A SIMPLE PROOF OF ANDREWS'S $_5F_4$ EVALUATION

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Abstract. We give a simple proof of George Andrews's balanced $_5F_4$ evaluation using two fundamental principles: the nth difference of a polynomial of degree less than n is zero, and a polynomial of degree n that vanishes at $n+1$ points is identically zero.

1. Introduction

George Andrews [1], in his evaluation of the Mills-Robbins-Rumsey determinant, needed the balanced $_5F_4$ evaluation

$$ {}_5F_4\left(\begin{matrix} -2m-1, x+2m+2, x-z+\frac{1}{2}, x+m+1, z+m+1 \\ \frac{1}{2}x+\frac{1}{2}, \frac{1}{2}x+1, 2z+2m+2, 2x-2z+1 \end{matrix} \,\middle|\, 1\right) = 0, \tag{1} $$

where m is a nonnegative integer. Here the hypergeometric series is defined by

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, t\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{k!\,(b_1)_k \cdots (b_q)_k} t^k $$

and $(a)_k$ is the rising factorial $a(a+1)\cdots(a+k-1)$. Andrews's proof of (1) used Pfaff's method, and required a complicated induction that proved 20 related identities. Andrews later discussed these identities and Pfaff's method in comparison with the WZ method [2], and a proof of (1) using the Gosper-Zeilberger algorithm was given by Ekhad and Zeilberger [5]. A completely different proof of (1) was given by Andrews and Stanton [3]. Generalizations of (1), proved using known transformations for hypergeometric series, have been given by Stanton [6], Chu [4], and Verma, Jain, and Jain [7].

We give here a simple self-contained proof of Andrews's identity, by using two fundamental principles: first, the nth difference of a polynomial of degree less than n is 0, and second, a polynomial of degree n that vanishes at $n+1$ points is identically 0.

To illustrate the method, we first use it to prove the Pfaff-Saalschütz identity. We then prove Andrews's identity.

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2. Lemmas

We first give two lemmas that we will need later on. Although they are well known, for completeness we include the short proofs.

Lemma 1. *If $p(k)$ is a polynomial of degree less than n then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}p(k)=0.$$

Proof. Since the polynomials $\binom{k}{i}$ form a basis for the vector space of all polynomials in k, it suffices by linearity to show that if $i<n$ then $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k}{i}=0$. But

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k}{i}=(-1)^i\binom{n}{i}\sum_{k=i}^{n}(-1)^{k-i}\binom{n-i}{k-i}=(-1)^i\binom{n}{i}(1-1)^{n-i}=0,$$

by the binomial theorem. □

Lemma 2. *If $\alpha-\beta=d$ is a nonnegative integer, then $(\alpha)_k/(\beta)_k$, as a function of k, is a polynomial of degree d.*

Proof. We first note the formula

$$(u)_{i+j}=(u)_i(u+i)_j,$$

which we will also use later. Then

$$(\beta)_d\frac{(\alpha)_k}{(\beta)_k}=\frac{(\beta)_d(\beta+d)_k}{(\beta)_k}=\frac{(\beta)_{d+k}}{(\beta)_k}=(\beta+k)_d.$$ □

We shall also use the fact that a polynomial of degree at most d is determined by its value at $d+1$ points, or by its leading coefficient and its value at d points.

3. The Pfaff-Saalschütz identity

As a warm-up we give a proof of the Pfaff-Saalschütz identity

$${}_3F_2\left(\begin{matrix}-m,\ a,\ b\\ c,1-m+a+b-c\end{matrix}\,\middle|\,1\right)=\frac{(c-a)_m(c-b)_m}{(c)_m(c-a-b)_m}. \tag{2}$$

We assume that $a-b$ is not an integer; it is easy to see that the identity with this restriction implies the general case. First we show that the left side of (2) vanishes if $c-a\in\{0,-1,\dots,-(m-1)\}$. With $c-a=-i$, we may write the left side of (2) as

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(c+i)_k}{(c)_k}\frac{(b)_k}{(1-m+i+b)_k}. \tag{3}$$

By Lemma 2,

$$\frac{(c+i)_k}{(c)_k}\frac{(b)_k}{(1-m+i+b)_k}$$

is a polynomial in k of degree $i+(m-i-1)=m-1$, so by Lemma 1, the sum (3) vanishes. By symmetry, (3) also vanishes if $c-b\in\{0,-1,\dots,-(m-1)\}$.

