right)\frac{(1/2-a+m/2)_n}{(1/2-a)_n} $$

is independent of n. Unlike the other cases we have seen, we cannot find out what this quantity is by setting $n=0$, although it is not hard to determine that it is 0 for m odd and $(\frac{1}{2})_{m/2}/(\frac{1}{2}-a)_{m/2}$ for m even. (This is, not surprisingly, equivalent to (4.6) and (4.7).) For simplicity, we shall not pursue identities of this type, and shall henceforth consider only those in which the terminating parameters are linear functions of n.

We may also replace k with $-k$ in (4.8), getting the associate

$$ \frac{(-c)_k(n-a-b-c+\frac{1}{2})_k}{(1+2n-2a-c)_k(1-2b-c)_k}2^k U_2(n). $$

We can get a WZ-function that will give a terminating sum of the type we want by replacing n with $-n$, and then setting $b=a-1/2$ and $c=1-2a$. After multiplying by a constant (which depends on a), we obtain the WZ-function

$$ F(n,k)=\frac{(2a-1)_k(-n)_k}{(1)_k(-2n)_k}2^k\frac{(\frac{1}{2})_n}{(a)_n}, \tag{4.9} $$

with mate

$$ G(n,k)=-\frac{1}{4}\frac{k(2n-k+1)}{(n+a)(n-k+1)}F(n,k). \tag{4.10} $$

We would like to evaluate the sum $\sum_{k=0}^{n} F(n,k)$ for n a nonnegative integer. Unfortunately, it is not true that $F(n,k)=0$ for all $k>n$ nor is $G(n,k)=0$ for all sufficiently large k. However, we do have $F(n,k)=0$ for $n+1\le k\le 2n$ and $G(n,k)=0$ for $n+2\le k\le 2n+1$, and thus we have

$$ \begin{aligned} \sum_{k=0}^{n+1} F(n+1,k)-\sum_{k=0}^{n} F(n,k) &= \sum_{k=0}^{n+1}\bigl(F(n+1,k)-F(n,k)\bigr)\\ &= \sum_{k=0}^{n+1}\bigl(G(n,k+1)-G(n,k)\bigr)=G(n,n+2)-G(n,0)=0 \end{aligned} $$

[1]It would be sufficient to make one of the denominator parameters a positive integer, possibly depending on n, but in that case we would need to start the sum at a point other than $k=0$. For simplicity we do not pursue these possibilities, although we may thereby miss some identities.

for $n \geq 1$. We deduce that $\sum_{k=0}^{n} F(n,k)$ is independent of n for $n \geq 1$, and checking the values of the sum for $n = 0$ and $n = 1$ yields the identity

$$ {}_2F_1\left(\begin{matrix} 2a-1, -n \\ -2n \end{matrix} \,\middle|\, 2\right) = \frac{(a)_n}{(\frac{1}{2})_n} \tag{4.11} $$

for all nonnegative integers n.

If we reverse the order of summation in the sum in (4.11), we obtain

$$ {}_2F_1\left(\begin{matrix} -n, n+1 \\ 2-2a-n \end{matrix} \,\middle|\, \frac{1}{2}\right) = \frac{(a)_n}{(2a-1)_n}, $$

which is a terminating case of the well-known identity [Bailey 1, p. 11, formula (3)]

$$ {}_2F_1\left(\begin{matrix} A, 1-A \\ C \end{matrix} \,\middle|\, \frac{1}{2}\right) = \Gamma\left(\begin{matrix} \frac{C}{2}, \frac{C}{2}+\frac{1}{2} \\ \frac{C}{2}+\frac{A}{2}, \frac{1}{2}+\frac{C}{2}-\frac{A}{2} \end{matrix}\right). \tag{4.12} $$

(The identity (4.12) is sometimes called "Bailey's theorem," but it can be found in Kummer's 1836 paper [8].) The identities (4.1) and (4.12) are associates in a certain sense; they sum (by shifting the starting point) the two special cases of the bilateral hypergeometric series

$$ \sum_{k=-\infty}^{\infty} \frac{(A+B-1)_k (B-A)_k}{(B)_k (C)_k} \left(\frac{1}{2}\right)^k \tag{4.13} $$

in which either B or C is a positive integer.

If we apply Theorem 1(vi) to (4.9) and (4.10), we obtain the dual WZ-function

$$ \frac{(1-a)_k (3/2-a-b-n)_k}{(2-a-n/2)_k (3/2-a-n/2)_k} U_3(n) $$

for some function $U_3(n)$. It is not difficult to see that any identity we obtain from this WZ-function will be an instance of Vandermonde's theorem, so we don't consider it further.

It was very convenient in our examples that when we adjusted the parameters so as to make $\sum_k F(n,k)$ a terminating hypergeometric series, $G(n,k)$ had the right properties to make the WZ method work. In fact, this is not a coincidence, and we state the following theorem without proof.

Theorem 2. *Let (F,G) be a WZ-pair such that for each nonnegative integer n, $F(n,k) = 0$ whenever k is a negative integer or a sufficiently large (depending on n) positive integer. Then for all nonnegative integers n, $G(n,0) = 0$, and for all but finitely many nonnegative integers n, $G(n,k) = 0$ for k sufficiently large.*

To see that we cannot replace "all but finitely many" with "all", consider the WZ function $F(n,k) = \binom{n}{2k}/2^{n-1}$, with mate

$$ G(n,k) = -F(n,k)k(k-1)/n(n-2k+1) = -2^{-n}\binom{n-1}{2k-2}. $$

We find that $G(0,k) = -1$ when k is a positive integer, but if n and k are positive integers then $G(n,k) = 0$ for $k > (n+1)/2$. Thus we may conclude that $\sum_{k=0}^{\infty} F(n,k) = 1$ for $n \geq 1$; but this identity is false for $n = 0$.

It is possible to find a more complicated generalization of Theorem 2 that covers cases such as that of (4.9) and (4.10), but it is easy enough in any particular case to verify that the right conditions are satisfied. We find that whenever F satisfies appropriate termination conditions, we do get an identity.

Next we return to (4.2) and apply a different substitution, $a := a+n$, $b := b+n$, $k := k-n$. We obtain the WZ-function

$$ F(n,k) = \Gamma\left(\begin{matrix} 2a+n+k, 2b+n+k, a+n+\frac{1}{2}, b+n+\frac{1}{2} \\ -n+1+k, a+b+n+\frac{1}{2}+k, 2a+2n, 2b+2n, \frac{1}{2} \end{matrix}\right), $$

with

$$G(n,k) = -\frac{1}{4}F(n,k)\frac{(3n+2a+2b+k)(k-n)}{(a+n)(b+n)}.$$

This WZ-function and its associates give us no new identities, but its duals do. Applying Theorem 1(vi), then substituting $n+c$ for n and dividing out the factors not involving n or k, gives the WZ-function

$$F(n,k) = \frac{(2a+n+c)_k(2b+n+c)_k(\frac{2b}{3}+\frac{2a}{3}+\frac{n}{3}+\frac{c}{3}+1)_k(-n-c+1)_k}{(a+b+\frac{1}{2}+n+c)_k(\frac{2b}{3}+\frac{2a}{3}+\frac{n}{3}+\frac{c}{3})_k(a+1)_k(b+1)_k}\left(-\frac{1}{8}\right)^k \times\frac{(2a+c)_n(2b+c)_n(2b+2a+c+1)_n}{(a+b+\frac{1}{2}+c)_n(2b+2a+c)_n(c)_n}\left(\frac{1}{2}\right)^n \tag{4.14}$$

To get proper termination in (4.14), we may set $c=1$ and $b=0$, and we obtain the identity

$${}_4F_3\left(\begin{matrix}2a+n+1, n+1, \frac{2a}{3}+\frac{n}{3}+\frac{4}{3}, -n\\ a+\frac{3}{2}+n, \frac{2a}{3}+\frac{n}{3}+\frac{1}{3}, a+1\end{matrix}\;\middle|\; -1/8\right) = \frac{(a+3/2)_n}{(2a+2)_n}2^n.$$

If we now consider the associate of (4.14) obtained by replacing n with $-n$, we find two further identities. To make $-n$ the terminating parameter, without loss of generality we set $c=-2a$ and we obtain the WZ-function

$$F(n,k) = \frac{(-n)_k(2b-n-2a)_k(\frac{2b}{3}-\frac{n}{3}+1)_k(2a+n+1)_k\,(-1/8)^k}{(b-a+\frac{1}{2}-n)_k(\frac{2b}{3}-\frac{n}{3})_k(a+1)_k(b+1)_k} \times\frac{(a-b+\frac{1}{2})_n(1-2b)_n(2a+1)_n2^n}{(1)_n(1-2b+2a)_n(-2b)_n} \tag{4.15}$$

To make one of the denominator parameter in (4.15) equal to 1, we may set $a=0$ or $b=0$. If we set $a=0$, we obtain the identity

$${}_4F_3\left(\begin{matrix}-n, 2b-n, \frac{2b}{3}-\frac{n}{3}+1, n+1\\ b+\frac{1}{2}-n, \frac{2b}{3}-\frac{n}{3}, b+1\end{matrix}\;\middle|\; -1/8\right) = \frac{(-2b)_n}{(\frac{1}{2}-b)_n}\left(\frac{1}{2}\right)^n.$$

If we try to set $b=0$ in (4.15), we get a factor of $(0)_n$ in the denominator. To avoid this, we multiply (4.15) by $-2b$ and then take the limit as b approaches 0, and we obtain the WZ-function

$$F(n,k) = \frac{(-n)_k(-n-2a)_k(1-\frac{n}{3})_k(2a+n+1)_k}{(1)_k(\frac{1}{2}-a-n)_k(-\frac{n}{3})_k(a+1)_k}\,(-1/8)^k\,\frac{(a+\frac{1}{2})_n}{(1)_{n-1}}2^n.$$

It then follows that for $n\ge 1$, $\sum_{k=0}^n F(n,k)$ is a constant,which turns out to be 0. Since $0\cdot(1)_{n-1}=(0)_n$, we write the identity obtained in this way in the cryptic-looking form

$${}_4F_3\left(\begin{matrix}-n, -n-2a, 1-\frac{n}{3}, 2a+n+1\\ \frac{1}{2}-a-n, -\frac{n}{3}, a+1\end{matrix}\;\middle|\; -1/8\right) = \frac{(0)_n}{(a+\frac{1}{2})_n}\left(\frac{1}{2}\right)^n$$

in order to indicate the WZ-function from which it is derived.

5. Integral forms and the multiplication formula. In order to extract a WZ-function from an identity involving several parameters, we must normalize it so that its parameters appear with integer coefficients. We call these normalizations the *integral forms* of the identity. In addition to the usual integral form of an identity, we can, in many cases, specialize the parameters and use the duplication or multiplication formula for the gamma function to obtain inequivalent integral forms whose duals give different identities. For example, Vandermonde's (Gauss's) theorem has the usual integral form

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\;\middle|\; 1\right) = \Gamma\left(\begin{matrix}c, c-a-b\\ c-a, c-b\end{matrix}\right). \tag{5.1}$$

If replace a by $a/2$ and b by $(a+1)/2$, then after applying the duplication formula for the gamma function, we obtain an additional integral form

$$ {}_2F_1\left(\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2} \\ c \end{matrix} \;\middle|\; 1\right) = \sum_{k=0}^{\infty} \frac{(a)_{2k}}{(1)_k (c)_k} 2^{2k} = 2^{2c-a-2}\Gamma\left(\begin{matrix} c, c-a-\frac{1}{2} \\ 2c-a-1, \frac{1}{2} \end{matrix}\right), $$

which may be written

$$ \sum_{k=0}^{\infty} \Gamma\left(\begin{matrix} a+2k, 2c-a-1, \frac{1}{2} \\ 1+k, c+k, a, c-a-\frac{1}{2} \end{matrix}\right) 2^{2-2k-2c+a} = 1. \tag{5.2} $$

For example, if we substitute $a+2n$ for a and $c+n$ for c in (5.2), one of the dual identities that we obtain is

$$ {}_4F_3\left(\begin{matrix} \frac{1}{2}-n, \frac{5}{4}-\frac{n}{2}, \frac{3}{2}-c, -n \\ \frac{1}{4}-\frac{n}{2}, c-n, \frac{3}{2} \end{matrix} \;\middle|\; -1\right) = \frac{(-\frac{1}{2})_n}{(1-b)_n}, $$

which is a special case of the very-well-poised ${}_4F_3(-1)$. (See, for example, Bailey [1, p. 28, equation (3)]).

