ON THE DESCENT NUMBERS AND MAJOR INDICES FOR THE HYPEROCTAHEDRAL GROUP

CHAK-ON CHOW AND IRA M. GESSEL

Abstract. Adin, Brenti, and Roichman [*Adv. in Appl. Math.* **27** (2001), 210–224], in answering a question posed by Foata, introduced two descent numbers and major indices for the hyperoctahedral group B_n, whose joint distribution generalizes an identity due to MacMahon and Carlitz. We shall show that yet another pair of statistics exists, and whose joint distribution constitutes a "natural" solution to Foata's problem.

1. Introduction

The classical Eulerian polynomial $\mathfrak{S}_n(t)$, which enumerates the symmetric group $\mathfrak{S}_n$ with respect to the (type A) descent number, is well studied (e.g., see [28, 29]).

Another well studied statistic on the symmetric group is the major index, which is equidistributed with the inversion number (a classical result due to MacMahon [24]). The latter is the same as the length, in the Coxeter group theoretic sense. Any statistic equidistributed with length, inversion number, or major index, on the symmetric group is then termed Mahonian. Equidistribution with the length function then serves as the defining condition for a statistic to be Mahonian for other Coxeter groups.

Generalizations of $\mathfrak{S}_n(t)$ by jointly enumerating the symmetric group by descent numbers and other statistics have also been made (e.g., see Garsia [18]). The joint distribution of descent number and major index for the symmetric group is described by a well-known identity of Carlitz, equation (4) below. One can similarly define the type B Eulerian polynomial $B_n(t)$, which counts the elements of the hyperoctahedral group B_n according to the number of (type B) descents. Properties of $B_n(t)$, analogous to their type A counterparts, are also well studied (see, e.g., [7, Theorem 3.4] with $q = 1$); for example, the rational generating function (5) below.

Among several candidates for the major index for B_n that have been proposed (e.g., see [12, 13, 14, 26, 27]), only the flag major index introduced by Adin and Roichman in [3] and the negative major index introduced by Adin, Brenti and Roichman in [2] are Mahonian. In considering further extensions to B_n of known results for the symmetric group, Dominique Foata posed the following problem [2, Problem 1.1].

Problem 1.1 (Foata). Extend the ("Euler-Mahonian") bivariate distribution of descent number and major index to the hyperoctahedral group B_n.

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Present address of C.-O. Chow: P.O. Box 91100, Tsimshatsui Post Office, Hong Kong; email: cchow@alum.mit.edu.

In response to this problem, Adin, Brenti, and Roichman [2] introduced and studied three new signed permutation statistics: the negative descent number (ndes), the negative major index (nmaj), and the flag descent number (fdes), which, when restricted to the symmetric group $\mathfrak{S}_n$, reduce to the descent number, the major index, and twice the descent number, respectively. Moreover, these statistics, together with the flag major index (fmaj) introduced in [3], solve Problem 1.1. More precisely, they show that the major indices are Mahonian (see (6)), and that the pair of statistics $(\mathrm{stat}_1, \mathrm{stat}_2) = (\mathrm{ndes}, \mathrm{nmaj})$ or $(\mathrm{fdes}, \mathrm{fmaj})$ satisfies a generalized Carlitz identity (see (7)).

The solutions by Adin, Brenti and Roichman were motivated by a wish to generalize Carlitz's identity. Comparing the right-hand sides of (1) and (4), the latter seems to be a natural q-analogue of the former. In contrast, a comparison of the right-hand sides of (5) and (7) reveals that (5) does not enjoy such an q-analogue. With the type A case (eqs. (1) and (4)) as a guiding example, we feel that, from the Eulerian polynomial point of view, a natural q-generalization of (5) ought to have a q-version of the multiplicative factor $(2k+1)^n$ in the summands. It is the purpose of the present work to show that a q-generalization with this feature exists. More precisely, we show that the joint distribution of $(\mathrm{des}_B, \mathrm{fmaj})$, where des_B is the usual type B descent number, yields an alternate solution, with a natural q-analogue of (5) (see Theorem 3.7), to Problem 1.1.

The organization of this paper is as follows. In Section 2, we introduce some notations and known results for subsequent sections. In Section 3, we introduce and prove several basic properties of the q-Eulerian polynomial $B_n(t,q)$, which enumerates B_n with respect to the pair $(\mathrm{des}_B, \mathrm{fmaj})$. In Section 4, we show further properties of $B_n(t,q)$. In Section 5, we give some concluding remarks concerning the defining conditions for a pair of statistics on B_n to be type B Euler-Mahonian; lines of further research are also mentioned. Section 6 is an addendum which contains a multivariate combinatorial identity (29), obtained by modifying the results of Adin *et al.* [1], a specialization of which yields the generalized Carlitz identity (17), thus generalizing the main result of Section 3.

2. Notations and preliminaries

In this section we collect some definitions, notations, and results that will be used in subsequent sections of this paper. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers, $\mathbb{P} = \{1, 2, \ldots\}$ the set of all positive integers, $\mathbb{Q}$ the set of all rational numbers, and $\mathbb{R}$ the set of all real numbers.

Let S be a finite set. Denote by $\#S$ the cardinality of S. Let a, b be integers. Define $[a, b]$ to be the interval of integers $\{a, a+1, \ldots, b\}$ if $a \leqslant b$, and $\varnothing$ otherwise. We write $[a, b] = [b]$ if $a = 1$ and $b \geqslant 1$.

Let P be a statement. The *characteristic function* χ of P is defined as $\chi(P) = 1$ if P is true, and is 0 otherwise.

The sign of a real quantity x is conveniently expressed in terms of the *signum function* $\operatorname{sgn} x$, which is defined as $\operatorname{sgn} x = 1$ if $x > 0$, 0 if $x = 0$, and -1 if $x < 0$.

Let q and t be two independent indeterminates. We denote by $\mathbb{Q}[q]$ the ring of polynomials in q with coefficient in $\mathbb{Q}$ and by $\mathbb{Q}[[q]]$ the corresponding ring of formal power series. For $i \in \mathbb{N}$ we let, as customary, $[i]_q = 1 + q + q^2 + \cdots + q^{i-1}$ (so $[0]_q = 0$). For generic i which is not necessarily a nonnegative integer, we let $[i]_q = (1-q^i)/(1-q)$. For $P \in \mathbb{Q}[q,t]$, define (as in [4]) the type A Eulerian differential operator $\delta_{A,t}\colon \mathbb{Q}[q,t] \to \mathbb{Q}[q,t]$ by

$$\delta_{A,t}P(q,t) = \frac{P(q,qt)-P(q,t)}{qt-t}.$$

Denote by $\mathfrak{S}_n$ the symmetric group of degree n, which is a Coxeter group of type A with the set of generators $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i, i+1)$ is the simple transposition exchanging i and $i+1$, $i = 1, 2, \ldots, n-1$. We represent the element σ of $\mathfrak{S}_n$ in one-line notation as the word $\sigma_1 \cdots \sigma_n$, where $\sigma_i = \sigma(i)$. Given $\sigma = \sigma_1 \cdots \sigma_n, \tau = \tau_1 \cdots \tau_n \in \mathfrak{S}_n$, we let $\sigma\tau = (\sigma\tau)_1 \cdots (\sigma\tau)_n$, where $(\sigma\tau)_i = \sigma(\tau_i)$, $i = 1, 2, \ldots, n$.

An element $\pi = \pi_1 \cdots \pi_n$ of $\mathfrak{S}_n$ is said to have an A-descent at position $i \in [n-1]$ if $\pi_i > \pi_{i+1}$. Define the A-descent set of π by

$$\mathrm{Des}_A(\pi) = \{i \in [n-1] \colon \pi_i > \pi_{i+1}\},$$

the A-descent number by

$$\mathrm{des}_A(\pi) = \#\,\mathrm{Des}_A(\pi),$$

and the major index by

$$\mathrm{maj}_A(\pi) = \sum_{i \in \mathrm{Des}_A(\pi)} i.$$

The classical Eulerian polynomial of type A is defined by

$$\mathfrak{S}_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}_A(\sigma)} = \sum_{k=0}^{n-1} \mathfrak{S}_{n,k} t^k,$$

which has the rational generating function

$$\frac{\mathfrak{S}_n(t)}{(1-t)^{n+1}} = \sum_{k \geqslant 0} (k+1)^n t^k, \tag{1}$$

and the q-Eulerian polynomial of type A is defined by

$$\mathfrak{S}_n(t,q) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}_A(\sigma)} q^{\mathrm{maj}(\sigma)} = \sum_{k=0}^{n-1} \mathfrak{S}_{n,k}(q) t^k, \tag{2}$$

where $\mathfrak{S}_{n,k}(q) = \sum_\sigma q^{\mathrm{maj}(\sigma)}$ where the sum is over all $\sigma \in \mathfrak{S}_n$ with k A-descents. Carlitz [9] showed that $\mathfrak{S}_n(t,q)$ satisfies the recurrence relation

$$\mathfrak{S}_n(t,q) = (1 + qt[n-1]_q)\mathfrak{S}_{n-1}(t,q) + qt(1-t)\delta_{A,t}(\mathfrak{S}_{n-1}(t,q)), \tag{3}$$

and gave the following generating function.

$$\frac{\mathfrak{S}_n(t,q)}{\prod_{i=0}^n (1-tq^i)} = \sum_{k \geqslant 0} [k+1]_q^n t^k. \tag{4}$$

The latter two results have also been studied by Garsia [18] and Gessel [19]. Formula (4) is in fact a special case of a result of MacMahon [23, Volume 2, Chapter IV, p. 211, §462] that gives the generating function for permutations of a multiset by descents and major index.