Multiplying the left side of (2) by $(c)_m(c-a-b)_m$ and simplifying gives

$$(c)_m(c-a-b)_m\,{}_3F_2\left(\begin{matrix}-m,\ a,\ b\\ c,1-m+a+b-c\end{matrix}\,\middle|\,1\right)$$
$$=\sum_{k=0}^{m}\binom{m}{k}(a)_k(b)_k(c+k)_{m-k}(c-a-b)_{m-k}. \tag{4}$$

Then (4) is a monic polynomial in c of degree $2m$ that vanishes for the $2m$ distinct (since $a-b$ is not an integer) values $c=a-i$ and $c=b-i$, for $i\in\{0,1,\dots m\}$. Thus (4) is equal to $(c-a)_m(c-b)_m$.

We note that the sum in the Pfaff-Saalschütz theorem is *balanced*; that is, the sum of the denominator parameters is one more than the sum of the numerator parameters. It is not hard to show that if a balanced hypergeometric series can be expressed in the form

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}p(k),$$

where $p(k)$ is a polynomial in k, then $p(k)$ must have degree $m-1$, and thus the sum vanishes by Lemma 1. For this reason, our method is especially applicable to balanced summation formulas.

4. Andrews's Identity

To prove (1), we start by making the substitution $x=y+2z$, obtaining the equivalent identity

$${}_5F_4\left(\begin{matrix}-2m-1,y+2z+2m+2,y+z+\frac{1}{2},y+2z+m+1,z+m+1\\ \frac{1}{2}y+z+\frac{1}{2},\frac{1}{2}y+z+1,2z+2m+2,2y+2z+1\end{matrix}\,\middle|\,1\right)=0. \tag{5}$$

We shall first show that (5) holds when $y\in\{0,1,\dots,2m+1\}$ by applying Lemma 1. We will then derive the general result by expressing the sum as a polynomial in y of degree $2m$.

Lemma 3. *Formula* (5) *holds for* $y\in\{0,1,\dots,2m+1\}$.

Proof. We write the sum in (5) as

$$\sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}P_1(k)P_2(k),$$

where

$$P_1(k)=\frac{(y+2z+2m+2)_k(y+2z+m+1)_k}{(2z+2m+2)_k(2y+2z+1)_k}$$

and

$$P_2(k)=\frac{(y+z+\frac{1}{2})_k(z+m+1)_k}{(\frac{1}{2}y+z+\frac{1}{2})_k(\frac{1}{2}y+z+1)_k}.$$

□

It will suffice to show that for each $y \in \{0, 1, \ldots, 2m+1\}$, $P_1(k)$ and $P_2(k)$ are polynomials in k. We do this by pairing up the numerator and denominator factors in $P_1(k)$ and $P_2(k)$ so that Lemma 2 applies.

For $0 \le y \le m$ we use

$$P_1(k) = \frac{(y+2z+2m+2)_k}{(2z+2m+2)_k} \cdot \frac{(y+2z+m+1)_k}{(2y+2z+1)_k},$$

and for $m+1 \le y \le 2m+1$, we use

$$P_1(k) = \frac{(y+2z+2m+2)_k}{(2y+2z+1)_k} \cdot \frac{(y+2z+m+1)_k}{(2z+2m+2)_k}.$$

For y even, we use

$$P_2(k) = \frac{(y+z+\frac{1}{2})_k}{(\frac{1}{2}y+z+\frac{1}{2})_k} \cdot \frac{(z+m+1)_k}{(\frac{1}{2}y+z+1)_k},$$

and for y odd we use

$$P_2(k) = \frac{(y+z+\frac{1}{2})_k}{(\frac{1}{2}y+z+1)_k} \cdot \frac{(z+m+1)_k}{(\frac{1}{2}y+z+\frac{1}{2})_k}.$$

It is easily checked that Lemma 2 applies in all cases. So for each y, $P_1(k)P_2(k)$ is a polynomial in k of degree $2m$, and the result follows from Lemma 1.

