# COUNTING ACYCLIC DIGRAPHS BY SOURCES AND SINKS


Ira M. Gessel*
Department of Mathematics
Brandeis University
Waltham, MA 02254-9110


January 3, 1995


Abstract. We count labeled acyclic digraphs according to the number sources, sinks, and edges.


**1. Counting acyclic digraphs by sources.** Let

$$A_n(t;\alpha) = \sum_D \alpha^{s(D)} t^{e(D)},$$

where the sum is over all acyclic digraphs $D$ on the vertex set $[n] = \{1, 2, \ldots, n\}$, $e(D)$ is the number of edges of $D$, and $s(D)$ is the number of sources of $D$; that is, the number of vertices of $D$ of indegree 0. Let $A_n(t) = A_n(t;1)$.

To find a recurrence for $A_n(t,\alpha)$, we take an acyclic digraph and add some new vertices as sources. In the digraph we obtain, the new vertices will be a subset of the set of sources.

**Lemma 1.**

$$\sum_{j=0}^{n} (1+t)^{j(n-j)} \binom{n}{j} \alpha^j A_{n-j}(t) = A_n(t;\alpha+1). \tag{1}$$

*Proof.* We count triples $(S, D, E)$, where $S$ is a subset of $[n]$, $D$ is an acyclic digraph on $[n]-S$, and $E$ is a subset of the set $S \times ([n] - S)$. We think of the elements of $E$ as edges from $S$ to $D$. To a triple $(S, D, E)$ we assign the weight $\alpha^j t^e$, where $j$ is the size of $S$ and $e$ is the total number of edges in $E$ and in $D$. It is clear that the sum of the weights of these triples in which $|S| = j$ is $(1+t)^{j(n-j)} \binom{n}{j} \alpha^j A_{n-j}(t)$, and summing on $j$ yields the left side of (1).

To a triple $(S, D, E)$ we may associate the pair $(S, D')$ in which $D'$ is the digraph on $[n]$ whose edges are those of $D$ together with the edges in $E$. Note that $S$ is a subset of the set of sources of $D'$. It is easily seen that this correspondence gives a bijection from triples $(S, D, E)$ to pairs $(S, D')$ in which $D'$ is an acyclic digraph on $[n]$ and $S$ is a subset of the set of sources of $D'$. This proves (1).

In working with recurrences like (1), generating functions of the form

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{(1+t)^{\binom{n}{2}} n!}$$


*partially supported by NSF grant DMS-9306297

are useful, since the convolution

$$\sum_{j=0}^{n}(1+t)^{j(n-j)}\binom{n}{j}a_j b_{n-j} = c_n$$

is equivalent to the generating function equation

$$\left[\sum_{n=0}^{\infty} a_n \frac{x^n}{(1+t)^{\binom{n}{2}}n!}\right]\left[\sum_{n=0}^{\infty} b_n \frac{x^n}{(1+t)^{\binom{n}{2}}n!}\right] = \sum_{n=0}^{\infty} c_n \frac{x^n}{(1+t)^{\binom{n}{2}}n!}.$$

For some further applications of these generating functions, which we call *graphic generating functions*, see [5] and [6].

We can now express a generating function for $A_n(t;\alpha)$ in terms of the power series

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{(1+t)^{\binom{n}{2}}n!}.$$

**Theorem 1.**

$$\sum_{n=0}^{\infty} A_n(t;\alpha)\frac{x^n}{(1+t)^{\binom{n}{2}}n!} = F\big((\alpha-1)x\big)/F(-x)$$

$$= \left[\sum_{n=0}^{\infty}(\alpha-1)^n \frac{x^n}{(1+t)^{\binom{n}{2}}n!}\right] \bigg/ \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(1+t)^{\binom{n}{2}}n!}\right]. \quad (2)$$