Denote by B_n the hyperoctahedral group of rank n (the group of signed permutations of $[n]$), which is a Coxeter group of type B with the set of generators $\{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = (1, -1)$ is the sign change, and s_i, $i = 1, 2, \ldots, n-1$, are simple transpositions as in the case of $\mathfrak{S}_n$. Given $\sigma = \sigma_1 \cdots \sigma_n, \tau = \tau_1 \cdots \tau_n \in B_n$, we let $\sigma\tau = (\sigma\tau)_1 \cdots (\sigma\tau)_n$, where $(\sigma\tau)_i = \sigma(\tau_i) = (\operatorname{sgn} \tau_i)\sigma(|\tau_i|)$ for $i = 1, 2, \ldots, n$.

An element π of B_n is said to have a B-descent at position $i \in [0, n-1]$ if $\pi_i > \pi_{i+1}$, where $\pi_0 = 0$. Define the B-descent set of π by

$$\operatorname{Des}_B(\pi) = \{i \in [0, n-1] \colon \pi_i > \pi_{i+1}\},$$

and the B-descent number by

$$\operatorname{des}_B(\pi) = \# \operatorname{Des}_B(\pi).$$

The length function l_B on B_n (see [7]) is given by

$$l_B(\pi) = \operatorname{inv}(\pi) - \sum_{\pi_i < 0} \pi_i,$$

where $\operatorname{inv}(\pi) = \#\{(i,j) \in [n] \times [n] \colon i < j, \pi_i > \pi_j\}$ is the inversion number of π. We also call this statistic the type B inversion number, denoted inv_B. The *number of negative entries* $N(\pi)$ of π is defined as $N(\pi) = \#\{i \colon \pi_i < 0\}$. Following [2], we define the *flag major index* by

$$\operatorname{fmaj}(\pi) = 2 \operatorname{maj}_A(\pi) + N(\pi).$$

Now define the Eulerian polynomial $B_n(t)$ of type B by

$$B_n(t) = \sum_{\pi \in B_n} t^{\operatorname{des}_B(\pi)} = \sum_{k=0}^{n} B_{n,k} t^k,$$

where $B_{n,k}$ is the Eulerian number of type B counting the elements of B_n with k B-descents. The properties of $B_n(t)$ are well studied. (See Theorem 3.4 in [7] with $q = 1$, and Corollary 3.9 in the next section.) In particular, $B_n(t)$ has the following rational generating function

$$\frac{B_n(t)}{(1-t)^{n+1}} = \sum_{k \geqslant 0} (2k+1)^n t^k, \tag{5}$$

of which a natural q-analogue, which solves Problem 1.1, will be obtained in the next section.

Although the present work does not involve the negative statistics and the flag descent number for B_n directly, we include their definitions here for the sake of completeness. The *negative descent multiset* is defined by

$$\operatorname{NDes}(\pi) = \operatorname{Des}_A(\pi) \biguplus \{-\pi_i \colon \pi_i < 0\},$$

where $\uplus$ denotes disjoint union, the *negative descent number* by

$$\operatorname{ndes}(\pi) = \# \operatorname{NDes}(\pi),$$

the *negative major index* by

$$\operatorname{nmaj}(\pi) = \sum_{i \in \operatorname{NDes}(\pi)} i,$$

and the *flag descent number* by

$$\mathrm{fdes}(\pi) = 2\,\mathrm{des}_A(\pi) + \chi(\pi_1 < 0).$$

It is shown in [2, Corollary 4.6] that nmaj and fmaj are Mahonian, i.e.,

$$\sum_{\pi\in B_n} q^{l_B(\pi)} = \sum_{\pi\in B_n} q^{\mathrm{nmaj}(\pi)} = \sum_{\pi\in B_n} q^{\mathrm{fmaj}(\pi)}. \tag{6}$$

and that [2, Theorem 4.2 and Corollary 4.5]) a generalized Carlitz's identity holds:

$$\frac{\sum_{\pi\in B_n} t^{\mathrm{stat}_1(\pi)} q^{\mathrm{stat}_2(\pi)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})} = \sum_{k\geqslant 0}[k+1]_q^n t^k, \tag{7}$$

in $\mathbb{Z}[q][[t]]$, where $(\mathrm{stat}_1, \mathrm{stat}_2) = (\mathrm{ndes}, \mathrm{nmaj})$, or $(\mathrm{fdes}, \mathrm{fmaj})$. For $P \in \mathbb{Q}[q,t]$, we define the type B Eulerian differential operator $\delta_{B,t}\colon\ \mathbb{Q}[q,t] \to \mathbb{Q}[q,t]$ by

$$\delta_{B,t}P(q,t) = \frac{P(q,q^2t) - P(q,t)}{q^2t - t}.$$

It is easy to see that

$$\delta_{B,t}t^n = (1 + q^2 + q^4 + \cdots + q^{2(n-1)})t^{n-1} = [n]_{q^2}t^{n-1},$$

and that as $q \to 1$, $\delta_{B,t}t^n$ approaches nt^{n-1}, the usual derivative of t^n. The differential operator $\delta_{B,t}$ satisfies a product rule, namely,

$$\delta_{B,t}(A(q,t)B(q,t)) = \delta_{B,t}(A(q,t))B(q,q^2t) + A(q,t)\delta_{B,t}(B(q,t)), \tag{8}$$

for any $A(q,t), B(q,t) \in \mathbb{Q}[q,t]$. By a straightforward induction and (8) we have the following lemma, whose proof we shall omit.

LEMMA 2.1. *Let $A_1(q,t),\ldots,A_n(q,t) \in \mathbb{Q}[q,t]$. Then*

$$\delta_{B,t}(A_1(q,t)\cdots A_n(q,t)) = \sum_{i=1}^{n} A_1(q,t)\cdots A_{i-1}(q,t)\delta_{B,t}(A_i(q,t))A_{i+1}(q,q^2t)\cdots A_n(q,q^2t).$$

The q-binomial theorem [5, Theorem 2.1] provides the expansion

$$\sum_{k\geqslant 0}\begin{bmatrix} k+n \\ n \end{bmatrix}_q t^k = \frac{1}{(1-t)(1-tq)\cdots(1-tq^n)}, \tag{9}$$

which for $q = 1$ reduces to the binomial theorem

$$\sum_{k\geqslant 0}\binom{k+n}{n}t^k = \frac{1}{(1-t)^{n+1}}.$$

We shall investigate the zeros of certain polynomials in Section 4. Some relevant definitions and results, taken from [6], are as follows.

A sequence $\{a_0, a_1, \ldots, a_d\}$ of real numbers is called *log-concave* if $a_{i-1}a_{i+1} \leqslant a_i^2$ for $i = 1,\ldots,d-1$. It is *unimodal* if there exists an index $0 \leqslant j \leqslant d$ such that $a_i \leqslant a_{i+1}$ for $i = 0,\ldots,j-1$ and $a_i \geqslant a_{i+1}$ for $i = j,\ldots,d-1$. It has *no internal zeros* if there do not exist three indices $0 \leqslant i < j < k \leqslant d$ such that $a_i, a_k \neq 0$ but $a_j = 0$. It is *symmetric* if $a_i = a_{d-i}$ for $i = 0,\ldots,\lfloor d/2\rfloor$. It is a *Pólya frequency sequence of order* r (or a *PF$_r$ sequence*)

n	$B_n(t,q)$
1	$1+qt$
2	$1+(2q+2q^2+2q^3)t+q^4t^2$
3	$1+(3q+5q^2+7q^3+5q^4+3q^5)t$
	$+(3q^4+5q^5+7q^6+5q^7+3q^8)t^2+q^9t^3$
4	$1+(4q+9q^2+16q^3+18q^4+16q^5+9q^6+4q^7)t$
	$+(6q^4+16q^5+30q^6+40q^7+46q^8+40q^9+30q^{10}+16q^{11}+6q^{12})t^2$
	$+(4q^9+9q^{10}+16q^{11}+18q^{12}+16q^{13}+9q^{14}+4q^{15})t^3+q^{16}t^4$

TABLE 1. $B_n(t,q)$ for $n=1,2,3,4$.

if the determinant of any $r\times r$ submatrix of the matrix $M=(M_{ij})_{i,j\in\mathbb{N}}$ defined by $M_{ij}=a_{j-i}$ for all $i,j\in\mathbb{N}$ (where $a_k=0$ if $k<0$ or $k>d$) is nonnegative. It is a *Pólya frequency sequence of infinite order* (or a *PF sequence*) if it is a PF_r sequence for all $r\geqslant 1$.