6. Specialization and factorization. There is one more technique that we shall use to find additional identities. Let us make the substitution $a := a+n$, $c := c+2n$ in (5.2). Applying Gosper's algorithm to

$$ F(n,k) = \Gamma\left(\begin{matrix} a+n+2k, \frac{1}{2}, 2c+3n-1-a \\ 1+k, c+2n+k, a+n, c+n-a-\frac{1}{2} \end{matrix}\right) 2^{2-2k-2c-3n+a} $$

yields $G(n,k) = F(n,k)Q(n,k)$ where

$$ Q(n,k) = \frac{k\left(-4c^2-10cn+6ca+2c-a^2+10na-5n^2+3n-a+2ak+2nk\right)}{(a+n)(2c+2n-2a-1)(c+2n+k)} \tag{6.1} $$

Although dualization of (F,G) can yield identities, the complicated numerator in (6.1) will cause them to be inelegant. We may hope to find specializations of the parameters a and c that will make $Q(n,k)$ factor into linear factors. One way to proceed is to find a specialization that will cause the numerator to be divisible by one of the factors in the denominator. For example, if the numerator is to be divisible by $a+n$, then setting $n := -a$ in the numerator must cause the numerator to vanish. We find that setting $n := -a$ in the numerator yields $-2k(c-2a)(2c-4a-1)$, and this vanishes if $c = 2a$ or $c = 2a+1/2$. Substituting $c := 2a$ yields the WZ-pair

$$ F(n,k) = \Gamma\left(\begin{matrix} \frac{1}{2}, 3a+3n-1, a+n+2k \\ 1+k, a+n, a+n-\frac{1}{2}, 2a+2n+k \end{matrix}\right) 2^{2-2k-3a-3n}, $$

$$ G(n,k) = F(n,k)\frac{k(-5a-5n+2k+3)}{(2a+2n+k)(2a+2n-1)}, $$

and dualization yields, for example, the identity

$$ {}_4F_3\left(\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \frac{11}{5}+\frac{2n}{5}, \frac{5}{6} \\ \frac{5}{3}, -2n, \frac{6}{5}+\frac{2n}{5} \end{matrix} \;\middle|\; \frac{32}{27}\right) = \frac{(2/3)_n (3)_n}{(1/2)_n (4)_n} $$

The other specialization $c := 2a+1/2$ yields similar identities.

Example 3 of Wilf and Zeilberger [10] involves this type of specialization. They dualize the WZ-function

$$ F(n,k) = \binom{n}{k}^2 \Big/ \binom{2n}{n}, \tag{6.2} $$

which corresponds to a special case of Vandermonde's theorem. In fact, if we make the substitution $a := a-n$, $b := b-n$ in the Vandermonde summand

$$ \Gamma\left(\begin{matrix} a+k, b+k, c-a, c-b \\ 1+k, c+k, a, b, c-a-b \end{matrix}\right) $$

then the case $a = 0$, $b = 0$, $c = 1$ yields the WZ-function of Wilf and Zeilberger's example. We find that without restrictions on a, b, and c we obtain a quadratic factor in $G(n,k)$, but the procedure described above shows that it factors if $c = 1 + a - b$ (or $c = 1 - a + b$). We obtain by dualization a number of identities with one more free parameter than those in [10]. A typical example is

$$ {}_4F_3\left(\begin{matrix} -n, \frac{2b}{3} - \frac{2n}{3} + 1, b - n, b + \frac{1}{2} \\ \frac{2b}{3} - \frac{2n}{3}, b + 1, 2b + 1 \end{matrix} \;\Big|\; 4\right) = \frac{(-b)_n}{(b+1)_n}. $$

7. Nonterminating identities. The WZ method can also be used to prove general nonterminating forms of hypergeometric summation formulas. (Wilf and Zeilberger [10] proved some nonterminating identities but these all had integrality restrictions on the parameters.) We give only one simple example, but the same approach can be used for many of the identities derived here. The basic idea is that we prove (2.2) as usual, but instead of evaluating $\sum_{k=0}^{\infty} F(n,k)$ at $n = 0$, which is no easier than for any other value of n, we evaluate the limit as $n \to \infty$. A similar approach has been used by Karlsson [7] to prove some nonterminating ${}_2F_1$ evaluations conjectured by Gosper, but using the contiguous relations for the ${}_2F_1$ rather than the WZ method to obtain the basic recurrence.

In this way we prove Gauss's theorem (the nonterminating form of Vandermonde's theorem),

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \;\Big|\; 1\right) = \Gamma\left(\begin{matrix} c, c - a - b \\ c - a, c - b \end{matrix}\right), $$

for $\operatorname{Re}(c - a - b) > 0$. The identity may be written

$$ \sum_{k=0}^{\infty} V(k) = 1, $$

where

$$ V(k) = V(k; a, b, c) = \Gamma\left(\begin{matrix} a + k, b + k, c - a, c - b \\ 1 + k, c + k, a, b, c - a - b \end{matrix}\right). $$

Let us set $F(n,k) = V(k; a, b, c + n)$. We find that $F(n,k)$ is a WZ-function with mate $G(n,k) = F(n,k)k/(c + n - a - b)$. Clearly $G(n, 0) = 0$, and by Stirling's formula we find that

$$ G(n,k) \sim k^{a+b-c-n}\Gamma\left(\begin{matrix} c + n - a, c + n - b \\ a, b, c + n - a - b + 1 \end{matrix}\right) $$

as $k \to \infty$. So if $\operatorname{Re}(c - a - b) > 0$ and $n \geq 0$ then $\lim_{k\to\infty} G(n,k) = 0$, and we may conclude that for $n \geq 0$,

$$ \sum_{k=0}^{\infty} F(n,k) = \sum_{k=0}^{\infty} F(n + 1, k). $$

It follows that for every nonnegative integer n, $\sum_{k=0}^{\infty} F(0,k) = \sum_{k=0}^{\infty} F(n,k)$, and thus $\sum_{k=0}^{\infty} F(0,k) = \lim_{n\to\infty} \sum_{k=0}^{\infty} F(n,k)$. By Stirling's formula, we find that $F(n,k) = n^{-k}\big(1 + O(1/n)\big)$ uniformly in k as $n \to \infty$. Thus

$$ \sum_{k=0}^{\infty} V(k) = \sum_{k=0}^{\infty} F(0,k) = \lim_{n\to\infty} \sum_{k=0}^{\infty} F(n,k) = 1. $$

8. Computer generation of identities. Given a WZ-function, the derivation of the terminating hypergeometric series corresponding to its dual is essentially mechanical, and the last sections of this paper consist of lists of identities obtained in this way, using programs written in Maple.

We start with a set of known hypergeometric series summation formulas. Here we have used the major summation formulas found in Bailey's book [1], but many other known summation formulas could also be used. For each summation formula we attempt to find all of its inequivalent integral forms. (This was done by hand, but with some help from the computer, and it is possible that some may have been missed.) Each integral form may be written as $\sum_k f(k, a, b, \dots) = 1$ for some function $f(k, a, b, \dots)$ expressed as a quotient of gamma functions. (In some cases $\sum_k f(k,a,b\dots)$ will not converge to 1 unless it terminates, but this is irrelevant to the application of the method.) The program is given a list of substitutions that replace some of the parameters with linear functions of n. (The substitutions are chosen so as to include nearly all possibilities involving integers small enough to give nice results.) Each substitution yields a potential WZ-function $F(n,k)$.

The program starts by making a substitution such as $a := a+n$, $b := b-n$, $k-n$ in the function f, yielding a potential WZ-function $F(n,k)$. Since one of the variables is now redundant, we set one of these variables equal to 0. (In our example we might set $a=0$; i.e., the substitution is now $a := n$. Eliminating a redundant variable makes Gosper's algorithm run faster.) The program next applies Gosper's algorithm to find the mate $G(n,k)$, if it exists. If G/F does not factor into linear factors, the program tries to find specializations of the parameters that will cause it to factor. Next the program takes the shifted dual WZ-pair $\big(\tilde{F}(n,k),\tilde{G}(n,k)\big) = \big(G(k+\beta,n+\alpha),F(k+\beta,n+\alpha)\big)$, where α and β are new parameters, and finds specializations of the parameters in $\tilde{F}(\pm n,\pm k)$ which lead to terminating hypergeometric series. Finally the identities corresponding to these specializations are constructed. Duplicates and identities that are special cases of the most frequently occurring "classical identities" are removed, and the remaining identities are converted into TEX.

A large number of identities were found, and those presented in this paper are only a selection, though they include the most interesting identities. Many of the identities included are different terminating forms of the same nonterminating sum, and if we consider bilateral sums, the number of "different" identities may be decreased further, as in (4.13), though it is clear that the number of truly different identities is still large.

The problem of finding an algorithm for determining when two identities are equivalent (i.e., up to a change of variables) does not seem to be easy. An abstract version of the problem is: given a finite-dimensional vector space V with a distinguished basis and two subspaces V_1 and V_2, does there exist a permutation of the basis elements that takes V_1 to V_2?

9. The 2-3-5 rule. An interesting observation suggested by our results is the "2-3-5 rule": in nearly every identity, the argument z has the property that the numerator and denominator are (up to sign) powers of 2, 3, or 5, and moreover, $1-z$ has the same property. Thus we find identities with $z=4$, 16/25, 27/32, and 2/27 (and Gosper has found some with $z=3/128$), but we do not find identities with $z=6$, 8, -9, or 4/27. There are a few exceptions to this rule: the only one occurring in this paper is $z=16/27$, but a few identities with $z=-27$ are given in [4] and Kummer [8] found some ${}_2F_1$ evaluations at irrational values. Moreover, some possibilities allowed by the rule, such as 2/3, 3/8 and 4/9, do not seem to occur.

10. The identities. The next sections contain a listing of some of the identities found by the program, together with their "proof certificates" $Q(n,k)=G(n,k)/F(n,k)$, where $F(n,k)$ is the WZ-function corresponding to the identity, and $G(n,k)$ is its mate. Each subsection begins with a function $f(k,a,b,\dots)$ obtained as described above from an integral form of one of the standard hypergeometric summation formulas. The names assigned to these summation formulas are (for simplicity but not necessarily complete historical accuracy) those used by Bailey [1], with the following clarifications: "Bailey's theorem" and "Gauss's second theorem" have already been mentioned (our equations (4.1) and (4.12)) and "Dougall's ${}_5F_4$ theorem" is the very-well poised ${}_5F_4(1)$ evaluation [1, p. 25, equation (3), and p. 27, equation (1)]. Next comes a list of identities obtained from this function by the procedure described above. These identities were selected from a much larger collection of identities (not all inequivalent) produced by the program. A few classical and trivial identities have been included to give an idea of the range of identities that appear. The TEX output produced by the program has been edited only minimally, so the order in which the parameters appear is random.

No attempt has been made here to identify the output of the program with identities in the literature. However, it should be noted that some of them are cases of unpublished identities of Gosper, and some can be found in [2, 3, and 4].