*Proof.* Multiplying (1) by $x^n/(1+t)^{\binom{n}{2}}n!$ and summing on $n$ yields

$$F(\alpha x)\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{(1+t)^{\binom{n}{2}}n!} = \sum_{n=0}^{\infty} A_n(t;\alpha+1)\frac{x^n}{(1+t)^{\binom{n}{2}}n!}. \quad (3)$$

Now let us set $\alpha = -1$ in (3). Since every digraph with at least one vertex has a source, $A_n(t;0) = 0$ for $n > 0$. Thus we obtain

$$F(-x)\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{(1+t)^{\binom{n}{2}}n!} = 1,$$

and so

$$\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{(1+t)^{\binom{n}{2}}n!} = 1/F(-x) = \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(1+t)^{\binom{n}{2}}n!}\right]^{-1}. \quad (4)$$

Replacing $\alpha$ with $\alpha - 1$ in (3), and using (4), yields (2).

Formula (4) was found (with $t = 1$) by Robinson [9] (see also Stanley [11] and Rodionov [10]). If we equate coefficients of $\alpha^k$ in (2), we find that the graphic generating function for acyclic digraphs with $k$ sources is

$$\frac{x^k}{(1+t)^{\binom{k}{2}}k!}\frac{F\big(-x/(1+t)^k\big)}{F(-x)},$$

as shown by Liskovets [7].

The approach used to derive Lemma 1 can also be applied to other types of acyclic digraphs. For example, in counting acyclic functional digraphs, a new vertex can be added as a source to a digraph with $m$ vertices in $m$ ways. We obtain the following analogue of Lemma 1:

**Proposition.** *Let $R$ be a set of size $r$ disjoint from the set of positive integers. Let*

$$T_n(\alpha) = \sum_V \alpha^{s(V)},$$

*where the sum is over all acyclic functional digraphs $V$ corresponding to functions from $[n]$ to $[n] \cup R$, and $s(V)$ is the number of elements of $[n]$ which are sources of $V$. (Thus $T_n(\alpha)$ counts forests of $r$ rooted trees by the number of non-root leaves.) Let $T_n = T_n(1)$. Then*

$$\sum_{j=0}^{n} (n-j+r)^j \binom{n}{j} \alpha^j T_{n-j} = T_n(\alpha+1). \tag{5}$$

If we multiply (5) by $x^n/n!$ and sum on $n$, we get

$$\sum_{n=0}^{\infty} T_n(\alpha+1)\frac{x^n}{n!} = e^{\alpha r x} \sum_{k=0}^{\infty} T_k \frac{(xe^{\alpha x})^k}{k!}.$$

Setting $\alpha = -1$ and using the fact that $T_n(0)$ is 1 for $n = 0$ and 0 for $n > 0$, we get a functional equation that can be solved by Lagrange inversion to get $T_n = r(n+r)^{n-1}$, as is well known; thus

$$\begin{aligned} T_n(\alpha) &= \sum_{j=0}^{n} \binom{n}{j} (\alpha-1)^j r(n-j+r)^{n-1} \\ &= \sum_{k=0}^{n} \alpha^k \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} r(j+r)^{n-1} \\ &= \sum_{k=0}^{n} \sum_{l=1}^{n} \alpha^k r^l (n-k)! \binom{n}{k} \binom{n-1}{l-1} S(n-l, n-k), \quad \text{for } n \geq 1, \end{aligned}$$

where $S(n,k)$ is the Stirling number of the second kind.

For a closely related approach to counting trees by leaves, see Bergeron, Labelle, and Leroux [1]. The same approach can also be used to count $k$-trees [8].

**2. Counting acyclic digraphs by sources and sinks.** We now count acyclic graphs by sources and sinks (vertices of outdegree zero). We use the same idea as before: starting with an acyclic digraph, we add some additional vertices as sources and some additional vertices as sinks. There is a small complication that arises for vertices that are both sources and sinks, but it is easy to deal with.