It is clear that a positive sequence $\{a_i\}$ is PF_1, and a log-concave (which is also unimodal and internal-zero free) sequence $\{a_i\}$ is PF_2.

A polynomial $\sum_{i=0}^d a_ix^i$ is called *symmetric* if the sequence $\{a_0,a_1,\ldots,a_d\}$ is symmetric, and similarly for unimodal, log-concave, and with no internal zeros. If $p(x)$ is a symmetric unimodal polynomial, then its *center of symmetry* $C(p)=(\deg(p)+\mathrm{mult}(0,p))/2$, where $\mathrm{mult}(0,p)$ is the multiplicity of 0 as a zero of p. If we write $p(x)=x^np(x^{-1})$, then $C(p)=n/2$.

An important result concerning *PF* sequences and polynomials having only real zeros is the following [6, Theorem 2.2.2].

THEOREM 2.2. *Let $A(x)=\sum_{i=0}^d a_ix^i$ be a polynomial with nonnegative coefficients. Then $A(x)$ has only real zeros if and only if $\{a_0,\ldots,a_d\}$ is a PF sequence.*

3. BASIC PROPERTIES

In the sequel, a descent means a B-descent. Now define the q-Eulerian polynomial, which is central to this work, by

$$B_n(t,q)=\sum_{\pi\in B_n} t^{\mathrm{des}_B(\pi)}q^{\mathrm{fmaj}(\pi)}=\sum_{k=0}^n B_{n,k}(q)t^k, \tag{10}$$

where $B_{n,k}(q)=\sum_\sigma q^{\mathrm{fmaj}(\sigma)}$ summed over all elements σ of B_n with k descents. It is clear from the definition that $B_n(t,1)=B_n(t)$, so that $B_n(t,q)$ can be regarded as a q-analogue of $B_n(t)$. The first four polynomials $B_n(t,q)$ are given in Table 1, from which a symmetry of $B_{n,k}(q)$ is evident.

PROPOSITION 3.1. *The q-Eulerian number $B_{n,k}(q)$ satisfies $q^{n^2}B_{n,k}(q^{-1})=B_{n,n-k}(q)$, where $k=0,1,\ldots,n$.*

Proof. The map $B_n\to B_n$ sending $\pi=\pi_1\cdots\pi_n\in B_n$ to $\bar\pi=\bar\pi_1\cdots\bar\pi_n$, where $\bar\pi_i=-\pi_i$, is a bijection mapping a permutation π with k descents to a permutation $\bar\pi$ with $n-k$ descents. It is easy to see that $\mathrm{maj}(\bar\pi)=\frac{1}{2}n(n-1)-\mathrm{maj}(\pi)$ and $N(\bar\pi)=n-N(\pi)$, so that

$\operatorname{fmaj}(\bar{\pi}) = 2\operatorname{maj}(\bar{\pi}) + N(\bar{\pi}) = n^2 - \operatorname{fmaj}(\pi)$. Thus,

$$\begin{aligned} B_{n,k}(q) &= \sum_{\operatorname{des}_B(\pi)=k} q^{\operatorname{fmaj}(\pi)} \\ &= \sum_{\operatorname{des}_B(\bar{\pi})=n-k} q^{n^2-\operatorname{fmaj}(\bar{\pi})} \\ &= q^{n^2} B_{n,n-k}(q^{-1}), \end{aligned}$$

which is equivalent to $q^{n^2} B_{n,k}(q^{-1}) = B_{n,n-k}(q)$. □

Proposition 3.2. *We have*

$$B_{n,k}(q) = [2k+1]_q B_{n-1,k}(q) + q^{2k-1}[2n-2k+1]_q B_{n-1,k-1}(q), \quad 1 \leqslant k \leqslant n-1.$$

Proof. The above recurrence relation is obtained by an insertion process. Let $\sigma = \sigma_1 \cdots \sigma_{n-1} \in B_{n-1}$. Denote by $\sigma_{\pm i}$ the element $\sigma_1 \cdots \sigma_i(\pm n)\sigma_{i+1} \cdots \sigma_{n-1}$ of B_n obtained by inserting $\pm n$ at position i. The descent number of $\sigma_{\pm i}$ is related to that of σ in the following way:

(i) Inserting $\pm n$ in a descent position of σ, or appending n to the end of σ, yields an element of B_n with the same number of descents;
(ii) inserting $\pm n$ in a non-descent positions of σ, or appending $-n$ at the end of σ, yields an element of B_n with one more descent.

Next we examine the relation of $\operatorname{fmaj}(\sigma_{\pm i})$ to $\operatorname{fmaj}(\sigma)$.

In case (i), let $\operatorname{Des}_B(\sigma) = \{i_1 < i_2 < \cdots < i_k\}$ be the descent set of σ. Inserting n at position i_l gives $\operatorname{Des}_B(\sigma_{+i_l}) = \{i_1, \ldots, i_{l-1}, i_l + 1, \ldots, i_k + 1\}$ so that $\operatorname{maj}_A(\sigma_{+i_l}) = \operatorname{maj}_A(\sigma) + k - l + 1$. Since $N(\sigma_{+i_l}) = N(\sigma)$, it follows that $\operatorname{fmaj}(\sigma_{+i_l}) = \operatorname{fmaj}(\sigma) + 2(k-l+1)$.

Inserting $-n$ at position i_l gives $\operatorname{Des}_B(\sigma_{-i_l}) = \{i_1, \ldots, i_{l-1}, i_l, i_{l+1} + 1, \ldots, i_k + 1\}$ so that $\operatorname{maj}_A(\sigma_{-i_l}) = \operatorname{maj}_A(\sigma) + k - l$. Since $N(\sigma_{-i_l}) = N(\sigma) + 1$, it follows that $\operatorname{fmaj}(\sigma_{-i_l}) = \operatorname{fmaj}(\sigma) + 2(k-l) + 1$.

Appending n to the end of σ gives $\operatorname{Des}_B(\sigma_{+(n-1)}) = \{i_1, \ldots, i_k\} = \operatorname{Des}_B(\sigma)$ so that $\operatorname{maj}_A(\sigma_{+(n-1)}) = \operatorname{maj}_A(\sigma)$. Since $N(\sigma_{+(n-1)}) = N(\sigma)$, it follows that $\operatorname{fmaj}(\sigma_{+(n-1)}) = \operatorname{fmaj}(\sigma)$.

Taking all these into account, we have the following number of elements of B_n with k descents arising from case (i):

$$\sum_{\sigma} q^{\operatorname{fmaj}(\sigma)} \left\{ \sum_{l=1}^{k} (q^{2(k-l+1)} + q^{2(k-l)+1}) + 1 \right\} = [2k+1]_q B_{n-1,k}(q). \tag{11}$$

In case (ii), let $\operatorname{Des}_B(\sigma) = \{j_1 < j_2 < \cdots < j_{k-1}\}$ be the descent set of σ. Define $j_0 = -1$ and $j_k = n-1$.

The set of non-descent positions is

$$\bigcup_{l=0}^{k-1} \{j_l + 1, \ldots, j_{l+1} - 1\}.$$

If $j_1 = 0$, the first member of the above union is empty; the case $j_{k-1} = n-2$ is similar.

For $l = 0, 1, \ldots, k-1$, insertion of n at position $m \in \{j_l+1, \ldots, j_{l+1}-1\}$ gives $\mathrm{Des}_B(\sigma_{+m}) = \{j_1, \ldots, j_l, m+1, j_{l+1}+1, \ldots, j_{k-1}+1\}$ so that $\mathrm{maj}_A(\sigma_{+m}) = \mathrm{maj}_A(\sigma) + k + m - l$. Since $N(\sigma_{+m}) = N(\sigma)$, we have $\mathrm{fmaj}(\sigma_{+m}) = \mathrm{fmaj}(\sigma) + 2(k+m-l)$.

Insertion of $-n$ at position $m \in \{j_l+1, \ldots, j_{l+1}-1\}$ gives $\mathrm{Des}_B(\sigma_{-m}) = \{j_1, \ldots, j_l, m, j_{l+1}+1, \ldots, j_{k-1}+1\}$ so that $\mathrm{maj}_A(\sigma_{-m}) = \mathrm{maj}_A(\sigma) + k + m - l - 1$. Since $N(\sigma_{-m}) = N(\sigma) + 1$, we have $\mathrm{fmaj}(\sigma_{-m}) = \mathrm{fmaj}(\sigma) + 2(k+m-l) - 1$.