11. From Bailey's theorem:

$$\frac{\Gamma(2\,a+k)\Gamma(1-2\,a+k)\Gamma(2\,a+2\,b)\Gamma(2\,a+b)\Gamma(b+1/2)\,(1/2)^k}{\Gamma(k+1)\Gamma(2\,a+2\,b+k)\Gamma(2\,a)\Gamma(1-2\,a)\Gamma(a+b)\Gamma(a+b+1/2)}$$

$${}_4F_3\left(\begin{matrix} n+1,-n,\frac{4\,a}{3}+1+\frac{n}{3},4\,a+n \\ 2\,a+n+1,\frac{4\,a}{3}+\frac{n}{3},1/2+2\,a\end{matrix}\;\Big|\;-1/8\right) = \frac{(1+2\,a)_n}{(4\,a+1)_n\,(1/2)^n} \qquad (11.1a)$$

$$Q = 2\frac{(4\,a-1+2\,k)\,k}{(4\,a+3\,k+n)\,(-n-1+k)} \tag{11.1b}$$

$$ {}_4F_3\left(\begin{matrix} 1-\frac{n}{3}, -n, 4\,a-1-n, 2-4\,a+n \\ -\frac{n}{3}, 2\,a-n, 3/2-2\,a \end{matrix} \,\Big|\, -1/8\right) = \frac{(0)_n}{(1-2\,a)_n 2^n} \tag{11.2a}$$

$$Q = 4\frac{(2\,a-n-1+k)\,k\,(1-4\,a+2\,k)}{(3\,k-n)\,(-n-1+k)\,(4\,a-2-n+k)} \tag{11.2b}$$

$$ {}_4F_3\left(\begin{matrix} n+1, -n, \frac{4\,a}{3}+2/3-\frac{n}{3}, 4\,a-1-n \\ \frac{4\,a}{3}-1/3-\frac{n}{3}, 1/2+2\,a, 2\,a-n \end{matrix} \,\Big|\, -1/8\right) = \frac{(1-4\,a)_n}{(1-2\,a)_n 2^n} \tag{11.3a}$$

$$Q = 4\frac{(2\,a-n-1+k)\,(4\,a-1+2\,k)\,k}{(4\,a-1-n+3\,k)\,(-n-1+k)\,(4\,a-2-n+k)} \tag{11.3b}$$

12. From Dixon's theorem:

$$\frac{\Gamma(2\,a+k)\sqrt{\pi}\Gamma(b+2\,k)\Gamma(1+2\,a)2^{2\,b-2\,a}\Gamma(1+4\,a-2\,b)}{\Gamma(1/2+a-b)\Gamma(2\,a)\Gamma(b)\Gamma(1+4\,a-b+2\,k)\Gamma(1+a)\Gamma(k+1)}$$

$$ {}_5F_4\left(\begin{matrix} -n, 1-2\,b, 1-2\,b+n, 4/3-\frac{4\,b}{3}, 1/2-2\,b \\ -2\,n, 1-b, 2-4\,b+2\,n, 1/3-\frac{4\,b}{3} \end{matrix} \,\Big|\, 4\right) = \frac{(3/2-2\,b)_n}{(1/2)_n} \tag{12.1a}$$

$$Q = -2\frac{(k-2\,n-1)\,k\,(k-b)}{(k-4\,b+2+2\,n)\,(3\,k-4\,b+1)\,(-n-1+k)} \tag{12.1b}$$

$$ {}_5F_4\left(\begin{matrix} -3/2, 1-\frac{6\,n}{11}, 1/2-\frac{n}{2}, -\frac{n}{2}, 4-2\,n \\ -\frac{6\,n}{11}, 2/3-\frac{2\,n}{3}, 1-\frac{2\,n}{3}, 4/3-\frac{2\,n}{3} \end{matrix} \,\Big|\, \frac{16}{27}\right) = 1/6\frac{(-3/2)_n}{(-1/2)_n}, \quad n \geq 3 \tag{12.2a}$$

$$Q = 2\frac{(3\,k+1-2\,n)\,(3\,k-2\,n)\,(-1-2\,n+3\,k)\,k}{(11\,k-6\,n)\,(k+3-2\,n)\,(-n-1+2\,k)\,(2-2\,n+k)} \tag{12.2b}$$

$$ {}_5F_4\left(\begin{matrix} 3/4, 1/4-2\,n, \frac{47}{44}-\frac{6\,n}{11}, 1/2-\frac{n}{2}, -\frac{n}{2} \\ \frac{7}{12}-\frac{2\,n}{3}, \frac{11}{12}-\frac{2\,n}{3}, 5/4-\frac{2\,n}{3}, \frac{3}{44}-\frac{6\,n}{11} \end{matrix} \,\Big|\, \frac{16}{27}\right) = \frac{(3/8)_n(-1/8)_n}{(1/8)_n(-3/8)_n} \tag{12.3a}$$

$$Q = 2\frac{(12\,k-5-8\,n)\,(12\,k-1-8\,n)\,(12\,k+3-8\,n)\,k}{(44\,k+3-24\,n)\,(4\,k-3-8\,n)\,(-n-1+2\,k)\,(4\,k-7-8\,n)} \tag{12.3b}$$

13. From Dougall's ${}_5F_4$ theorem:

$$\frac{(a+2\,k)\,\Gamma(a+k)\Gamma(c+k)\Gamma(d+k)\Gamma(e+k)\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)}{a\Gamma(k+1)\Gamma(1+a-c+k)\Gamma(1+a-d+k)\Gamma(1+a-e+k)\Gamma(a)\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(1+a-c-d-e)}$$

$$ {}_5F_4\left(\begin{matrix} -3\,b, -n, -2\,b-n, -b-n, 1-\frac{6\,b}{5}-\frac{3\,n}{5} \\ 1/2-\frac{3\,b}{2}-\frac{n}{2}, 1-\frac{3\,b}{2}-\frac{n}{2}, -\frac{6\,b}{5}-\frac{3\,n}{5}, 1-2\,b+n \end{matrix} \,\Big|\, -1/4\right) = \frac{(1-2\,b)_n(2\,b)_n}{(1-b)_n(3\,b)_n} \tag{13.1a}$$

$$Q = \frac{k\,(k-b-2\,n-2)\,(2\,k-3\,b-n-1)\,(-3\,b-n+2\,k)}{(-6\,b-3\,n+5\,k)\,(-n-1+k)\,(-b-n-1+k)\,(-2\,b-n-1+k)} \tag{13.1b}$$

$$ {}_5F_4\left(\begin{matrix} -n, 1-b-n, 2-2\,b-n, 3\,b-3, 1-\frac{3\,n}{5} \\ 1/2+\frac{b}{2}-\frac{n}{2}, 4\,b-3+n, \frac{b}{2}-\frac{n}{2}, -\frac{3\,n}{5} \end{matrix}\ \Big|\ -1/4\right) = \frac{(0)_n(4\,b-3)_n}{(3\,b-2)_n(1-b)_n} \tag{13.2a} $$

$$ Q = \frac{(1-3\,b+k-2\,n)\,k\,(b-2+2\,k-n)\,(b-1+2\,k-n)}{(5\,k-3\,n)\,(-n-1+k)\,(-b-n+k)\,(1-2\,b-n+k)} \tag{13.2b} $$

$$ {}_5F_4\left(\begin{matrix} 3-3\,e, 1/2-\frac{n}{2}, n-4\,e+4, 1-\frac{3\,n}{5}, -\frac{n}{2} \\ 1-n, e-n, 2\,e-1-n, -\frac{3\,n}{5} \end{matrix}\ \Big|\ -4\right) = 1/3\frac{(3-3\,e)_n}{(1-e)_n}, \quad n \geq 1 \tag{13.3a} $$

$$ Q = \frac{k\,(e-n-1+k)\,(4\,e-5+k-2\,n)\,(2\,e-2+k-n)\,(k-n)}{(5\,k-3\,n)\,(3\,e-3-n)\,(2\,e-n-2)\,(4\,e-4-n)\,(-n-1+2\,k)} \tag{13.3b} $$

14. From Dougall's ${}_5F_4$ theorem:

$$ \frac{\Gamma(c+2\,k)\Gamma(1/2+a-c)\,(a+2\,k)\,4^b\Gamma(b+k)\Gamma(a+k)\Gamma(1+2\,a-c-2\,b)}{\Gamma(b)\Gamma(k+1)\Gamma(1+2\,a-c+2\,k)\Gamma(1/2+a-c-b)\Gamma(c)\Gamma(1+a-b+k)} $$

$$ {}_5F_4\left(\begin{matrix} -n, d+n, 4/3-\frac{c}{3}, 1-c, 3/2-d-c \\ 2-c-2\,d-2\,n, 1/3-\frac{c}{3}, 2\,d-1+c, 2-c+2\,n \end{matrix}\ \Big|\ 4\right) = \frac{(3/2-\frac{c}{2})_n(1-\frac{c}{2})_n}{(\frac{c}{2}-1/2+d)_n(\frac{c}{2}+d)_n} \tag{14.1a} $$

$$ Q = -\frac{(1+d+2\,n)\,(2\,d-2+c+k)\,(1-c+k-2\,d-2\,n)\,k}{(2-c+2\,n+k)\,(d+n)\,(1-c+3\,k)\,(-n-1+k)} \tag{14.1b} $$

$$ {}_5F_4\left(\begin{matrix} b+1/2, b-n, -n, 2\,d-2\,n, 1+\frac{2\,b}{3}-\frac{2\,n}{3} \\ -2\,n, 1-d+b, 1+2\,b, \frac{2\,b}{3}-\frac{2\,n}{3} \end{matrix}\ \Big|\ 4\right) = \frac{(1/2-d)_n(-b)_n}{(1/2)_n(1-d+b)_n} \tag{14.2a} $$

$$ Q = -\frac{(d+k-2\,n-2)\,(k-d+b)\,k\,(2\,b+k)\,(k-1-2\,n)}{(3\,k+2\,b-2\,n)\,(-n-1+k)\,(b-n-1+k)\,(2\,d-2\,n-2+k)\,(2\,d-2\,n+k-1)} \tag{14.2b} $$

$$ {}_5F_4\left(\begin{matrix} -2\,n, b-2\,d+1/2, 1-\frac{2\,n}{3}, d-n, d-b-n \\ 1-2\,d, 4\,d-2\,b-2\,n, 2\,b-2\,d+1, -\frac{2\,n}{3} \end{matrix}\ \Big|\ 4\right) = \frac{(1/2)_n(0)_n}{(b-2\,d+1/2)_n(1+b-2\,d)_n} \tag{14.3a} $$

$$ Q = -\frac{(4\,d-2\,b+k-2\,n-1)\,(2\,b-2\,d+k)\,(k-2\,d)\,k\,(k-b+2\,d-2\,n-2)}{(3\,k-2\,n)\,(k-b+d-n-1)\,(d-n-1+k)\,(k-2-2\,n)\,(k-1-2\,n)} \tag{14.3b} $$

$$ {}_5F_4\left(\begin{matrix} -2\,n, 1+2\,a+2\,b, b+1, 2\,n-2\,a+2, \frac{2\,b}{3}+5/3 \\ 1+a+b-n, \frac{2\,b}{3}+2/3, 2+b+n, 3/2-a \end{matrix}\ \Big|\ 1/4\right) = \frac{(1/2)_n(2+b)_n}{(-a-b)_n(3/2-a)_n} \tag{14.4a} $$

$$ Q = 2\frac{k\,(1-2\,a+2\,k)\,(a-2\,n-2)\,(a+k+b-n)}{(2\,b+2+3\,k)\,(a-n-1)\,(k-2-2\,n)\,(k-1-2\,n)} \tag{14.4b} $$