Let

$$B_n(t; \alpha, \beta, \gamma) = \sum_D \alpha^{s_0(D)} \beta^{s_1(D)} \gamma^{i(D)} t^{e(D)},$$

where the sum is over all acyclic digraphs $D$ on $[n]$; here $s_0(D)$ is the number of sources of $D$ that are not sinks, $s_1(D)$ is the number of sinks of $D$ that are not sources, and $i(D)$ is the number of isolated vertices of $D$, i.e., vertices that are both sources and sinks.

By the same reasoning as in the proof of Lemma 1, we get a formula for $B_n$ analogous to (1) that yields the following generating function:

**Lemma 2.**

$$F(\alpha x)\left[\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{(1+t)^{\binom{n}{2}} n!}\right] F(\beta x) = \sum_{n=0}^{\infty} B_n(t;\alpha+1,\beta+1,\alpha+\beta+1)\frac{x^n}{(1+t)^{\binom{n}{2}} n!}. \quad (6)$$

From Lemma 2, we derive a generating function for $B_n(t;\alpha,\beta,\gamma)$:

**Theorem 2.**

$$\sum_{n=0}^{\infty} B_n(t;\alpha,\beta,\gamma)\frac{x^n}{n!} = e^{(\gamma-\alpha-\beta+1)x}\sum_{n=0}^{\infty} C_n(t;\alpha,\beta)\frac{x^n}{n!}, \quad (7)$$

*where*

$$\sum_{n=0}^{\infty} C_n(t;\alpha,\beta)\frac{x^n}{(1+t)^{\binom{n}{2}} n!} = \frac{F\big((\alpha-1)x\big)F\big((\beta-1)x\big)}{F(-x)}. \quad (8)$$

*Proof.* Since any acyclic digraph consists of an acyclic digraph without isolated vertices together with a set of isolated vertices, we have the recurrence

$$B_n(t;\alpha,\beta,\gamma) = \sum_{j=0}^{n}\binom{n}{j}\gamma^j B_{n-j}(t;\alpha,\beta,0),$$

or equivalently,

$$\sum_{n=0}^{\infty} B_n(t;\alpha,\beta,\gamma)\frac{x^n}{n!} = e^{\gamma x}\sum_{n=0}^{\infty} B_n(t;\alpha,\beta,0)\frac{x^n}{n!}. \quad (9)$$

Now let $C_n(t;\alpha,\beta) = B_n(t;\alpha,\beta,\alpha+\beta-1)$. Then (7) follows from (9), and (8) follows from (6) and (4).

The polynomials $B_n(t;\alpha,\beta,\gamma)$ are easily computed from (7) and (8). Note that (7) is an exponential generating function, whereas (8) is a graphic generating function.

The first few values of the polynomials $B_n = B_n(t;\alpha,\beta,\gamma)$ are

$$\begin{aligned}
B_0 &= 1\\
B_1 &= \gamma\\
B_2 &= \gamma^2 + 2\alpha\beta t\\
B_3 &= 3\alpha\beta^2t^2 + 3\alpha^2\beta t^2 + 6\alpha\beta t\gamma + \gamma^3 + 6\alpha\beta t^3 + 6\alpha\beta t^2\\
B_4 &= 12\alpha^2\beta^2t^2 + 24\alpha\beta t^3 + 4\alpha^3\beta t^3 + 36\alpha^2\beta t^3 + 24\alpha^2\beta^2t^3 + 6\alpha^2\beta^2t^4 + 48\alpha^2\beta t^4\\
&\quad + 36\alpha\beta^2t^3 + 48\alpha\beta^2t^4 + 84\alpha\beta t^4 + 4\alpha\beta^3t^3 + 84\alpha\beta t^5 + 12\alpha\beta^2t^5 + 12\alpha^2\beta t^5\\
&\quad + 24\alpha\beta t^6 + \gamma^4 + 12\alpha\beta t\gamma^2 + 12\alpha\beta^2t^2\gamma + 12\alpha^2\beta t^2\gamma + 24\alpha\beta t^3\gamma + 24\alpha\beta t^2\gamma
\end{aligned}$$