Appending $-n$ to the end of σ gives $\mathrm{Des}_B(\sigma_{-(n-1)}) = \{j_1, \ldots, j_{k-1}, n-1\}$ so that $\mathrm{maj}_A(\sigma_{-(n-1)}) = \mathrm{maj}_A(\sigma)+n-1$. Since $N(\sigma_{-(n-1)}) = N(\sigma)+1$, we have $\mathrm{fmaj}(\sigma_{-(n-1)}) = \mathrm{fmaj}(\sigma)+2(n-1)+1$.

Taking all these into account, we have the following number of elements of B_n with k descents arising from (ii):

$$\begin{aligned}(12)\qquad &\sum_{\sigma} q^{\mathrm{fmaj}(\sigma)} \left\{ \sum_{l=0}^{k-1} \sum_{m=j_l+1}^{j_{l+1}-1} \left(q^{2(k+m-l)} + q^{2(k+m-l)-1}\right) + q^{2(n-1)+1} \right\} \\ &= q^{2k-1}[2n-2k+1]_q B_{n-1,k-1}(q).\end{aligned}$$

Summing (11) and (12) yields the proposition. □

Lemma 3.3. *We have*

$$[2r+1]_q [n]_{q^2} = [2k+1]_q [r+n-k]_{q^2} + q^{2k+1}[2n-2k-1]_q [r-k]_{q^2}.$$

Proof. It is an easy exercise to verify that

$$(q^{2r+1}-1)(q^{2n}-1) = (q^{2k+1}-1)(q^{2r+2n-2k}-1) + q^{2k+1}(q^{2n-2k-1}-1)(q^{2r-2k}-1).$$

The lemma now follows upon dividing both sides by $(q-1)(q^2-1)$. □

The proof of the next lemma is a straightforward calculation which we omit.

Lemma 3.4. *We have*

$$\begin{bmatrix} n \\ k+1 \end{bmatrix}_q = \frac{[n-k]_q}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q}{[k]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Proposition 3.5. *The q-Eulerian number $B_{n,k}(q)$ satisfies the q-Worpitzky identity*

$$[2r+1]_q^n = \sum_{k=0}^{n} B_{n,k}(q) \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2}.$$

Proof. The proof is by induction on n. Since $B_{1,0}(q) = 1$ and $B_{1,1}(q) = q$,

$$\begin{aligned} B_{1,0}(q) \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_{q^2} + B_{1,1}(q) \begin{bmatrix} r \\ 1 \end{bmatrix}_{q^2} &= \frac{(1-q^{2(r+1)}) + q(1-q^{2r})}{1-q^2} \\ &= [2r+1]_q, \end{aligned}$$

from which the case $n = 1$ is true. Now assume that the identity holds for $B_{n-1,k}(q)$, i.e.,

$$[2r+1]_q^{n-1} = \sum_{k=0}^{n-1} B_{n-1,k}(q) \begin{bmatrix} r+n-1-k \\ n-1 \end{bmatrix}_{q^2}.$$

Multiplying both sides by $[2r+1]_q$, we have

$$\begin{aligned}
[2r+1]_q^n &= \sum_{k=0}^{n-1} B_{n-1,k}(q) \begin{bmatrix} r+n-1-k \\ n-1 \end{bmatrix}_{q^2} [2r+1]_q \\
&= \sum_{k=0}^{n-1} B_{n-1,k}(q) \begin{bmatrix} r+n-1-k \\ n-1 \end{bmatrix}_{q^2} \frac{[2k+1]_q[r+n-k]_{q^2} + q^{2k+1}[2n-2k-1]_q[r-k]_{q^2}}{[n]_{q^2}} \\
&= \sum_{k=0}^{n-1} B_{n-1,k}(q) \left\{ [2k+1]_q \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2} + q^{2k+1}[2n-2k-1]_q \begin{bmatrix} r+n-1-k \\ n \end{bmatrix}_{q^2} \right\} \\
&= \sum_{k=0}^{n-1} B_{n-1,k}(q)[2k+1]_q \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2} + \sum_{k=1}^{n} B_{n-1,k-1}(q) q^{2k-1}[2n-2k+1]_q \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2} \\
&= \sum_{k=0}^{n} B_{n,k}(q) \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2},
\end{aligned}$$

where the second equality follows from Lemma 3.3, the third from Lemma 3.4, and the last from Proposition 3.2. This completes the induction. □

Proposition 3.5 actually holds for any r, not necessarily a nonnegative integer, thanks to the definition $[i]_q = (1-q^i)/(1-q)$ for generic i.

For a fixed n, it is more convenient to compute all the $B_{n,k}(q)$ in one stroke by computing $B_n(t,q)$. Towards this end, a recurrence relation for $B_n(t,q)$ is useful.

Theorem 3.6. *We have*

$$B_n(t,q) = (qt[2n-1]_q + 1)B_{n-1}(t,q) + q(1+q)t(1-t)\delta_{B,t}(B_{n-1}(t,q)), \quad n \geqslant 1.$$

Proof. Multiplying Proposition 3.2 by t^k and summing on k from 1 to n yields

$$\begin{aligned}
\sum_{k=1}^{n} B_{n,k}(q)t^k &= \sum_{k=1}^{n} \{[2k+1]_q B_{n-1,k}(q) + q^{2k-1}[2n-2k+1]_q B_{n-1,k-1}(q)\} t^k \\
&= \sum_{k=1}^{n} [2k+1]_q B_{n-1,k}(q)t^k + qt \sum_{k=0}^{n-1} q^{2k}[2n-2k-1]_q B_{n-1,k}(q)t^k.
\end{aligned} \tag{13}$$

It is easy to see that

$$[2k+1]_q = 1 + q(1+q)[k]_{q^2} \tag{14}$$

and that

$$q^{2k}[2n-2k-1]_q = [2n-1]_q - (1+q)[k]_{q^2}. \tag{15}$$

Since $\delta_{B,t} t^k = [k]_{q^2} t^{k-1}$, we have

$$\delta_{B,t} B_{n-1}(t,q) = \sum_{k=0}^{n-1} B_{n-1,k}(q)[k]_{q^2} t^{k-1}. \tag{16}$$

The theorem now follows from (13), (14), (15), (16), and the fact that $B_{n,0}(q) = 1$ for all $n \geqslant 0$. □

We now prove the main theorem of this section.

THEOREM 3.7. *We have*

$$\frac{B_n(t,q)}{\prod_{i=0}^n (1-tq^{2i})} = \sum_{k\geqslant 0} [2k+1]_q^n t^k. \tag{17}$$

Proof. From Proposition 3.5, we have

$$\begin{aligned}
\sum_{r\geqslant 0} [2r+1]_q^n t^r &= \sum_{r\geqslant 0}\sum_{k=0}^n B_{n,k}(q) \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2} t^r \\
&= \sum_{k=0}^n B_{n,k}(q) t^k \sum_{r\geqslant k} \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2} t^{r-k} \\
&= B_n(t,q) \sum_{s\geqslant 0} \begin{bmatrix} s+n \\ n \end{bmatrix}_{q^2} t^s \\
&= \frac{B_n(t,q)}{(1-t)(1-tq^2)\cdots(1-tq^{2n})},
\end{aligned}$$

where the last equality follows from (9). □

Theorem 3.7 shows that the pair of statistics $(\mathrm{des}_B, \mathrm{fmaj})$ constitutes a solution to Problem 1.1, from the Eulerian polynomial point of view. Note that Theorem 3.7 is *false* if fmaj is replaced by nmaj.

THEOREM 3.8. *We have*

$$\sum_{n\geqslant 0} \frac{B_n(t,q)}{\prod_{i=0}^n (1-tq^{2i})} \frac{x^n}{n!} = \sum_{k\geqslant 0} t^k \exp([2k+1]_q x).$$

Proof. Multiplying Theorem 3.7 by $x^n/n!$ followed by summing over n, we have

$$\begin{aligned}
\sum_{n\geqslant 0} \frac{B_n(t,q)}{\prod_{i=0}^n (1-tq^{2i})} \frac{x^n}{n!} &= \sum_{n\geqslant 0} \frac{x^n}{n!} \sum_{k\geqslant 0} [2k+1]_q^n t^k \\
&= \sum_{k\geqslant 0} t^k \sum_{n\geqslant 0} \frac{([2k+1]_q x)^n}{n!} \\
&= \sum_{k\geqslant 0} t^k \exp([2k+1]_q x).
\end{aligned}$$

□

We have, by specializing $q = 1$ in Proposition 3.1, 3.2, 3.5, Theorem 3.6, 3.7, 3.8 the following properties of Eulerian numbers and Eulerian polynomials of type B. (*cf.* Theorem 3.4 in [7] with $q = 1$.)