15. From Dougall's $_5F_4$ theorem:

$$\frac{(a+2\,k)\,3^{b+1}\Gamma(a+k)\Gamma(3\,k+b)\Gamma(3\,a-2\,b)}{\Gamma(k+1)\Gamma(1+3\,a-b+3\,k)\Gamma(b)\Gamma(a-b)}$$

$$ {}_3F_2\left(\begin{matrix}-3\,n, 2/3-c, 1+3\,n\\ 1/2, 2-3\,c\end{matrix}\,\Big|\,3/4\right)=\frac{(c)_n}{(4/3-c)_n} \tag{15.1a}$$

$$Q=4/3\frac{(3\,n+2)\,(1-3\,c+k)\,(2\,k-1)\,k}{(k-3\,n-1)\,(c+n)\,(-3\,n-3+k)\,(k-3\,n-2)} \tag{15.1b}$$

$$ {}_3F_2\left(\begin{matrix}-3\,n, 2/3-c, 3\,n+2\\ 3/2, 1-3\,c\end{matrix}\,\Big|\,3/4\right)=\frac{(c+2/3)_n(1/3)_n}{(1-c)_n(4/3)_n} \tag{15.2a}$$

$$Q=2\frac{(5+6\,n)\,(k-3\,c)\,k\,(1+2\,k)}{(3\,c+2+3\,n)\,(-3\,n-3+k)\,(k-3\,n-2)\,(k-3\,n-1)} \tag{15.2b}$$

$$ {}_3F_2\left(\begin{matrix}-3\,b, -\frac{3\,n}{2}, 1/2-\frac{3\,n}{2}\\ -3\,n, 2/3-b-n\end{matrix}\,\Big|\,4/3\right)=\frac{(1/3-b)_n}{(1/3+b)_n} \tag{15.3a}$$

$$Q=\frac{k\,(k-3\,n-1)\,(-1-3\,b-3\,n+3\,k)\,(3\,k-5-6\,n)}{(-3\,n-1+2\,k)\,(-2-3\,n+2\,k)\,(-3\,n-3+2\,k)\,(3\,b-3\,n-1)} \tag{15.3b}$$

$$ {}_5F_4\left(\begin{matrix}-3\,n, 5/4-\frac{3\,b}{4}, 1/2-\frac{3\,b}{2}, 3\,n+2, -\frac{3\,b}{2}\\ 3/2, 4/3-b+n, 1/4-\frac{3\,b}{4}, 2/3-b-n\end{matrix}\,\Big|\,1/9\right)=\frac{(1/3)_n(4/3-b)_n}{(1/3+b)_n(4/3)_n} \tag{15.4a}$$

$$Q=-2\frac{(-1-3\,b-3\,n+3\,k)\,k\,(1+2\,k)\,(5+6\,n)}{(1-3\,b+4\,k)\,(-3\,n-3+k)\,(k-3\,n-2)\,(k-3\,n-1)} \tag{15.4b}$$

$$ {}_5F_4\left(\begin{matrix}-3\,n, 1+3\,n, 3/4-\frac{3\,b}{4}, -\frac{3\,b}{2}, -1/2-\frac{3\,b}{2}\\ 1/2, 1/3-b-n, 2/3-b+n, -\frac{3\,b}{4}-1/4\end{matrix}\,\Big|\,1/9\right)=\frac{(2/3-b)_n}{(2/3+b)_n} \tag{15.5a}$$

$$Q=-4\frac{k\,(2\,k-1)\,(-3\,b-2-3\,n+3\,k)\,(3\,n+2)}{(-3\,b-1+4\,k)\,(-3\,n-3+k)\,(k-3\,n-2)\,(k-3\,n-1)} \tag{15.5b}$$

$$ {}_5F_4\left(\begin{matrix}-b, -b-1/3-n, 3/4-\frac{3\,b}{4}-\frac{3\,n}{4}, -\frac{3\,n}{2}, 1/2-\frac{3\,n}{2}\\ -3\,n, -\frac{3\,b}{4}-1/4-\frac{3\,n}{4}, 1/2-\frac{3\,b}{2}, -\frac{3\,b}{2}\end{matrix}\,\Big|\,9\right)=\frac{(1/3+b)_n}{(1/3-b)_n} \tag{15.6a}$$

$$Q=-\frac{(k-3\,n-1)\,(-1-3\,b+2\,k)\,(-3\,b-2+2\,k)\,k\,(3\,k-5-6\,n)}{(-3\,b-1-3\,n+4\,k)\,(-2-3\,n+2\,k)\,(-3\,n-1+2\,k)\,(-3\,n-3+2\,k)\,(-3\,b-4-3\,n+3\,k)} \tag{15.6b}$$

$$ {}_5F_4\left(\begin{matrix}-b, -n, 1-\frac{3\,n}{4}, 1/2+\frac{3\,b}{2}-\frac{3\,n}{2}, 1+\frac{3\,b}{2}-\frac{3\,n}{2}\\ 1/2-\frac{3\,b}{2}, -\frac{3\,b}{2}, -\frac{3\,n}{4}, 1+3\,b-3\,n\end{matrix}\,\Big|\,9\right)=\frac{(0)_n}{(-2\,b)_n} \tag{15.7a}$$

$$Q=-\frac{(3\,b+k-3\,n)\,(2\,b-1+k-2\,n)\,k\,(-3\,b-2+2\,k)\,(-1-3\,b+2\,k)}{(3\,b-3\,n+2\,k)\,(4\,k-3\,n)\,(3\,b-1-3\,n+2\,k)\,(3\,b-2-3\,n+2\,k)\,(-n-1+k)} \tag{15.7b}$$

$$ {}_5F_4\left(\begin{matrix}1+a, -n, 1+\frac{3\,a}{2}-\frac{3\,n}{2}, 3/2+\frac{3\,a}{2}-\frac{3\,n}{2}, 3/2+\frac{3\,a}{4}-\frac{3\,n}{4}\\ 1/2, 3\,a+3, 1/2+\frac{3\,a}{4}-\frac{3\,n}{4}, -3\,n\end{matrix}\,\Big|\,9\right)=\frac{(-1/3-a)_n(-2/3-a)_n}{(1/3)_n(2/3)_n} \tag{15.8a}$$

$$Q=-2\frac{(a-1+k-2\,n)\,(3\,a+2+k)\,k\,(2\,k-1)\,(k-3\,n-1)}{(1+3\,a-3\,n+2\,k)\,(3\,a-3\,n+2\,k)\,(2+3\,a-3\,n+4\,k)\,(3\,a-1-3\,n+2\,k)\,(-n-1+k)} \tag{15.8b}$$

16. From Dougall's theorem:

$$\frac{\Gamma(2a+1/2-b-d-c+k)\Gamma(b+2k)\Gamma(c+k)\Gamma(a+k)\Gamma(d+k)\Gamma(\frac{a}{2}+1+k)}{\Gamma(k+1)\Gamma(b-a+1/2+d+c+k)\Gamma(1+a-c+k)\Gamma(\frac{a}{2}+k)\Gamma(1+2a-b+2k)\Gamma(1+a-d+k)}$$
$$\times\frac{\Gamma(b-a+1/2+d+c)\Gamma(\frac{a}{2})\Gamma(1+2a-b-2c)\Gamma(1+a)}{\Gamma(c)\Gamma(2a+1/2-b-d-c)\Gamma(1+2a-b-2d-2c)\Gamma(a)}$$
$$\times\frac{\Gamma(1+2a-b-2d)\Gamma(1+a-c-d)\Gamma(1/2+a-b-c-d)\Gamma(1/2+a-b)}{\Gamma(\frac{a}{2}+1)\Gamma(d)\Gamma(1/2+a-b-d)\Gamma(b)\Gamma(1/2+a-c-b)}$$

$$ {}_7F_6\left(\begin{matrix} b+1,-n,3/2-e+b,3-2e+b-2d,4/3+\frac{b}{3},2e-2-b+2d,e+n \\ 2+b-2e-2n,-1-b+2e,2+b+2n,5/2-e+b-d,1/3+\frac{b}{3},d+e \end{matrix}\;\middle|\;1\right)$$
$$=\frac{(3/2+\frac{b}{2})_n(2e-3/2-b+d)_n(1-d)_n(1+\frac{b}{2})_n}{(e-\frac{b}{2})_n(5/2-e+b-d)_n(d+e)_n(-\frac{b}{2}-1/2+e)_n} \qquad (16.1a)$$

$$Q=-\frac{(3-2e+2k+2b-2d)(k-b-2+2e)k(e-1+d+k)(k+b+1-2e-2n)(1+e+2n)}{(1+b+3k)(e+n)(3-4e+2b-2d-2n)(d-n-1)(2+k+b+2n)(-n-1+k)} \qquad (16.1b)$$

17. From Dougall's theorem:

$$\frac{\Gamma(3k+b)\Gamma(2a-b-c+k)\Gamma(c+k)\Gamma(a+k)\Gamma(\frac{a}{2}+1+k)}{\Gamma(k+1)\Gamma(b-a+1+c+k)\Gamma(1+3a-b+3k)\Gamma(1+a-c+k)\Gamma(\frac{a}{2}+k)}$$
$$\times\frac{\Gamma(3a-2b)\Gamma(a-b-c)\Gamma(\frac{a}{2})\Gamma(1+a)\Gamma(1+3a-b-3c)\Gamma(1-a+c+b)}{\Gamma(c)\Gamma(a)\Gamma(\frac{a}{2}+1)\Gamma(b)\Gamma(2a-b-c)\Gamma(3a-2b-3c)\Gamma(a-b)}$$

$$ {}_7F_6\left(\begin{matrix} 4/3-d,-3n,\frac{3c}{4}+5/4,\frac{3c}{2},1/2+\frac{3c}{2},c-1/3+d,3n+2 \\ 3/2,3-3d,2/3+c-n,4/3+n+c,1/4+\frac{3c}{4},3d-2+3c \end{matrix}\;\middle|\;1\right)$$
$$=\frac{(4/3+c)_n(d)_n(5/3-d-c)_n(1/3)_n}{(4/3)_n(c+d)_n(5/3-d)_n(1/3-c)_n} \qquad (17.1a)$$

$$Q=2/3\frac{(2-3d+k)(1+2k)k(3d-3+k+3c)(5+6n)(3k-1+3c-3n)}{(3c+1+4k)(d+n)(3d-5+3c-3n)(-3n-3+k)(k-3n-1)(k-3n-2)} \qquad (17.1b)$$

$$ {}_7F_6\left(\begin{matrix} 1/3-d,-3n,\frac{3c}{2},c-1/3+d,\frac{3c}{4}+3/4,\frac{3c}{2}-1/2,1+3n \\ 1/2,2/3+n+c,1-3d,3d-1+3c,1/3+c-n,\frac{3c}{4}-1/4 \end{matrix}\;\middle|\;1\right)=\frac{(1-d-c)_n(d+1/3)_n(c+2/3)_n}{(2/3-c)_n(1-d)_n(1/3+d+c)_n} \qquad (17.2a)$$

$$Q=4/3\frac{(3d-2+k+3c)(2k-1)k(k-3d)(3k-2+3c-3n)(3n+2)}{(3c-1+4k)(1+3d+3n)(-n-1+c+d)(-3n-3+k)(k-3n-2)(k-3n-1)} \qquad (17.2b)$$

$$ {}_7F_6\left(\begin{matrix} 1-\frac{b}{4},-n,2b-3a+3,\frac{3a}{2}-1-b,\frac{3a}{2}-1/2-b,-b,1-a+n \\ a-b,1-b+3n,\frac{b}{2}-\frac{3a}{2}+2,-\frac{b}{4},\frac{b}{2}-\frac{3a}{2}+3/2,3a-2-b-3n \end{matrix}\;\middle|\;1\right)$$
$$=\frac{(1-\frac{b}{3})_n(2/3-\frac{b}{3})_n(1/3-\frac{b}{3})_n(2-2a+b)_n}{(1-a+\frac{b}{3})_n(5/3-a+\frac{b}{3})_n(4/3-a+\frac{b}{3})_n(a-b)_n} \qquad (17.3a)$$

$$Q=\frac{(3a-b+k-3n-3)(a-2n-2)(a-b-1+k)(b-3a+2k+1)(2-3a+b+2k)k}{(4k-b)(2a-b-2-n)(2-b+k+3n)(1-b+k+3n)(-n-1+k)(a-n-1)} \qquad (17.3b)$$

18. From Dougall's theorem:

$$\frac{\Gamma(a+k)\Gamma(2\,a-1/2-b+k)\Gamma(\frac{a}{2}+1+k)\Gamma(b+4\,k)}{\Gamma(b-a+3/2+k)\Gamma(k+1)\Gamma(\frac{a}{2}+k)\Gamma(1+4\,a-b+4\,k)} \times \frac{2^{6\,a-3\,b-2}\Gamma(b-a+3/2)\Gamma(1/2+2\,a-b)\Gamma(1+a)\Gamma(4\,a-1-2\,b)\Gamma(a-1/2-b)\Gamma(\frac{a}{2})}{\Gamma(a)\Gamma(\frac{a}{2}+1)\Gamma(4\,a-2-3\,b)\Gamma(b)\pi\,\Gamma(2\,a-1/2-b)}$$

$$ {}_6F_5\left(\begin{matrix}2+4\,n,2/3-\frac{4\,c}{3},7/5-\frac{4\,c}{5},1/3-\frac{4\,c}{3},1-\frac{4\,c}{3},-4\,n\\ 3/2,1-c-n,3/2-c+n,2/5-\frac{4\,c}{5},1-4\,c\end{matrix}\,\middle|\,\frac{27}{32}\right)=\frac{(1/4)_n(c+1/2)_n(3/2-c)_n}{(c)_n(1-c)_n(5/4)_n} \tag{18.1a}$$

$$Q=8\frac{(3+4\,n)\,(1+2\,k)\,(k-4\,c)\,k\,(2\,k-1)\,(-n-c+k)}{(2-4\,c+5\,k)\,(1+2\,c+2\,n)\,(k-4-4\,n)\,(k-3-4\,n)\,(k-2-4\,n)\,(k-1-4\,n)} \tag{18.1b}$$

$$ {}_6F_5\left(\begin{matrix}6/5-\frac{4\,c}{5},1+4\,n,2/3-\frac{4\,c}{3},1/3-\frac{4\,c}{3},1-\frac{4\,c}{3},-4\,n\\ 1/2,1-c-n,5/4-c+n,1/5-\frac{4\,c}{5},2-4\,c\end{matrix}\,\middle|\,\frac{27}{32}\right)=1 \tag{18.2a}$$

$$Q=4\frac{k\,(1-4\,c+k)\,(2\,k-1)\,(k-1)\,(-n-c+k)\,(5+8\,n)}{(1-4\,c+5\,k)\,(c+n)\,(k-1-4\,n)\,(k-2-4\,n)\,(k-3-4\,n)\,(k-4-4\,n)} \tag{18.2b}$$