Lemma 4. *The series in* (5)*, after multiplication by* $(y+z+1)_m(y+2z+1)_m$*, is a polynomial in* y *of degree at most* $2m$*.*

Proof. We show that each term in the sum, when multiplied by $(y+z+1)_m(y+2z+1)_m$, is a polynomial in y of degree at most $2m$. Ignoring factors that do not contain y, we see that we must show that for $0 \le k \le 2m+1$,

$$(y+z+1)_m(y+2z+1)_m \frac{(y+2z+2m+2)_k(y+z+\frac{1}{2})_k(y+2z+m+1)_k}{(\frac{1}{2}y+z+\frac{1}{2})_k(\frac{1}{2}y+z+1)_k(2y+2z+1)_k}$$

is a polynomial in y of degree at most $2m$. To do this we define

$$Q_1(y) = (y+z+1)_m \frac{(y+z+\frac{1}{2})_k}{(2y+2z+1)_k} \tag{6}$$

and

$$Q_2(y) = (y+2z+1)_m \frac{(y+2z+2m+2)_k(y+2z+m+1)_k}{(\frac{1}{2}y+z+\frac{1}{2})_k(\frac{1}{2}y+z+1)_k} \tag{7}$$

and we show that $Q_1(y)$ and $Q_2(y)$ are both polynomials in y of degree m. We will use the formulas

$$(a)_{2n} = 2^{2n}(\tfrac{1}{2}a)_n(\tfrac{1}{2}a+\tfrac{1}{2})_n,$$
$$(a)_{2n+1} = 2^{2n+1}(\tfrac{1}{2}a)_{n+1}(\tfrac{1}{2}a+\tfrac{1}{2})_n.$$

For $k \le m$, we have

$$\begin{aligned}
Q_1(y) &= (y+z+1)_k (y+z+1+k)_{m-k} \frac{(y+z+\frac{1}{2})_k}{(2y+2z+1)_k} \\
&= 2^{-2k}(y+z+1+k)_{m-k} \frac{(2y+2z+1)_{2k}}{(2y+2z+1)_k} \\
&= 2^{-2k}(y+z+1+k)_{m-k}(2y+2z+1+k)_k,
\end{aligned}$$

and for $m+1 \le k \le 2m+1$ we have

$$\begin{aligned}
Q_1(y) &= (y+z+1)_m \frac{(y+z+\frac{1}{2})_{m+1}(y+z+m+\frac{3}{2})_{k-m-1}}{(2y+2z+1)_k} \\
&= 2^{-2m-1} \frac{(2y+2z+1)_{2m+1}}{(2y+2z+1)_k}(y+z+m+\tfrac{3}{2})_{k-m-1} \\
&= 2^{-2m-1}(2y+2z+1+k)_{2m+1-k}(y+z+m+\tfrac{3}{2})_{k-m-1},
\end{aligned}$$

so in both cases, $Q_1(y)$ is a polynomial in y of degree m.

We have

$$Q_2(y) = 2^{2k} \frac{(y+2z+1)_{m+k}(y+2z+2m+2)_k}{(y+2z+1)_{2k}}.$$

For $k \le m$ we have

$$Q_2(y) = 2^{2k}(y+2z+1+2k)_{m-k}(y+2z+2m+2)_k.$$

For $m+1 \le k \le 2m+1$, we have

$$\begin{aligned}
Q_2(y) &= 2^{2k} \frac{(y+2z+1)_{m+k}(y+2z+2m+2)_k}{(y+2z+1)_{2k}} \\
&= 2^{2k} \frac{(y+2z+1)_{m+k}}{(y+2z+1)_{2m+1}} \cdot \frac{(y+2z+1)_{2m+1}(y+2z+2m+2)_k}{(y+2z+1)_{2k}} \\
&= 2^{2k}(y+z+2m+2)_{k-m-1} \frac{(y+2z+1)_{2m+1+k}}{(y+2z+1)_{2k}} \\
&= 2^{2k}(y+z+2m+2)_{k-m-1}(y+2z+1+2k)_{2m+1-k}.
\end{aligned}$$

Thus in both cases, $Q_2(y)$ is also a polynomial in y of degree m.

As an alternative, we could have expressed $Q_1(y)$ and $Q_2(y)$ as rising factorials with respect to y,

$$\begin{aligned}
Q_1(y) &= C_1 \frac{(z+m+1)_y (z+k+\frac{1}{2})_y}{(z+\frac{1}{2}k+\frac{1}{2})_y (z+\frac{1}{2}k+1)_y} \\
Q_2(y) &= C_2 \frac{(2z+m+k+1)_y (2z+2m+k+2)_y}{(2z+2k+1)_y (2z+2m+2)_y},
\end{aligned}$$

where C_1 and C_2 do not contain y, and applied Lemma 2. □

We can now finish the proof of (5). By Lemmas 3 and 4, $(y+z+1)_m(y+2z+1)_m$ times the sum in (5) is a polynomial in y of degree at most $2m$ that vanishes for $y = 0, 1, \ldots, 2m+1$. Therefore this polynomial is identically zero.

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