COROLLARY 3.9. *We have*

(i) $B_{n,k} = B_{n,n-k}$, $k = 0, 1, \ldots, n$;
(ii) $B_{n,k} = (2k+1)B_{n-1,k} + (2n-2k+1)B_{n-1,k-1}$, $k = 0, 1, \ldots, n$;
(iii) $(2r+1)^n = \sum_{k=0}^{n} B_{n,k}\binom{r+n-k}{n}$;
(iv) $B_n(t) = ((2n-1)t+1)B_{n-1}(t) + 2t(1-t)B'_{n-1}(t)$;
(v) $\dfrac{B_n(t)}{(1-t)^{n+1}} = \sum_{k\geqslant 0}(2k+1)^n t^k$;
(vi) $\sum_{n\geqslant 0} B_n(t)\dfrac{x^n}{n!} = \dfrac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}$.

Proof. Parts (i)–(v) follow immediately from Proposition 3.1, 3.2, 3.5, Theorem 3.6, 3.7, respectively. With $q = 1$, Theorem 3.8 becomes

$$\sum_{n\geqslant 0} \frac{B_n(t)}{(1-t)^{n+1}}\frac{x^n}{n!} = \sum_{k\geqslant 0} t^k e^{(2k+1)x} = \frac{e^x}{1-te^{2x}}.$$

Replacing x by $x(1-t)$, followed by multiplication by $(1-t)$, gives (vi). □

4. Further properties

In this section, we study further properties, including expansion, symmetry, and reality of zeros, of $B_n(t, q)$. The first expansion property of $B_n(t, q)$ is a Frobenius-like formula (*cf.* [11, Theorem E, p. 244], [7], and [18]), as follows.

Proposition 4.1. *We have*

$$\frac{B_n(t, q)}{\prod_{i=0}^{n}(1-tq^{2i})} = \sum_{k=0}^{n} \frac{[k]_{q^2}!S_{n,k}(q)t^k}{\prod_{i=0}^{k}(1-tq^{2i})},$$

where the polynomial $S_{n,k}(q)$ in q satisfies the recurrence relation

$$S_{n,k}(q) = [2k+1]_q S_{n-1,k}(q) + q^{2k-1}(1+q)S_{n-1,k-1}(q). \tag{18}$$

Proof. The polynomials

$$\theta_k(t) = t^k(1-tq^{2k+2})\cdots(1-tq^{2n}), \quad k = 0, 1, \ldots, n,$$

are linearly independent and of degree n, so that $B_n(t, q) = \sum_{k=0}^{n} C_{n,k}(q)\theta_k(t)$ for some polynomials $C_{n,k}(q)$ in q, $k = 0, 1, \ldots, n$. Now define $S_{n,k}(q) = C_{n,k}(q)/[k]_{q^2}!$, so that

$$\frac{B_n(t, q)}{(1-t)(1-tq^2)\cdots(1-tq^{2n})} = \sum_{k=0}^{n} \frac{[k]_{q^2}!S_{n,k}(q)t^k}{(1-t)(1-tq^2)\cdots(1-tq^{2k})}.$$

We shall show that $S_{n,k}(q)$ satisfies the recurrence relation (18). We first note that

$$\delta_{B,t}\frac{1}{(1-t)\cdots(1-tq^{2n})} = \frac{[2n+2]_q}{(1+q)(1-t)\cdots(1-tq^{2n+2})}.$$

We also have, by (8), that

$$\delta_{B,t}\frac{B_n(t,q)}{(1-t)\cdots(1-tq^{2n})} = \frac{\delta_{B,t}B_n(t,q)}{(1-tq^2)\cdots(1-tq^{2n+2})} + \frac{[2n+2]_qB_n(t,q)}{(1+q)(1-t)\cdots(1-tq^{2n+2})}$$

$$= \frac{[2n+2]_qB_n(t,q)+(1+q)(1-t)\delta_{B,t}B_n(t,q)}{(1+q)(1-t)\cdots(1-tq^{2n+2})}$$

so that

$$q(1+q)t\delta_{B,t}\frac{B_n(t,q)}{(1-t)\cdots(1-tq^{2n})} = \frac{qt[2n+2]_qB_n(t,q)+q(1+q)t(1-t)\delta_{B,t}B_n(t,q)}{(1-t)\cdots(1-q^{2n+2})}.$$

It is easy to see that

$$1+qt[2n+1]_q = (1-tq^{2n+2})+qt[2n+2]_q.$$

This, together with Theorem 3.6, implies that

$$\frac{B_{n+1}(t,q)}{(1-t)\cdots(1-tq^{2n+2})} = \frac{B_n(t,q)}{(1-t)\cdots(1-tq^{2n})} + q(1+q)t\delta_{B,t}\frac{B_n(t,q)}{(1-t)\cdots(1-tq^{2n})}.$$

The left hand side is

$$\frac{B_{n+1}(t,q)}{(1-t)\cdots(1-tq^{2n+2})} = \sum_{k=0}^{n+1}\frac{[k]_{q^2}!S_{n+1,k}(q)t^k}{(1-t)\cdots(1-tq^{2k})}.$$

The right-hand side is

$$\begin{aligned}
&\frac{B_n(t,q)}{(1-t)\cdots(1-tq^{2n})} + q(1+q)t\delta_{B,t}\frac{B_n(t,q)}{(1-t)\cdots(1-tq^{2n})}\\
&\quad= \sum_{k=0}^{n}\frac{[k]_{q^2}!S_{n,k}(q)t^k}{(1-t)\cdots(1-tq^{2k})} + q(1+q)t\delta_{B,t}\sum_{k=0}^{n}\frac{[k]_{q^2}!S_{n,k}(q)t^k}{(1-t)\cdots(1-tq^{2k})}\\
&\quad= \sum_{k=0}^{n}\frac{[k]_{q^2}!S_{n,k}(q)t^k}{(1-t)\cdots(1-tq^{2k})}\\
&\qquad+ q(1+q)t\left\{\sum_{k=0}^{n}\frac{[k]_{q^2}![2k+2]_qq^{2k}S_{n,k}(q)t^k}{(1+q)(1-t)\cdots(1-tq^{2k+2})} + \sum_{k=0}^{n}\frac{[k]_{q^2}![k]_{q^2}S_{n,k}(q)t^{k-1}}{(1-t)\cdots(1-tq^{2k})}\right\}\\
&\quad= \sum_{k=0}^{n}\frac{[k]_{q^2}!S_{n,k}(q)t^k}{(1-t)\cdots(1-tq^{2k})}\\
&\qquad+ \sum_{k=0}^{n}\frac{[k+1]_{q^2}!q^{2k+1}(1+q)S_{n,k}(q)t^{k+1}}{(1-t)\cdots(1-tq^{2k+2})} + \sum_{k=0}^{n}\frac{[k]_{q^2}!q(1+q)S_{n,k}(q)t^k}{(1-t)\cdots(1-tq^{2k})}\\
&\quad= \sum_{k=0}^{n}\frac{[k]_{q^2}!(1+q(1+q)[k]_{q^2})S_{n,k}(q)t^k}{(1-t)\cdots(1-tq^{2k})} + \sum_{k=1}^{n+1}\frac{[k]_{q^2}!q^{2k-1}(1+q)S_{n,k-1}(q)t^k}{(1-t)\cdots(1-tq^{2k})}.
\end{aligned}$$

Equating coefficients of $t^k/(1-t)\cdots(1-tq^{2k})$, we get

$$\begin{aligned}S_{n+1,k}(q) &= (1+q(1+q)[k]_{q^2})S_{n,k}(q) + q^{2k-1}(1+q)S_{n,k-1}(q)\\ &= [2k+1]_q S_{n,k}(q) + q^{2k-1}(1+q)S_{n,k-1}(q).\end{aligned}$$

Since $S_{n,0}(q)=1$ for $n \geqslant 0$, the above recurrence relation implies that $S_{n,k}(q)$ is a polynomial in q. This finishes the proof. □

The polynomials $S_{n,k}(q)$ have a combinatorial characterization given in Proposition 4.2 below. We need some notations from [7] and [18]. Let $S \subset \mathbb{P}$. A *signed partition* of S is a collection $\pi = (B_1, \ldots, B_k)$ of subsets of $-S \cup S$ with $\min|B_1| \leqslant \cdots \leqslant \min|B_k|$ and such that $\{B_1, \ldots, B_k, -B_1, \ldots, -B_k\}$ is a partition of $-S \cup S$, where $-S = \{-s\colon s \in S\}$ and $|S| = \{|s|\colon s \in S\}$.