19. From Dougall's theorem:

$$\frac{\Gamma(2\,a-b-c+k)\Gamma(a+k)\Gamma(c+2\,k)\Gamma(b+2\,k)\Gamma(\frac{a}{2}+1+k)}{\Gamma(\frac{a}{2}+k)\Gamma(1+2\,a-c+2\,k)\Gamma(b-a+1+c+k)\Gamma(k+1)\Gamma(1+2\,a-b+2\,k)} \times \frac{2^{4\,a-2\,b-2\,c-1}\Gamma(1/2+a-c)\Gamma(a-b-c)\Gamma(1/2+a-b)\Gamma(\frac{a}{2})\Gamma(1+2\,a-b-c)\Gamma(1+a)\Gamma(1-a+c+b)}{\Gamma(a)\Gamma(2\,a-2\,b-c)\Gamma(2\,a-b-2\,c)\Gamma(b)\Gamma(c)\pi\,\Gamma(\frac{a}{2}+1)}$$

$$ {}_6F_5\left(\begin{matrix}-n,d+1,4/3-\frac{d}{3},a+d+1/2,-a-2\,d+n+1,-d\\ 3/2,-2\,a-3\,d+1,1/3-\frac{d}{3},2\,n-d+2,2\,a+3\,d-2\,n\end{matrix}\,\middle|\,1\right) =\frac{(1-a-d)_n(3/2-\frac{d}{2})_n(1-\frac{d}{2})_n(1/2-a-2\,d)_n}{(1-d)_n(1/2-a-\frac{3\,d}{2})_n(-a-\frac{3\,d}{2}+1)_n(3/2)_n} \tag{19.1a}$$

$$Q=-\frac{(2\,a+k+3\,d-1-2\,n)\,(k-d)\,k\,(1+2\,k)\,(k-2\,a-3\,d)\,(a+2\,d-2\,n-2)}{(1-d+3\,k)\,(2\,a+4\,d-1-2\,n)\,(a+d-n-1)\,(k-d+2\,n+2)\,(a+2\,d-n-1)\,(-n-1+k)} \tag{19.1b}$$

$$ {}_6F_5\left(\begin{matrix}-n,d-1+a,2/3+\frac{d}{3}+\frac{a}{3},2\,d-1+a+n,1/2-d,1-a-d\\ 1/2,d+a+2\,n,a-1+3\,d,\frac{d}{3}-1/3+\frac{a}{3},2-3\,d-a-2\,n\end{matrix}\,\middle|\,1\right) =\frac{(a-1/2+2\,d)_n(d)_n(\frac{d}{2}+\frac{a}{2}+1/2)_n(\frac{a}{2}+\frac{d}{2})_n}{(a+d)_n(\frac{a}{2}-1/2+\frac{3\,d}{2})_n(\frac{a}{2}+\frac{3\,d}{2})_n(1/2)_n} \tag{19.2a}$$

$$Q=-\frac{(a-2+3\,d+k)\,(2\,k-1)\,k\,(1-3\,d-a+k-2\,n)\,(d-1+a+k)\,(a+2\,d+2\,n)}{(2\,d-1+a+n)\,(d+a+2\,n+k)\,(d+n)\,(2\,a+4\,d-1+2\,n)\,(d-1+a+3\,k)\,(-n-1+k)} \tag{19.2b}$$

$$ {}_6F_5\left(\begin{matrix} b, -n, b-1/2+d, 2\,d-1, 2/3+\frac{2\,d}{3}+\frac{2\,b}{3}, b+d+n \\ 2\,b+2\,d+2\,n, -2\,n, \frac{2\,d}{3}-1/3+\frac{2\,b}{3}, \frac{b}{2}+d, \frac{b}{2}+d+1/2 \end{matrix} \;\Big|\; 1\right) = \frac{(1+\frac{b}{2})_n(d)_n(1/2+b+d)_n(\frac{b}{2}+1/2)_n}{(b+1)_n(1/2)_n(\frac{b}{2}+d)_n(\frac{b}{2}+d+1/2)_n} \quad (19.3a) $$

$$ Q = -\frac{k\,(2\,d+b-2+2\,k)\,(2\,d+b-1+2\,k)\,(k-1-2\,n)\,(b+k)\,(d+b+2\,n+1)}{(b+1+2\,n)\,(k+2\,b+2\,d+2\,n)\,(d+n)\,(2+b+2\,n)\,(2\,d-1+2\,b+3\,k)\,(-n-1+k)} \quad (19.3b) $$

$$ {}_6F_5\left(\begin{matrix} 1-2\,d, -2\,n, a+d-1/2, d-1+a, 1/3+\frac{2\,a}{3}+\frac{2\,d}{3}, 4\,d-2+2\,a+2\,n \\ a-1/2+2\,d, 1-d-n, \frac{2\,a}{3}-2/3+\frac{2\,d}{3}, 2\,a+2\,d-1, a+d+n \end{matrix} \;\Big|\; 1\right) = 1 \quad (19.4a) $$

$$ Q = \frac{(a+2\,d+2\,n)\,k\,(-d-n+k)\,(2\,a+2\,d-2+k)\,(k-1)\,(2\,a+4\,d-3+2\,k)}{(2\,d-1+a+n)\,(2\,a+4\,d-1+2\,n)\,(d+n)\,(2\,a-2+2\,d+3\,k)\,(k-2-2\,n)\,(k-1-2\,n)} \quad (19.4b) $$

$$ {}_6F_5\left(\begin{matrix} -2\,n, 1-c, c-1+2\,d, 3/2-d-c, 3/2-\frac{c}{2}, 2\,d+2\,n \\ 1/2-\frac{c}{2}, 2-c+2\,n, 3-2\,d-2\,c, 1/2+d, 2-c-2\,d-2\,n \end{matrix} \;\Big|\; -1\right) = \frac{(3/2-\frac{c}{2})_n(1-\frac{c}{2})_n(c-1+2\,d)_n(1/2)_n}{(\frac{c}{2}-1/2+d)_n(\frac{c}{2}+d)_n(2-d-c)_n(1/2+d)_n} \quad (19.5a) $$

$$ Q = 1/2\frac{(d+1+2\,n)\,(c-2+2\,d)\,(k-c+1-2\,d-2\,n)\,(k+2-2\,d-2\,c)\,k\,(2\,d-1+2\,k)}{(2\,d-1+c+n)\,(2-c+k+2\,n)\,(d+n)\,(1-c+2\,k)\,(k-2-2\,n)\,(k-1-2\,n)} \quad (19.5b) $$

$$ {}_6F_5\left(\begin{matrix} b+1, -2\,n, -b, b-2\,d+2, 2\,n+1, \frac{b}{2}-d+2 \\ 3-2\,d+2\,b, 2-2\,d, 2+b-2\,d-2\,n, \frac{b}{2}-d+1, b-2\,d+2\,n+3 \end{matrix} \;\Big|\; -1\right) = \frac{(\frac{b}{2}-d+2)_n(d)_n(-1/2-b+d)_n(3/2+\frac{b}{2}-d)_n}{(3/2-d)_n(2-d+b)_n(d-\frac{b}{2})_n(-1/2-\frac{b}{2}+d)_n} \quad (19.6a) $$

$$ Q = \frac{(b-2\,d+2)\,k\,(1-2\,d+k)\,(2-2\,d+k+2\,b)\,(1+k+b-2\,d-2\,n)\,(3+4\,n)}{(2-2\,d+b+2\,k)\,(d+n)\,(k-2-2\,n)\,(k-1-2\,n)\,(1+2\,b-2\,d-2\,n)\,(b-2\,d+2\,n+k+3)} \quad (19.6b) $$

20. From Dougall's theorem:

$$ \frac{5^{1-a}\Gamma(5/2-a)\,(1+2\,a)\,\Gamma(5\,k-2\,a)\Gamma(1/2-a+k)\,(1-2\,a+4\,k)}{4\,\Gamma(7/2-3\,a+5\,k)\Gamma(k+1)\Gamma(-2\,a)\Gamma(a+3/2)} $$

$$ {}_5F_4\left(\begin{matrix} 1/2, 1-\frac{5\,n}{4}, 2/3-\frac{5\,n}{3}, 1/3-\frac{5\,n}{3}, -\frac{5\,n}{3} \\ 1-\frac{5\,n}{2}, -\frac{5\,n}{4}, 4/5-n, 1/2-\frac{5\,n}{2} \end{matrix} \;\Big|\; -\frac{27}{5}\right) = -1, \quad n \geq 1 \quad (20.1a) $$

$$Q = \frac{(-1-5\,n+5\,k)\,(2\,k-1-5\,n)\,(k-1)\,(5\,k-7-10\,n)\,k\,(k-2)\,(2\,k-5\,n)}{(4\,k-5\,n)\,(1+5\,n)\,(-1-5\,n+3\,k)\,(3\,k-4-5\,n)\,(3\,k-3-5\,n)\,(3\,k-2-5\,n)\,(-5\,n-5+3\,k)} \tag{20.1b}$$

$${}_5F_4\left(\begin{matrix}1/2,-n,5/8-\frac{5\,n}{4},-3/4-\frac{5\,n}{2},-5/4-\frac{5\,n}{2}\\ 1/3-\frac{5\,n}{3},-1/3-\frac{5\,n}{3},-\frac{5\,n}{3},-3/8-\frac{5\,n}{4}\end{matrix}\;\middle|\;-\frac{5}{27}\right) = \frac{(11/10)_n(9/10)_n(7/10)_n(3/10)_n}{(4/5)_n(3/5)_n(2/5)_n(6/5)_n} \tag{20.2a}$$

$$Q = 32\frac{(2\,k+3)\,(1+2\,k)\,k}{(8\,k-3-10\,n)\,(-n-1+k)} \times\frac{(3\,k-2-5\,n)\,(3\,k-4-5\,n)\,(3\,k-3-5\,n)\,(2\,k-5-4\,n)}{(4\,k-7-10\,n)\,(4\,k-9-10\,n)\,(4\,k-11-10\,n)\,(4\,k-13-10\,n)\,(4\,k-15-10\,n)} \tag{20.2b}$$

$${}_5F_4\left(\begin{matrix}1/2,5/4-\frac{5\,n}{4},1/2-\frac{5\,n}{2},1/2-n,-\frac{5\,n}{2}\\ 1/4-\frac{5\,n}{4},7/6-\frac{5\,n}{3},1/2-\frac{5\,n}{3},5/6-\frac{5\,n}{3}\end{matrix}\;\middle|\;-\frac{5}{27}\right) = \frac{(-1/5)_n(1/5)_n(2/5)_n(3/5)_n}{(7/10)_n(-1/10)_n(3/10)_n(1/10)_n} \tag{20.3a}$$

$$Q = 1/8\frac{(2\,k+3)\,(1+2\,k)\,k\,(2\,k-3-4\,n)}{(4\,k+1-5\,n)\,(2\,k-1-2\,n)} \times\frac{(6\,k+1-10\,n)\,(6\,k-3-10\,n)\,(6\,k-1-10\,n)}{(2\,k-1-5\,n)\,(2\,k-2-5\,n)\,(2\,k-3-5\,n)\,(2\,k-4-5\,n)\,(2\,k-5-5\,n)} \tag{20.3b}$$