The polynomials $S_n(t)$ that count acyclic digraphs on $[n]$ with a single source, a single sink, and no isolated vertices are

$$
\begin{aligned}
S_2(t) &= 2t \\
S_3(t) &= 6t^2 + 6t^3 \\
S_4(t) &= 24t^3 + 84t^4 + 84t^5 + 24t^6 \\
S_5(t) &= 120t^4 + 960t^5 + 2660t^6 + 3500t^7 + 2400t^8 + 840t^9 + 120t^{10} \\
S_6(t) &= 720t^5 + 10800t^6 + 59280t^7 + 170250t^8 + 296010t^9 + 334680t^{10} + 253920t^{11} \\
&\quad + 129300t^{12} + 42660t^{13} + 8280t^{14} + 720t^{15} \\
S_7(t) &= 5040t^6 + 126000t^7 + 1175580t^8 + 5931240t^9 + 18958842t^{10} + 41833302t^{11} \\
&\quad + 67063080t^{12} + 80561460t^{13} + 73777620t^{14} + 51838080t^{15} + 27842220t^{16} \\
&\quad + 11261460t^{17} + 3327660t^{18} + 679140t^{19} + 85680t^{20} + 5040t^{21}
\end{aligned}
$$

It may be noted that Theorems 1 and 2 have analogues in the context of the Cartier-Foata theory of free partially commutative monoids [3]: the analogue of (4) is due to Cartier and Foata [3, Théorème 2.4], the analogue of Theorem 1 is due in the case of a single source to Foata [4, Theorem 3.1] and in the general case to Viennot [12, Proposition 5.3], and the analogue of Theorem 2 is due to Bousquet-Mélou [2, Lemme 1.2 and Théorème 1.3].

## References


1. F. Bergeron, G. Labelle, and P. Leroux, *Computation of the expected number of leaves in a tree having a given automorphism, and related topics*, Discrete Applied Math. **34** (1991), 49–66.
2. M. Bousquet-Mélou, *q-Énumeration de polyominos convexes*, J. Combin. Theory Ser. A **64** (1993), 265–288.
3. P. Cartier and D. Foata, *Problèmes combinatoires de commutation et réarrangement*, Lecture Notes in Mathematics 85, Springer-Verlag, Berlin, 1969.
4. D. Foata, *A noncommutative version of the matrix inversion formula*, Adv. Math. **31** (1979), 330–339.
5. I. M. Gessel, *Enumerative applications of a decomposition for graphs and digraphs*, Discrete Math. **139**, 257–271.
6. I. M. Gessel and B. E. Sagan, *The Tutte polynomial of a graph, depth-first search, and simplicial complex partitions*, Electronic J. Combinatorics, to be published.
7. V. A. Liskovets, *On the number of maximal vertices of a random acyclic digraph*, Theory Probab. Appl. **20** (1975), 401–409.
8. J. W. Moon, *The number of labeled $k$-trees*, J. Combin. Theory **6** (1969), 196–199.
9. R. W. Robinson, *Counting labeled acyclic digraphs*, New Directions in the Theory of Graphs, Proc. Third Ann Arbor Conference on Graph Theory, ed. F. Harary, Academic Press, New York, 1971, pp. 239–273.
10. V. I. Rodionov, *On the number of labeled acyclic digraphs*, Discrete Math. **105** (1992), 319–321.
11. R. P. Stanley, *Acyclic orientations of graphs*, Discrete Math. **5** (1973), 171–178.
12. G. X. Viennot, *Heaps of pieces, I: Basic definitions and combinatorial lemmas*, Combinatoire Énumerative. Proceedings, Montréal, Québec 1985, ed. G. Labelle and P. Leroux. Lecture Notes in Mathematics 1234, Springer-Verlag, Berlin, 1986, pp. 321–350.