We call $B_1, \ldots, B_k$ the *blocks* of π and say that π has k blocks. We also let

$$P(\pi) = \#\left\{x \in \bigcup_{i=1}^{k} B_i\colon x > 0\right\}.$$

A *partial signed partition* of S is a signed partition of some subset of S. Denote by $B\Pi_{\subseteq}(S,k)$ the set of all partial signed partitions of S having k blocks, and let

$$S_B(n,k,q) = \sum_{\pi \in B\Pi_{\subseteq}([n],k)} q^{m(\pi)},$$

where

$$m(\pi) = \left[2\sum_{i=1}^{k}(i-1)\sum_{\nu=1}^{n}\chi(\nu \in -B_i \cup B_i)\right] + n - P(\pi) + 1.$$

PROPOSITION 4.2. *We have $S_{n,k}(q) = S_B(n,k,q)$.*

Proof. It suffices to show that $S_B(n,k,q)$ satisfies (18), and the initial condition. The case $n=0$ is trivial. So, suppose that $n>0$ and $\pi = (B_1, \ldots, B_k) \in B\Pi_{\subseteq}([n],k)$. If $\{n\}$ (resp., $\{-n\}$) is a block of π, then $\{n\} = B_k$ (resp., $\{-n\} = B_k$) and removing it from π yields a partial signed partition τ of $[n-1]$ into $k-1$ blocks, with $P(\tau) = P(\pi) - 1$ (resp., $P(\tau) = P(\pi)$) so that $m(\pi) = m(\tau) + 2k - 1$ (resp., $m(\pi) = m(\tau) + 2k$).

If n is an element of B_j for some $j \in [k]$, then removing it from B_j yields a partial signed partition τ' of $[n-1]$ into k blocks with $P(\tau') = P(\pi) - 1$ so that $m(\pi) = m(\tau') + 2j - 1$. Similarly, If $-n$ is an element of B_j for some $j \in [k]$, then removing it from B_j yields a partial signed partition τ' of $[n-1]$ into k blocks with $P(\tau') = P(\pi)$ so that $m(\pi) = m(\tau') + 2j$. If neither n nor $-n$ is in any block of π, then $\pi \in B\Pi_{\subseteq}([n-1],k)$.

Thus,

$$\begin{aligned}S_B([n],k,q) &= q^{2k-1}(1+q)\sum_{\tau \in B\Pi_{\subseteq}([n-1],k-1)} q^{m(\tau)} + (1+q)\sum_{j=1}^{k} q^{2j-1}\sum_{\tau' \in B\Pi_{\subseteq}([n-1],k)} q^{m(\tau')}\\ &\quad + \sum_{\pi \in B\Pi_{\subseteq}([n-1],k)} q^{m(\pi)}\\ &= [2k+1]_q S_B([n-1],k,q) + q^{2k-1}(1+q)S_B([n-1],k-1,q).\end{aligned}$$

This finishes the proof. □

The polynomials $S_{n,k}(q)$ are type B q-Stirling numbers. For details of other q- (and p,q-) Stirling numbers, see [15, 20, 25, 31].

When $q=1$, the recurrence relation (18) becomes

$$S_{n,k}(1) = 2S_{n-1,k-1}(1) + (2k+1)S_{n-1,k}(1).$$

It is an easy exercise to check that

$$S_{n,k}(1) = \sum_{m=k}^{n} \binom{n}{m} 2^m S(m,k), \tag{19}$$

where $S(m,k)$ is the usual Stirling number of the second kind, which satisfies

$$\sum_{m\geqslant k} S(m,k)\frac{x^m}{m!} = \frac{(e^x-1)^k}{k!}. \tag{20}$$

For other properties of $S(m,k)$, see [29, p. 34]. With (19) and (20), the exponential generating function for $S_{n,k}(1)$ is readily computed:

$$\sum_{n\geqslant k} S_{n,k}(1)\frac{x^n}{n!} = \frac{(e^{2x}-1)^k e^x}{k!}.$$

Lemma 4.3. *For $n \geqslant 1$, we have*

$$\sum_{\sigma\in B_n} t^{\mathrm{des}_A(\sigma)} q^{\mathrm{fmaj}(\sigma)} = (1+q)^n \mathfrak{S}_n(t,q^2),$$

where $\mathfrak{S}_n(t,q)$ is defined by (2).

Proof. Let $T = \{\sigma \in B_n\colon \mathrm{des}_A(\sigma) = 0\}$. Any $\sigma = \sigma_1\cdots\sigma_n \in T$ satisfies $\sigma_1 < \cdots < \sigma_n$ so that $\mathrm{fmaj}(\sigma) = N(\sigma)$, and that

$$\sum_{\sigma\in T} q^{\mathrm{fmaj}(\sigma)} = \sum_{\sigma\in T} q^{N(\sigma)} = (1+q)^n.$$

Let $u = u_1\cdots u_n \in \mathfrak{S}_n$. Note that $u_i > 0$ for all i.

We have $\mathrm{Des}_A(\sigma u) = \mathrm{Des}_A(u)$ because

$$i \in \mathrm{Des}_A(\sigma u) \iff (\sigma u)_i = \sigma(u_i) > \sigma(u_{i+1}) = (\sigma u)_{i+1} \iff i \in \mathrm{Des}_A(u).$$

It is immediate that $\mathrm{des}_A(\sigma u) = \mathrm{des}_A(u)$ and that $\mathrm{maj}_A(\sigma u) = \mathrm{maj}_A(u)$.

We also have $N(\sigma u) = N(\sigma)$ because

$$(\sigma u)_i < 0 \iff \sigma(u_i) < 0.$$

Thus, $\mathrm{fmaj}(\sigma u) = 2\,\mathrm{maj}_A(\sigma u) + N(\sigma u) = 2\,\mathrm{maj}_A(u) + N(\sigma)$.

By virtue of the well-known decomposition

$$B_n = \biguplus_{u\in\mathfrak{S}_n} \{\sigma u\colon \sigma \in T\},$$

where $\uplus$ denotes disjoint union, it follows that

$$\begin{aligned}\sum_{\sigma\in B_n} t^{\mathrm{des}_A(\sigma)} q^{\mathrm{fmaj}(\sigma)} &= \sum_{u\in\mathfrak{S}_n}\sum_{\sigma\in T} t^{\mathrm{des}_A(\sigma u)} q^{\mathrm{fmaj}(\sigma u)}\\ &= \sum_{u\in\mathfrak{S}_n}\sum_{\sigma\in T} t^{\mathrm{des}_A(u)} q^{2\,\mathrm{maj}_A(u)+N(\sigma)}\\ &= \sum_{\sigma\in T} q^{N(\sigma)} \sum_{u\in\mathfrak{S}_n} t^{\mathrm{des}_A(u)} q^{2\,\mathrm{maj}_A(u)}\\ &= (1+q)^n \mathfrak{S}_n(t,q^2),\end{aligned}$$

and the lemma follows. □

The proof just given is similar to the proof of [2, Theorem 5.2].

The second expansion property of $B_n(t,q)$ is a convolution-type recurrence involving $\mathfrak{S}_n(t,q)$. (*Cf.* [16, p. 70] and [7, Theorem 3.6].)

Theorem 4.4. *We have, for $n \geqslant 1$,*

$$B_n(t,q) = \sum_{i=1}^{n} \binom{n-1}{i-1} q^{2i-1}(1+q)^{n-i+1} t B_{i-1}(t,q)\mathfrak{S}_{n-i}(tq^{2i},q^2) + (1-tq^{2n})B_{n-1}(t,q).$$

Proof. Let $\sigma = \sigma_1\cdots\sigma_{n-1} \in B_{n-1}$. Denote by $\sigma_{\pm i}$ the element $\sigma_1\cdots\sigma_{i-1}(\pm n)\sigma_i\cdots\sigma_n$ of B_n obtained by inserting $\pm n$ at the ith position, where $i = 1,2,\ldots,n$.