$${}_5F_4\left(\begin{matrix}1/4,2/3,-1/4,-5\,n,1+5\,n\\ 1/2,-1/3,3/5+n,2/5-n\end{matrix}\;\middle|\;\frac{16}{25}\right) = 1 \tag{20.4a}$$

$$Q = -4\frac{(k-1)\,(2\,k-3)\,(2\,k-1)\,(-3-5\,n+5\,k)\,k\,(3+5\,n)}{(-3-5\,n+k)\,(k-4-5\,n)\,(k-5\,n-1)\,(k-5\,n-2)\,(3\,k-1)\,(k-5\,n-5)} \tag{20.4b}$$

$${}_5F_4\left(\begin{matrix}1/4,3/4,7/6,-5\,n,5\,n+2\\ 3/2,1/6,6/5+n,4/5-n\end{matrix}\;\middle|\;\frac{16}{25}\right) = 1 \tag{20.5a}$$

$$Q = -4\frac{(2\,k-1)\,(k-1)\,k\,(-1-5\,n+5\,k)\,(1+2\,k)\,(7+10\,n)}{(-3-5\,n+k)\,(k-4-5\,n)\,(k-5\,n-1)\,(k-5\,n-2)\,(1+6\,k)\,(k-5\,n-5)} \tag{20.5b}$$

$${}_5F_4\left(\begin{matrix}1/4,3/4,-n,\frac{19}{12},n+3/2\\ 3/2,9/2+5\,n,-3-5\,n,\frac{7}{12}\end{matrix}\;\middle|\;\frac{25}{16}\right) = \frac{(17/10)_n(11/10)_n(9/10)_n(13/10)_n}{(6/5)_n(7/5)_n(8/5)_n(4/5)_n} \tag{20.6a}$$

$$Q = -128\frac{(3+4\,k)\,(1+4\,k)\,k\,(1+2\,k)\,(5+4\,n)\,(k-4-5\,n)}{(12\,k+7)\,(11+2\,k+10\,n)\,(9+2\,k+10\,n)\,(15+2\,k+10\,n)\,(13+2\,k+10\,n)\,(-n-1+k)} \tag{20.6b}$$

$${}_5F_4\left(\begin{matrix}1/4,-1/4,-n,\frac{13}{12},n+1/2\\ 1/2,1/12,-5\,n-1,3/2+5\,n\end{matrix}\;\middle|\;\frac{25}{16}\right) = \frac{(11/10)_n(9/10)_n(7/10)_n(3/10)_n}{(2/5)_n(4/5)_n(3/5)_n(6/5)_n} \tag{20.7a}$$

$$Q = -128\frac{(1+4\,k)\,(4\,k-1)\,(2\,k-1)\,k\,(3+4\,n)\,(k-5\,n-2)}{(1+12\,k)\,(2\,k+5+10\,n)\,(3+2\,k+10\,n)\,(9+2\,k+10\,n)\,(7+2\,k+10\,n)\,(-n-1+k)} \tag{20.7b}$$

21. From Gauss's second theorem:

$$\frac{\Gamma(2\,a+k)\Gamma(2\,b+k)\Gamma(a+1/2)\Gamma(b+1/2)\,(1/2)^k}{\Gamma(k+1)\Gamma(a+b+1/2+k)\Gamma(2\,a)\Gamma(2\,b)\sqrt{\pi}}$$

$$ {}_2F_1\left(\begin{matrix} n+1,-n \\ 2-2\,b+n \end{matrix}\;\Big|\; 1/2\right) = \frac{(2-2\,b)_n}{(3/2-b)_n 2^n} \tag{21.1a}$$

$$Q = 2\frac{k}{-n-1+k} \tag{21.1b}$$

$$ {}_3F_2\left(\begin{matrix} 1/2-\frac{n}{2},-\frac{n}{2},1-\frac{n}{6} \\ 1/2,-\frac{n}{6} \end{matrix}\;\Big|\; 1/4\right) = 2^{1-n}, \quad n \geq 1 \tag{21.2a}$$

$$Q = 8\frac{k\,(2\,k-1)}{(6\,k-n)\,(-n-1+2\,k)} \tag{21.2b}$$

$$ {}_4F_3\left(\begin{matrix} 1+2\,b+n,1-\frac{n}{3},-n,-2\,b-n \\ -\frac{n}{3},b+1,1/2-b-n \end{matrix}\;\Big|\; -1/8\right) = \frac{(0)_n}{(b+1/2)_n 2^n} \tag{21.3a}$$

$$Q = 4\frac{(-1-2\,b-2\,n+2\,k)\,(b+k)\,k}{(3\,k-n)\,(-n-1+k)\,(-2\,b-n-1+k)} \tag{21.3b}$$

$$ {}_4F_3\left(\begin{matrix} n+1,-n,-\frac{2\,b}{3}-\frac{n}{3}+1,-2\,b-n \\ 1-b,-\frac{2\,b}{3}-\frac{n}{3},1/2-b-n \end{matrix}\;\Big|\; -1/8\right) = \frac{(2\,b)_n}{(b+1/2)_n 2^n} \tag{21.4a}$$

$$Q = 4\frac{(k-b)\,k\,(-1-2\,b-2\,n+2\,k)}{(-2\,b-n+3\,k)\,(-n-1+k)\,(-2\,b-n-1+k)} \tag{21.4b}$$

22. From Kummer's theorem:

$$\frac{\Gamma(2\,a+k)\Gamma(b+k)\Gamma(1+2\,a)\Gamma(a-b+1)(-1)^k}{\Gamma(k+1)\Gamma(1+2\,a-b+k)\Gamma(2\,a)\Gamma(b)\Gamma(1+a)}$$

$$ {}_4F_3\left(\begin{matrix} -n,-b-\frac{n}{2},-\frac{2\,b}{3}-\frac{n}{3}+1,-b-\frac{n}{2}+1/2 \\ 1-b,-\frac{2\,b}{3}-\frac{n}{3},1-2\,b-n \end{matrix}\;\Big|\; 4\right) = (-1)^n \tag{22.1a}$$

$$Q = -2\frac{(k-2\,b-n)\,k\,(k-b)}{(-2\,b-n+3\,k)\,(-n-1+k)\,(-2\,b-n-1+2\,k)} \tag{22.1b}$$

$$ {}_4F_3\left(\begin{matrix} -n,-\frac{n}{2}-a,1/2-\frac{n}{2}-a,1-\frac{2\,a}{3}-\frac{n}{3} \\ 1-a,1-n-2\,a,-\frac{2\,a}{3}-\frac{n}{3} \end{matrix}\;\Big|\; 4\right) = (-1)^n \tag{22.2a}$$

$$Q = -2\frac{(k-2\,a-n)\,(k-a)\,k}{(-n-1+k)\,(-1-n+2\,k-2\,a)\,(3\,k-2\,a-n)} \tag{22.2b}$$

23. From Saalschütz's theorem:

$$\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c+k)\Gamma(1+a+b+c-d)\Gamma(d-c)\Gamma(d-a-b-c)\Gamma(d-a)\Gamma(d-b)}{\Gamma(k+1)\Gamma(d+k)\Gamma(1+a+b+c-d+k)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d-a-c)\Gamma(d-b-c)\Gamma(d-a-b)}$$

$$ {}_5F_4\left(\begin{matrix} n, 1+\frac{5n}{4}, -n, 1/12, -\frac{5}{12} \\ 1/3, \frac{5n}{4}, 1/4+\frac{n}{2}, 3/4+\frac{n}{2} \end{matrix} \,\Big|\, 1\right) = \frac{(1/2)_n(2/3)_n}{(5/6)_n(1/3)_n} \tag{23.1a}$$

$$Q = \frac{(4k-1+2n)(3k-2)k}{(4k+5n)(2+3n)(-n-1+k)} \tag{23.1b}$$

$$ {}_5F_4\left(\begin{matrix} \frac{3d}{4}+1/4-\frac{3n}{4}, -n, d-1-n, 3d-3, d-1/2 \\ \frac{3d}{2}-\frac{n}{2}-1/2, d, \frac{3d}{2}-1-\frac{n}{2}, \frac{3d}{4}-3/4-\frac{3n}{4} \end{matrix} \,\Big|\, 1\right) = \frac{(2-d)_n(1-d)_n}{(d)_n(3-3d)_n} \tag{23.2a}$$

$$Q = -\frac{(3d-3-n+2k)(2k-4+3d-n)(d+k-1)k}{(3d-3-3n+4k)(d-2-n)(d-2-n+k)(-n-1+k)} \tag{23.2b}$$

$$ {}_5F_4\left(\begin{matrix} 1/4, 3/4, -n, 5/6-\frac{5n}{4}, 4/3+n \\ 5/3, \frac{13}{12}-\frac{n}{2}, \frac{7}{12}-\frac{n}{2}, -1/6-\frac{5n}{4} \end{matrix} \,\Big|\, 1\right) = \frac{(5/6)_n(2/15)_n}{(-1/6)_n(17/15)_n} \tag{23.3a}$$

$$Q = \frac{(1-6n+12k)(3k+2)k(12k-5-6n)}{(-2-15n+12k)(5+6n)(4+3n)(-n-1+k)} \tag{23.3b}$$

$$ {}_5F_4\left(\begin{matrix} \frac{3b}{2}-1, 2-\frac{3b}{2}, -\frac{n}{2}, 1/2-\frac{n}{2}, 3/4-\frac{3n}{4} \\ 3/2, \frac{b}{2}-n, 1-\frac{b}{2}-n, -1/4-\frac{3n}{4} \end{matrix} \,\Big|\, 1\right) = \frac{(1/3)_n(b)_n(2-b)_n}{(4/3)_n(\frac{b}{2})_n(1-\frac{b}{2})_n} \tag{23.4a}$$

$$Q = 1/2\frac{(1+2k)k(2k-b-2n)(2k-2+b-2n)}{(4k-1-3n)(b+n)(b-2-n)(-n-1+2k)} \tag{23.4b}$$

$$ {}_5F_4\left(\begin{matrix} -\frac{n}{2}, 1/2-\frac{n}{2}, \frac{3b}{2}, -\frac{3b}{2}, 1-\frac{3n}{4} \\ 1/2, 1-\frac{b}{2}-n, 1+\frac{b}{2}-n, -\frac{3n}{4} \end{matrix} \,\Big|\, 1\right) = 1/4\frac{(b)_n(-b)_n}{(\frac{b}{2})_n(-\frac{b}{2})_n}, \quad n \geq 1 \tag{23.5a}$$

$$Q = 1/2\frac{k(2k-1)(2k-b-2n)(2k+b-2n)}{(4k-3n)(b+n)(b-n)(-n-1+2k)} \tag{23.5b}$$

$$ {}_5F_4\left(\begin{matrix} -\frac{n}{2}, 1/2-\frac{n}{2}, 1-\frac{3n}{4}, 2-2d, 3-3d \\ 1-n, d-n, -\frac{3n}{4}, 5/2-2d \end{matrix} \,\Big|\, 1\right) = 2/3\frac{(3-3d)_n(2-2d)_n}{(4-4d)_n(1-d)_n}, \quad n \geq 1 \tag{23.6a}$$

$$Q = -2\frac{(2k-4d+3)k(d-n-1+k)(k-n)}{(4k-3n)(3d-3-n)(2d-2-n)(-n-1+2k)} \tag{23.6b}$$

$$ {}_5F_4\left(\begin{matrix} 7/6, 5/6, -\frac{n}{2}, 1/2-\frac{n}{2}, 9/4-\frac{5n}{4} \\ 1/6, \frac{11}{6}-n, 5/4-\frac{5n}{4}, 3/2+n \end{matrix} \,\Big|\, 1\right) = \frac{(3/2)_n(-1)_n}{(4/3)_n(-5/6)_n}, \quad n \geq 2 \tag{23.7a}$$