It is not hard to see that

$$\mathrm{des}_B(\sigma_{\pm i}) = \mathrm{des}_B(\sigma_1\cdots\sigma_{i-1}) + \mathrm{des}_A(\sigma_i\cdots\sigma_{n-1}) + 1, \tag{21}$$

and that $\mathrm{fmaj}(\sigma_{+n}) = \mathrm{fmaj}(\sigma)$ and $\mathrm{fmaj}(\sigma_{-n}) = \mathrm{fmaj}(\sigma) + 2n - 1$. Therefore,

$$\begin{aligned}B_n(t,q) &= \sum_{\sigma\in B_{n-1}} \big(t^{\mathrm{des}_B(\sigma_{+n})} q^{\mathrm{fmaj}(\sigma_{+n})} + t^{\mathrm{des}_B(\sigma_{-n})} q^{\mathrm{fmaj}(\sigma_{-n})}\big)\\ &\quad + \sum_{i=1}^{n-1}\sum_{\sigma\in B_{n-1}} \big(t^{\mathrm{des}_B(\sigma_{+i})} q^{\mathrm{fmaj}(\sigma_{+i})} + t^{\mathrm{des}_B(\sigma_{-i})} q^{\mathrm{fmaj}(\sigma_{-i})}\big)\\ &= (1+tq^{2n-1})B_{n-1}(t,q) + \sum_{i=1}^{n-1}\sum_{\sigma\in B_{n-1}} \big(t^{\mathrm{des}_B(\sigma_{+i})} q^{\mathrm{fmaj}(\sigma_{+i})} + t^{\mathrm{des}_B(\sigma_{-i})} q^{\mathrm{fmaj}(\sigma_{-i})}\big)\end{aligned} \tag{22}$$

It is easy to see that

$$\mathrm{maj}_A(\sigma_{+i}) = \mathrm{maj}_A(\sigma_1\cdots\sigma_{i-1}) + i + i\,\mathrm{des}_A(\sigma_i\cdots\sigma_{n-1}) + \mathrm{maj}_A(\sigma_i\cdots\sigma_{n-1})$$

and that $N(\sigma_{+i}) = N(\sigma_1\cdots\sigma_{i-1}) + N(\sigma_i\cdots\sigma_{n-1})$ so that

$$\mathrm{fmaj}(\sigma_{+i}) = \mathrm{fmaj}(\sigma_1\cdots\sigma_{i-1}) + \mathrm{fmaj}(\sigma_i\cdots\sigma_{n-1}) + 2i + 2i\,\mathrm{des}_A(\sigma_i\cdots\sigma_{n-1}).$$

This, together with (21), implies that

$$
\begin{aligned}
&\sum_{i=1}^{n-1} \sum_{\sigma\in B_{n-1}} t^{\mathrm{des}_B(\sigma_{+i})} q^{\mathrm{fmaj}(\sigma_{+i})} \\
&\quad = \sum_{i=1}^{n-1} \binom{n-1}{i-1} \sum_{\sigma\in B_{i-1}} \sum_{\tau\in B_{n-i}} t^{\mathrm{des}_B(\sigma)+\mathrm{des}_A(\tau)+1} q^{\mathrm{fmaj}(\sigma)+\mathrm{fmaj}(\tau)+2i+2i\,\mathrm{des}_A(\tau)} \\
&\quad = \sum_{i=1}^{n-1} \binom{n-1}{i-1} q^{2i} t \sum_{\sigma\in B_{i-1}} t^{\mathrm{des}_B(\sigma)} q^{\mathrm{fmaj}(\sigma)} \sum_{\tau\in B_{n-i}} (tq^{2i})^{\mathrm{des}_A(\tau)} q^{\mathrm{fmaj}(\tau)} \\
&\quad = \sum_{i=1}^{n-1} \binom{n-1}{i-1} q^{2i} t B_{i-1}(t,q)(1+q)^{n-i} \mathfrak{S}_{n-i}(tq^{2i},q^2),
\end{aligned}
\tag{23}
$$

where the last equality follows from Lemma 4.3. Similarly, we have

$$
\mathrm{maj}_A(\sigma_{-i}) = \mathrm{maj}_A(\sigma_1\cdots\sigma_{i-1}) + i - 1 + i\,\mathrm{des}_A(\sigma_i\cdots\sigma_{n-1}) + \mathrm{maj}_A(\sigma_i\cdots\sigma_{n-1})
$$

and

$$
N(\sigma_{-i}) = N(\sigma_1\cdots\sigma_{i-1}) + N(\sigma_i\cdots\sigma_{n-1}) + 1
$$

so that

$$
\mathrm{fmaj}(\sigma_{-i}) = \mathrm{fmaj}(\sigma_1\cdots\sigma_{i-1}) + \mathrm{fmaj}(\sigma_i\cdots\sigma_{n-1}) + 2i - 2 + 2i\,\mathrm{des}_A(\sigma_i\cdots\sigma_{n-1}).
$$

This, together with (21), imply that

$$
\sum_{i=1}^{n-1} \sum_{\sigma\in B_{n-1}} t^{\mathrm{des}_B(\sigma_{-i})} q^{\mathrm{fmaj}(\sigma_{-i})} = \sum_{i=1}^{n-1} \binom{n-1}{i-1} q^{2i-1} t B_{i-1}(t,q)(1+q)^{n-i} \mathfrak{S}_{n-i}(tq^{2i},q^2).
\tag{24}
$$

Now combining (23) and (24) with (22), we have

$$
\begin{aligned}
B_n(t,q) &= (1+tq^{2n-1})B_{n-1}(t,q) + \sum_{i=1}^{n-1} \binom{n-1}{i-1} q^{2i-1}(1+q)^{n-i+1} t B_{i-1}(t,q) \mathfrak{S}_{n-i}(tq^{2i},q^2) \\
&= (1-tq^{2n})B_{n-1}(t,q) + \sum_{i=1}^{n} \binom{n-1}{i-1} q^{2i-1}(1+q)^{n-i+1} t B_{i-1}(t,q) \mathfrak{S}_{n-i}(tq^{2i},q^2),
\end{aligned}
$$

and the theorem follows. □

The next result is a symmetry property of $B_n(t,q)$.

PROPOSITION 4.5. *The q-Eulerian polynomial* $B_n(t,q)$ *satisfies* $q^{n^2} B_n(t,q^{-1}) = t^n B_n(t^{-1},q)$.

Proof. By Proposition 3.1, we have that

$$\begin{aligned} B_n(t, q^{-1}) &= \sum_{k=0}^{n} B_{n,k}(q^{-1}) t^k \\ &= \sum_{k=0}^{n} q^{-n^2} B_{n,n-k}(q) t^k \\ &= q^{-n^2} \sum_{k=0}^{n} B_{n,k}(q) t^{n-k} \\ &= q^{-n^2} t^n B_n(t^{-1}, q), \end{aligned}$$

from which the proposition follows. □

Proposition 4.5 implies that if $q, t \neq 0$, then $B_n(t, q^{-1}) = 0$ if and only if $B_n(t^{-1}, q) = 0$.

We now look at the nature of zeros of $B_n(t, q)$. Suppose that for a fixed $n > 0$ and $1 \neq q > 0$, $B_n(t, q)$ is simply real-rooted, and let $t_{n,1}(q) < t_{n,2}(q) < \cdots < t_{n,n}(q) < 0$ be these real zeros. Setting $t = t_{n,i}(q)$ in Theorem 3.6 yields that

$$B_{n+1}(t_{n,i}(q), q) = q(1+q) t_{n,i}(q)(1 - t_{n,i}(q)) \delta_{B,t}(B_n(t, q))\Big|_{t=t_{n,i}(q)}.$$

On the other hand, it is easy to see that

$$\delta_{B,t}(B_n(t, q))\Big|_{t=t_{n,i}(q)} = \frac{B_n(q^2 t_{n,i}(q), q)}{(q^2 - 1) t_{n,i}(q)}. \tag{25}$$

It is clear that if $q \gtrless 1$, then $q^2 t_{n,i}(q) \lessgtr t_{n,i}(q)$ for each $i = 1, 2, \ldots, n$. Extensive computer evidence supports that if $q > 1$, then $t_{n,i}(q) < q^2 t_{n,i+1}(q)$, and that if $0 < q < 1$, then $q^2 t_{n,i}(q) < t_{n,i+1}(q)$. We have not been able to prove these inequalities, which are summarized in the following conjecture.

Conjecture 4.6. *Let $n \geqslant 1$ and $q > 0$. Suppose that $B_n(t, q)$ is simply real-rooted and let $t_{n,1}(q) < t_{n,2}(q) < \cdots < t_{n,n}(q) < 0$ be these real zeros. Then $t_{n,i}(q)$'s satisfy the following separation property:*

$$t_{n,i+1}(q) > \min(q^2, q^{-2}) t_{n,i}(q), \quad i = 1, 2, \ldots, n-1.$$

Conjecture 4.6, if true, would imply the following results.