$$Q = 2/9\frac{(6k+5-6n)(6k-5)k}{(-n-1+2k)n(4k+5-5n)} \tag{23.7b}$$

$$ {}_5F_4\left(\begin{matrix} -n, 1/4-\frac{3a}{4}-\frac{3n}{4}, -n-a-1, -3/2-\frac{3a}{2}, -1-\frac{3a}{2} \\ -\frac{3a}{4}-3/4-\frac{3n}{4}, -a-1/2-\frac{n}{2}, -\frac{n}{2}-a, -2-3a \end{matrix} \,\Big|\, 1\right) = \frac{(-a)_n(1+a)_n}{(-1-2a)_n(2+2a)_n} \tag{23.8a}$$

$$Q=-\frac{(-2a-2-n+2k)(-3-3a+k)k(2k-3-2a-n)}{(a-n)(-3a-3-3n+4k)(-n-2-a+k)(-n-1+k)} \tag{23.8b}$$

$${}_5F_4\left(\begin{matrix}n,-n,1+\frac{3n}{5},3b-3,3-3b\\1/2,\frac{3n}{5},2b-1+n,3-2b+n\end{matrix}\,\middle|\,-1/4\right)=\frac{(3-2b)_n(2b-1)_n}{(b)_n(2-b)_n} \tag{23.9a}$$

$$Q=2\frac{k(2k-1)}{(5k+3n)(-n-1+k)} \tag{23.9b}$$

$${}_5F_4\left(\begin{matrix}n,1-\frac{3n}{5},-n,-3b,3b\\1/2,-\frac{3n}{5},1-2b-n,1+2b-n\end{matrix}\,\middle|\,-1/4\right)=\frac{(b)_n(-b)_n}{(-2b)_n(2b)_n} \tag{23.10a}$$

$$Q=-2\frac{(2b-n+k)(2k-1)k(k-2b-n)}{(b-n)(b+n)(5k-3n)(-n-1+k)} \tag{23.10b}$$

24. From Saalschütz's theorem:

$$\frac{2^{-2k-2c}\Gamma(3/2+a+c-b)\Gamma(c+k)\Gamma(b-a-1/2-c)\Gamma(b-c)\Gamma(a+2k)\Gamma(2b-1-a)}{\Gamma(2b-a-1-2c)\Gamma(a)\Gamma(3/2+a+c-b+k)\Gamma(k+1)\Gamma(b-a-1/2)\Gamma(c)\Gamma(b+k)}$$

$${}_5F_4\left(\begin{matrix}-2n,1-2d-c,5/3-\frac{2c}{3}-2d,1-3d-c,1/2-d\\1-2d,2-4d-2c,-c-3d+n+2,2/3-\frac{2c}{3}-2d\end{matrix}\,\middle|\,4\right)=\frac{(1/2)_n(2-c-3d)_n}{(3/2-c-2d)_n(1-d)_n} \tag{24.1a}$$

$$Q=-\frac{k(k-2d)(1-4d+k-2c)}{(3k-2c+2-6d)(k-2n-2)(k-2n-1)} \tag{24.1b}$$

$${}_5F_4\left(\begin{matrix}-n,2-2b+2c,5/3-\frac{2b}{3}-\frac{2n}{3},1-b-n,3/2-b\\-2n,2/3-\frac{2b}{3}-\frac{2n}{3},3-2b,1-c-n\end{matrix}\,\middle|\,4\right)=\frac{(b-1)_n(3/2+c-b)_n}{(c)_n(1/2)_n} \tag{24.2a}$$

$$Q=-1/2\frac{(2-2b+k)k(k-c-n)(k-2n-1)}{(3+2c-2b+2n)(3k-2b+2-2n)(-b-n+k)(-n-1+k)} \tag{24.2b}$$

25. From Saalschütz's theorem:

$$\frac{3^{-1-3k-a}\Gamma(a-b+2)\Gamma(b-a-1)\Gamma(a+3k)\Gamma(3b-a-2)}{\Gamma(k+1)\Gamma(3b-2a-3)\Gamma(2+a-b+k)\Gamma(a)\Gamma(b+k)}$$

$${}_3F_2\left(\begin{matrix}-n,2-3c,3c-1\\1/2,-3n\end{matrix}\,\middle|\,3/4\right)=\frac{(c)_n(1-c)_n}{(1/3)_n(2/3)_n} \tag{25.1a}$$

$$Q=\frac{2}{27}\frac{(k-3n-1)k(2k-1)}{(c+n)(c-n-1)(-n-1+k)} \tag{25.1b}$$

$${}_3F_2\left(\begin{matrix}-n,4-3c,3c-2\\3/2,-3n-1\end{matrix}\,\middle|\,3/4\right)=\frac{(c)_n(2-c)_n}{(2/3)_n(4/3)_n} \tag{25.2a}$$

$$Q=\frac{2}{27}\frac{(k-3n-2)(1+2k)k}{(c+n)(c-2-n)(-n-1+k)} \tag{25.2b}$$

$${}_5F_4\left(\begin{matrix}-n,b-n-1,\frac{3b}{4}+1/4-\frac{3n}{4},\frac{3b}{2}-1/2,\frac{3b}{2}-1\\3b-2,-\frac{3n}{2},1/2-\frac{3n}{2},\frac{3b}{4}-3/4-\frac{3n}{4}\end{matrix}\,\middle|\,9\right)=\frac{(1-b)_n(b)_n}{(1/3)_n(2/3)_n} \tag{25.3a}$$

$$Q=-1/27\frac{(3b-3+k)k(2k-1-3n)(-2-3n+2k)}{(-n-1+k)(k-2+b-n)(3b+4k-3-3n)(b+n)} \tag{25.3b}$$

26. From Saalschütz's theorem:

$$\frac{\Gamma(1-c+2\,a+2\,b)\Gamma(b+k)\Gamma(c-2\,a)2^{2\,k}\Gamma(a+k)\Gamma(c-1/2-a-b+k)}{\Gamma(c+2\,k)\Gamma(k+1)\Gamma(a)\Gamma(c-1/2-a-b)\Gamma(1-c+2\,b)\Gamma(b)}$$

$$ {}_5F_4\left(\begin{matrix}-a, 1-2\,d, -n, -n-a-1/2, 2/3-\frac{2\,a}{3}-\frac{2\,n}{3}\\ d-n-a, -2\,a, -2\,n, -1/3-\frac{2\,a}{3}-\frac{2\,n}{3}\end{matrix}\;\middle|\; 4\right)=\frac{(1-d)_n(a+1/2)_n}{(1/2)_n(1+a-d)_n} \tag{26.1a}$$

$$Q=1/2\frac{(k-1-2\,n)\,(d-n-1+k-a)\,k\,(-1-2\,a+k)}{(-1-2\,a-2\,n+3\,k)\,(d-n-1)\,(-n-1+k)\,(-2\,a-3-2\,n+2\,k)} \tag{26.1b}$$

27. From Saalschütz's theorem:

$$\frac{\Gamma(b-a)\Gamma(a+2\,k)\Gamma(b-a-1+k)}{\Gamma(k+1)\Gamma(a)\Gamma(b+2\,k)\Gamma(b-2\,a-1)}$$

$$ {}_5F_4\left(\begin{matrix}-1, 1/2, -n, 1/2-n, \frac{9}{8}-\frac{3\,n}{4}\\ 3/2, 1/4-\frac{n}{2}, 3/4-\frac{n}{2}, 1/8-\frac{3\,n}{4}\end{matrix}\;\middle|\; 1\right)=\frac{(3/2)_n(-1/6)_n}{(1/2)_n(5/6)_n} \tag{27.1a}$$

$$Q=\frac{(-3-2\,n+4\,k)\,(4\,k-1-2\,n)\,k\,(1+2\,k)}{(8\,k+1-6\,n)\,(3+2\,n)\,(-n-1+k)\,(-1-2\,n+2\,k)} \tag{27.1b}$$

$$ {}_4F_3\left(\begin{matrix}1/2-c, -2\,n, 1-\frac{2\,c}{3}, -c\\ 1-c+n, -\frac{2\,c}{3}, 1-2\,c\end{matrix}\;\middle|\; 4\right)=1 \tag{27.2a}$$

$$Q=-\frac{k\,(k-1)\,(k-2\,c)}{(3\,k-2\,c)\,(k-2\,n-2)\,(k-1-2\,n)} \tag{27.2b}$$

$$ {}_4F_3\left(\begin{matrix}a, -a, -n, \frac{a}{3}+1\\ 1/2, 2\,n+a+1, \frac{a}{3}\end{matrix}\;\middle|\; 1/4\right)=\frac{(\frac{a}{2}+1)_n(1/2+\frac{a}{2})_n}{(1/2)_n(1+a)_n} \tag{27.3a}$$

$$Q=2\frac{(2\,k-1)\,k\,(a+k)}{(a+3\,k)\,(a+1+2\,n+k)\,(-n-1+k)} \tag{27.3b}$$

28. From Saalschütz's theorem:

$$\frac{(-1)^a 3^{-1-3\,k-a}2^{1+2\,k}\Gamma(a+3\,k)\Gamma(a+5/2)}{\Gamma(k+1)\Gamma(a)\,(1+2\,a)\,\Gamma(3/2+a+2\,k)}$$

$$ {}_2F_1\left(\begin{matrix}1/2, -3\,n\\ 1/2-2\,n\end{matrix}\;\middle|\; -1/3\right)=\frac{(2/3)_n(1/3)_n}{(3/4)_n(1/4)_n} \tag{28.1a}$$

$$Q=3/16\frac{(-4\,n-1+2\,k)\,(3+2\,k)\,k}{(-3\,n-3+k)\,(k-3\,n-2)\,(k-3\,n-1)} \tag{28.1b}$$

$$ {}_3F_2\left(\begin{matrix}-1/4, 5/4, -n\\ 1/2, -3\,n\end{matrix}\;\middle|\; 3/4\right)=\frac{(3/4)_n(1/4)_n}{(2/3)_n(1/3)_n} \tag{28.2a}$$

$$Q=-\frac{32}{27}\frac{k\,(2\,k-1)\,(k-3\,n-1)}{(3+4\,n)\,(1+4\,n)\,(-n-1+k)} \tag{28.2b}$$

$$ {}_3F_2\left(\begin{matrix}1/4, -n, 7/4\\ 3/2, -3\,n-1\end{matrix} \,\Big|\, 3/4\right) = \frac{(3/4)_n(5/4)_n}{(4/3)_n(2/3)_n} \tag{28.3a} $$

$$ Q = -\frac{32}{27}\frac{(1+2\,k)\,k\,(k-3\,n-2)}{(5+4\,n)\,(3+4\,n)\,(-n-1+k)} \tag{28.3b} $$

$$ {}_3F_2\left(\begin{matrix}1/4, -1/4, -3\,n\\ -1/2, 1/2-n\end{matrix} \,\Big|\, 4/3\right) = \frac{(2/3)_n(1/3)_n}{(1/2)_n^2} \tag{28.4a} $$

$$ Q = -3/4\frac{k\,(2\,k-3)\,(-1-2\,n+2\,k)}{(-3\,n-3+k)\,(k-3\,n-2)\,(k-3\,n-1)} \tag{28.4b} $$

$$ {}_5F_4\left(\begin{matrix}5/4-\frac{3\,n}{4}, 1/4, -\frac{3\,n}{2}, 1/2-\frac{3\,n}{2}, 7/4\\ 3/2, 1/4-\frac{3\,n}{4}, 3/4-n, 5/4-n\end{matrix} \,\Big|\, 1/9\right) = \frac{(1/3)_n(-1/3)_n}{(1/4)_n(-1/4)_n} \tag{28.5a} $$

$$ Q = \frac{9}{8}\frac{(1+4\,k-4\,n)\,k\,(1+2\,k)\,(-1-4\,n+4\,k)}{(1-3\,n+4\,k)\,(2\,k-2-3\,n)\,(2\,k-1-3\,n)\,(-3\,n-3+2\,k)} \tag{28.5b} $$

$$ {}_5F_4\left(\begin{matrix}5/4, \frac{11}{8}-\frac{3\,n}{4}, -n, 7/4, 1/2-n\\ 5/2, 3/8-\frac{3\,n}{4}, -\frac{3\,n}{2}, 1/2-\frac{3\,n}{2}\end{matrix} \,\Big|\, 9\right) = \frac{(-1/2)_n(3/2)_n}{(2/3)_n(1/3)_n} \tag{28.6a} $$

$$ Q = -\frac{4}{27}\frac{(3+2\,k)\,k\,(2\,k-1-3\,n)\,(2\,k-2-3\,n)}{(3+8\,k-6\,n)\,(3+2\,n)\,(-1-2\,n+2\,k)\,(-n-1+k)} \tag{28.6b} $$

$$ {}_4F_3\left(\begin{matrix}1/2, -n, \frac{7}{10}-\frac{6\,n}{5}, -2\,n-1/2\\ -3/10-\frac{6\,n}{5}, -\frac{3\,n}{2}, 1/2-\frac{3\,n}{2}\end{matrix} \,\Big|\, -\frac{9}{16}\right) = \frac{(1/4)_n(3/4)_n}{(2/3)_n(1/3)_n} \tag{28.7a} $$

$$ Q = -\frac{16}{27}\frac{(3+2\,k)\,k\,(2\,k-1-3\,n)\,(2\,k-2-3\,n)}{(2\,k-3-4\,n)\,(10\,k-3-12\,n)\,(2\,k-5-4\,n)\,(-n-1+k)} \tag{28.7b} $$

$$ {}_4F_3\left(\begin{matrix}1/2, -\frac{3\,n}{2}, 1/2-\frac{3\,n}{2}, 1-\frac{6\,n}{5}\\ -\frac{6\,n}{5}, 5/6-n, 2/3-2\,n\end{matrix} \,\Big|\, -\frac{16}{9}\right) = \frac{(1/3)_n(0)_n}{(1/6)_n^2} \tag{28.8a} $$

$$ Q = 1/8\frac{(-1-6\,n+6\,k)\,(-1-6\,n+3\,k)\,k\,(k-2)}{(5\,k-6\,n)\,(2\,k-2-3\,n)\,(2\,k-1-3\,n)\,(-3\,n-3+2\,k)} \tag{28.8b} $$

29. From Vandermonde's theorem:

$$ \frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c-a)\Gamma(c-b)}{\Gamma(k+1)\Gamma(c+k)\Gamma(a)\Gamma(b)\Gamma(c-a-b)} $$

$$ {}_4F_3\left(\begin{matrix}1-b, 1/2-\frac{n}{2}, 1-\frac{2\,n}{3}, -\frac{n}{2}\\ -\frac{2\,n}{3}, 1-n, 1/2+\frac{b}{2}-n\end{matrix} \,\Big|\, 4\right) = 1/2\frac{(1-b)_n}{(1/2-\frac{b}{2})_n} \quad n \geq 1 \tag{29.1a} $$

$$ Q = 1/2\frac{(b-1+2\,k-2\,n)\,k\,(k-n)}{(3\,k-2\,n)\,(b-n-1)\,(-n-1+2\,k)} \tag{29.1b} $$

$$ {}_4F_3\left(\begin{matrix}1/2-b, -n, -b-n, 1-\frac{2\,b}{3}-\frac{2\,n}{3}\\ 1-b, 1-2\,b, -\frac{2\,b}{3}-\frac{2\,n}{3}\end{matrix} \,\Big|\, 4\right) = \frac{(b)_n}{(1-b)_n} \tag{29.2a} $$

$$Q = -\frac{(k-2\,b)\,k\,(k-b)}{(-2\,b-2\,n+3\,k)\,(-b-n-1+k)\,(-n-1+k)} \tag{29.2b}$$

$$ {}_2F_1\left(\begin{matrix}1+\frac{n}{3},-n\\ \frac{n}{3}\end{matrix}\,\Big|\,1/4\right)=\frac{(0)_n}{(1/2)_n} \tag{29.3a}$$

$$Q = 2\frac{(2\,k-1)\,k}{(n+3\,k)\,(-n-1+k)} \tag{29.3b}$$

$$ {}_4F_3\left(\begin{matrix}b,n,1-\frac{n}{3},-n\\ 1/2,-\frac{n}{3},1-2\,b-n\end{matrix}\,\Big|\,1/4\right)=\frac{(b)_n}{(2\,b)_n} \tag{29.4a}$$

$$Q = -2\frac{(k-2\,b-n)\,(2\,k-1)\,k}{(-n-1+k)\,(b+n)\,(3\,k-n)} \tag{29.4b}$$

$$ {}_4F_3\left(\begin{matrix}-n,-n-a-1,-2-2\,a,1/3-\frac{2\,a}{3}-\frac{2\,n}{3}\\ -\frac{2\,a}{3}-2/3-\frac{2\,n}{3},-1/2-\frac{n}{2}-a,-\frac{n}{2}-a\end{matrix}\,\Big|\,1/4\right)=\frac{(1+a)_n}{(2+2\,a)_n} \tag{29.5a}$$

$$Q = \frac{k\,(-3-n+2\,k-2\,a)\,(-2-n+2\,k-2\,a)}{(3\,k-2\,a-2-2\,n)\,(-n-1+k)\,(k-2-n-a)} \tag{29.5b}$$

$$ {}_2F_1\left(\begin{matrix}-n,1-\frac{n}{5}\\ -\frac{n}{5}\end{matrix}\,\Big|\,-1/4\right)=\frac{(0)_n}{(1/2)_n} \tag{29.6a}$$

$$Q = 2\frac{(2\,k-1)\,k}{(5\,k-n)\,(-n-1+k)} \tag{29.6b}$$

30. From Vandermonde's theorem:

$$\frac{4\sqrt{\pi}\Gamma(2\,b-1-a)\Gamma(a+2\,k)2^{-2\,k-2\,b+a}}{\Gamma(k+1)\Gamma(a)\Gamma(b-a-1/2)\Gamma(b+k)}$$

$$ {}_3F_2\left(\begin{matrix}1/2-\frac{n}{2},1-\frac{n}{4},-\frac{n}{2}\\ 1/2,-\frac{n}{4}\end{matrix}\,\Big|\,1\right)=\frac{(-1)_n}{(1/2)_n}, n\geq 2 \tag{30.1a}$$

$$Q = \frac{(2\,k-1)\,k\,(4\,k-1)}{(4\,k-n)\,(n-1)\,(-n-1+2\,k)} \tag{30.1b}$$

$$ {}_3F_2\left(\begin{matrix}1/2-n,-n,1-\frac{2\,n}{3}\\ 1/2,-\frac{2\,n}{3}\end{matrix}\,\Big|\,4\right)=-1, \quad n\geq 1 \tag{30.2a}$$

$$Q = -\frac{(k-1)\,k\,(2\,k-1)}{(3\,k-2\,n)\,(-n-1+k)\,(2\,k-2\,n-1)} \tag{30.2b}$$

$$ {}_4F_3\left(\begin{matrix}-n,b-1/2,b-n-1,\frac{2\,b}{3}+1/3-\frac{2\,n}{3}\\ -2\,n,\frac{2\,b}{3}-\frac{2\,n}{3}-2/3,2\,b-1\end{matrix}\,\Big|\,-8\right)=\frac{(1-b)_n}{(1/2)_n} \tag{30.3a}$$

$$Q = 1/4\frac{k\,(k-2+2\,b)\,(k-2\,n-1)}{(2\,b-2+3\,k-2\,n)\,(b-2+k-n)\,(-n-1+k)} \tag{30.3b}$$

$$ {}_4F_3\left(\begin{matrix} 3/2+\frac{n}{5}, 2/3, -n, 2+2\,n \\ \frac{11}{6}+n, 4/3, 1/2+\frac{n}{5} \end{matrix} \;\Big|\; \frac{2}{27}\right) = \frac{(5/2)_n (11/6)_n}{(3/2)_n (7/2)_n} \tag{30.4a} $$

$$ Q = 9/2 \frac{(3\,k+1)\,k}{(-n-1+k)\,(10\,k+5+2\,n)} \tag{30.4b} $$

$$ {}_4F_3\left(\begin{matrix} 2\,n, -n, 1/6, 1+\frac{n}{5} \\ 1/3, 1/3+n, \frac{n}{5} \end{matrix} \;\Big|\; \frac{2}{27}\right) = \frac{(1/3)_n}{(1/2)_n} \tag{30.5a} $$

$$ Q = 9/4 \frac{k\,(3\,k-2)}{(-n-1+k)\,(5\,k+n)} \tag{30.5b} $$

$$ {}_4F_3\left(\begin{matrix} 1/3, -n, 1+2\,n, \frac{11}{10}+\frac{n}{5} \\ 2/3, 1/10+\frac{n}{5}, 7/6+n \end{matrix} \;\Big|\; \frac{2}{27}\right) = \frac{(7/6)_n}{(3/2)_n} \tag{30.6a} $$

$$ Q = 9/2 \frac{k\,(3\,k-1)}{(-n-1+k)\,(10\,k+1+2\,n)} \tag{30.6b} $$

$$ {}_4F_3\left(\begin{matrix} 5/6, -n, 2+2\,n, 9/5+\frac{n}{5} \\ 5/3, 4/5+\frac{n}{5}, 5/3+n \end{matrix} \;\Big|\; \frac{2}{27}\right) = \frac{(4)_n (5/3)_n}{(3/2)_n (5)_n} \tag{30.7a} $$

$$ Q = 9/4 \frac{(3\,k+2)\,k}{(-n-1+k)\,(5\,k+4+n)} \tag{30.7b} $$

$$ {}_4F_3\left(\begin{matrix} 1/3, 2/3, -2\,n, 7/5+\frac{2\,n}{5} \\ 3/4-\frac{n}{2}, \frac{2\,n}{5}+2/5, 5/4-\frac{n}{2} \end{matrix} \;\Big|\; \frac{27}{32}\right) = \frac{(1)_n (1/2)_n}{(2)_n (-1/2)_n} \tag{30.8a} $$

$$ Q = 2 \frac{(1-2\,n+4\,k)\,(-1-2\,n+4\,k)\,k}{(2\,n+2+5\,k)\,(k-2\,n-2)\,(k-2\,n-1)} \tag{30.8b} $$

$$ {}_4F_3\left(\begin{matrix} \frac{11}{5}+\frac{8\,n}{5}, n+1, 5/6, -2\,n \\ 4/3, 5/3, 6/5+\frac{8\,n}{5} \end{matrix} \;\Big|\; \frac{32}{27}\right) = \frac{(1/2)_n (3/4)_n}{(7/6)_n (7/4)_n} \tag{30.9a} $$

$$ Q = -3 \frac{k\,(3\,k+2)\,(3\,k+1)}{(6+8\,n+5\,k)\,(k-2\,n-2)\,(k-2\,n-1)} \tag{30.9b} $$

$$ {}_4F_3\left(\begin{matrix} 1/3, 1/2-\frac{n}{2}, -\frac{n}{2}, 1+\frac{2\,n}{5} \\ 2/3, 1-2\,n, \frac{2\,n}{5} \end{matrix} \;\Big|\; \frac{32}{27}\right) = 3\frac{(1/6)_n}{(1/2)_n}, n \geq 1 \tag{30.10a} $$

$$ Q = -\frac{27}{2} \frac{(k-2\,n)\,(3\,k-1)\,k}{(1+6\,n)\,(2\,n+5\,k)\,(-n-1+2\,k)} \tag{30.10b} $$

31. From Watson's theorem:

$$ \frac{\Gamma(2\,a+k)\Gamma(2\,b+k)\Gamma(c+k)\Gamma(2\,c)\Gamma(a+1/2)\Gamma(b+1/2)\Gamma(c-a+1/2)\Gamma(c-b+1/2)}{\Gamma(k+1)\Gamma(a+b+1/2+k)\Gamma(2\,c+k)\Gamma(2\,a)\Gamma(2\,b)\Gamma(c)\sqrt{\pi}\Gamma(c+1/2)\Gamma(1/2-a-b+c)} $$

$$ {}_5F_4\left(\begin{matrix} 1/2+a-c, -n, n+1, 2-2\,c+n, 5/3-\frac{2\,c}{3}+\frac{n}{3} \\ 2-c+n, 2/3-\frac{2\,c}{3}+\frac{n}{3}, n-2\,a+2, 3/2-c \end{matrix} \;\Big|\; 1/4\right) = \frac{(2-c)_n (2-2\,a)_n}{(3-2\,c)_n (3/2-a)_n} \tag{31.1a} $$

$$ Q = 2 \frac{k\,(1-2\,c+2\,k)}{(2-2\,c+n+3\,k)\,(-n-1+k)} \tag{31.1b} $$

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