THEOREM 4.7. *For $n \geqslant 1$ and $q > 0$,*

(i) *$\delta_{B,t}(B_n(t, q))$ is simply real-rooted and interlaces $B_n(t, q)$;*
(ii) *$B_n(t, q)$ has only simple negative real zeros, and $B_n(t, q)$ interlaces $B_{n+1}(t, q)$;*
(iii) *the sequence $\{B_{n,0}(q), B_{n,1}(q), \ldots, B_{n,n}(q)\}$ is a PF sequence. In particular, it is unimodal and log-concave.*

The real-rootedness of $\mathfrak{S}_n(t, q)$ in the $q = 1$ case is well known. It is given as an exercise in [11, p. 292, Exercise 3]. We briefly sketch an inductive argument here. Suppose that $\mathfrak{S}_{n-1}(t, 1)$ has simple zeros $t_{n-1,1} < t_{n-1,2} < \cdots < t_{n-1,n-2} < 0$. Let also $t_{n-1,0} := -\infty$ and

$t_{n-1,n-1} := 0$ so that $\operatorname{sgn} \mathfrak{S}_n(t_{n-1,0}, 1) = (-1)^{n-1}$ and $\operatorname{sgn} \mathfrak{S}_n(t_{n-1,n-1}, 1) = 1$. Then setting $t = t_{n-1,i}$ in (3) with $q = 1$, we have

$$\mathfrak{S}_n(t_{n-1,i}, 1) = t_{n-1,i}(1 - t_{n-1,i})\mathfrak{S}'_{n-1}(t_{n-1,i}, 1).$$

The simplicity of $t = t_{n-1,i}$ implies that $\operatorname{sgn} \mathfrak{S}'_{n-1}(t_{n-1,i}, 1) = (-1)^{n-2-i}$ so $\operatorname{sgn} \mathfrak{S}_n(t_{n-1,i}, 1) = (-1)^{n-1-i}$ for $i = 1, 2, \ldots, n-2$. It follows from the intermediate value theorem that there are $t_{n,i} \in (t_{n-1,i-1}, t_{n-1,i})$ such that $\mathfrak{S}_n(t_{n,i}, 1) = 0$, where $i = 1, 2, \ldots, n-1$. The above arguments are readily adapted to proving the real-rootedness of $B_n(t, 1)$. (A different proof has been given by Brenti [7].)

Proof of Theorem 4.7. As noted above, the case $q = 1$ is known. So we assume that $0 < q \neq 1$. Let $t_{n,1}(q) < t_{n,2}(q) < \cdots < t_{n,n}(q) < 0$ be the zeros of $B_n(t, q)$. Conjecture 4.6 implies that either $t_{n,i-1}(q) < q^2 t_{n,i}(q) < t_{n,i}(q)$ or $t_{n,i}(q) < q^2 t_{n,i}(q) < t_{n,i+1}(q)$ depending on whether $q > 1$ or $0 < q < 1$, so that from (25) we have

$$\operatorname{sgn} \delta_{B,t}(B_n(t, q))\Big|_{t=t_{n,i}(q)} = (-1)^{n-i}, \quad i = 1, 2, \ldots, n.$$

Thus, there exist $\eta_i \in (t_{n,i}(q), t_{n,i+1}(q))$ such that $\delta_{B,t}(B_n(t, q))\Big|_{t=\eta_i} = 0$, where $i = 1, 2, \ldots, n-1$, proving (i). We see from the preceding paragraph that $B'_n(t)$ interlacing $B_n(t)$ is crucial in the induction step. In the generic $0 < q \neq 1$ case, we have $\delta_{B,t}(B_n(t, q))$ interlacing $B_n(t, q)$ in place of $B'_n(t)$ interlacing $B_n(t)$, so that the same arguments as in the $q = 1$ case proof apply and (ii) follows. (iii) is an immediate consequence of (ii) and Theorem 2.2. ☐

In Theorem 4.7, the case $q = 0$ is excluded because $B_n(t, 0) = 1$ which has no real zeros. Theorem 4.7 generalizes the corresponding results for the usual type B Eulerian polynomials $B_n(t)$ and Eulerian numbers $B_{n,k}$. See, e.g., [7, Corollary 3.7 with $q = 1$] for an alternative proof. By similar consideration, one also has the following type A version of Conjecture 4.6.

Conjecture 4.8. *Let $n \geqslant 2$ and $q > 0$. Suppose that $\mathfrak{S}_n(t, q)$ is simply real-rooted and let $t_{n,1}(q) < t_{n,2}(q) < \cdots < t_{n,n-1}(q) < 0$ be these real zeros. Then $t_{n,i}(q)$'s satisfy the following separation property:*

$$t_{n,i+1}(q) > \min(q, q^{-1})t_{n,i}(q), \quad i = 1, 2, \ldots, n-2.$$

Conjecture 4.8, if true, would imply the following type A result.

THEOREM 4.9. *For $n \geqslant 2$ and $q > 0$,*

(i) *$\delta_{A,t}(\mathfrak{S}_n(t, q))$ is simply real-rooted and interlaces $\mathfrak{S}_n(t, q)$;*
(ii) *$\mathfrak{S}_n(t, q)$ has only simple negative real zeros, and $\mathfrak{S}_n(t, q)$ interlaces $\mathfrak{S}_{n+1}(t, q)$;*
(iii) *the sequence $\{\mathfrak{S}_{n,0}(q), \mathfrak{S}_{n,1}(q), \ldots, \mathfrak{S}_{n,n-1}(q)\}$ is a PF sequence. In particular, it is unimodal and log-concave.*

The proof of Theorem 4.9, being similar to that of Theorem 4.7, is omitted. Note that the case $n = 1$ is excluded because $\mathfrak{S}_1(t, q) = 1$ which has no zeros.

5. Concluding Remarks

Theorem 3.7 can be generalized to

$$\sum_{k\geqslant 0}([k+1]_q + z[k]_q)^n t^k = \frac{\sum_{\pi\in B_n} z^{N(\pi)} t^{\mathrm{des}_B(\pi)} q^{\mathrm{maj}_A(\pi)}}{\prod_{i=0}^{n}(1-tq^i)}, \tag{26}$$

where $N(\pi)$ is the number of negative entries of π. Replacing q by q^2 and then setting $z=q$ gives (17). This "explains", in a certain sense, why we should take $2\,\mathrm{maj}_A(\pi)+N(\pi) = \mathrm{fmaj}(\pi)$ as the Mahonian partner for des_B. Equation (26) can be proved in the same way as Theorem 3.7, though our original proof used a different approach.

Foata and Zeilberger [17] give four equivalent conditions, which are the type A analogues of Proposition 3.2, and Theorems 3.6, 3.7, and 3.8, defining a sequence of polynomials $\{A_n(t,q)\}$ in two variables to be Euler-Mahonian; they further define a pair of statistics $(\mathrm{stat}_1, \mathrm{stat}_2)$ on each symmetric group $\mathfrak{S}_n$, $n\geqslant 1$, to be Euler-Mahonian if

$$\sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{stat}_1(\pi)} q^{\mathrm{stat}_2(\pi)} = A_n(t,q),$$

where $A_n(t,q)$ satisfies any of the equivalent conditions.

We shall define, in an analogous manner, a sequence of polynomials $\{B_n(t,q)\}$ in two variables to be type B Euler-Mahonian if $B_n(t,q)$ satisfies any of Proposition 3.2, Theorem 3.6, Theorem 3.7, or Theorem 3.8 and we define a pair of statistics $(\mathrm{stat}_1, \mathrm{stat}_2)$ on each hyperoctahedral group B_n, $n\geqslant 1$, to be type B Euler-Mahonian if

$$\sum_{\pi\in B_n} t^{\mathrm{stat}_1(\pi)} q^{\mathrm{stat}_2(\pi)} = B_n(t,q).$$

Biagioli [8] defines negative statistics ndes and nmaj, analogous to those introduced in [2], for the even-signed permutation group D_n (the subgroup of B_n consisting of all signed permutations $\sigma=\sigma_1\cdots\sigma_n$ with even number of $\sigma_i<0$), and generalizes the Carlitz identity (4) to D_n:

$$\frac{\sum_{\sigma\in D_n} t^{\mathrm{ndes}(\sigma)} q^{\mathrm{nmaj}(\sigma)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})} = \sum_{k\geqslant 0}[k+1]_q^n t^k.$$

Brenti [7] proves that the Eulerian polynomial $D_n(t)$ of type D satisfies

$$\frac{D_n(t)}{(1-t)^{n+1}} = \sum_{k\geqslant 0}\{(2k+1)^n - n2^{n-1}[\mathscr{B}_n(k+1)-\mathscr{B}_n]\}t^k, \tag{27}$$

where $\mathscr{B}_n(x)$ is the nth Bernoulli polynomial and $\mathscr{B}_n$ the nth Bernoulli number.

In light of the present work, it is desirable, though more difficult (see, e.g., [10]), to have a q-generalization of (27), again from the Eulerian polynomial point of view. This will be the subject of further research.

6. Addendum (By Chak-On Chow)

In a paper sequel to [2], Adin, Brenti and Roichman [1] revisited the descent representations of types A and B and obtained several multivariate identities, of relevance to us here are

$$\frac{\sum_{\sigma\in B_n}\prod_{i=1}^n q_i^{2d_i(\sigma)+\varepsilon_i(\sigma)}}{\prod_{i=1}^n(1-q_1^2\cdots q_i^2)} = \sum_{l(\lambda)\leqslant n}\binom{n}{\bar{m}(\lambda)}\prod_{i=1}^n q_i^{\lambda_i} = \frac{\sum_{\sigma\in B_n}\prod_{i=1}^n q_i^{d_i(\sigma)+n_i(\sigma^{-1})}}{\prod_{i=1}^n(1-q_1^2\cdots q_i^2)}, \tag{28}$$

where $q_1, q_2, \ldots, q_n$ are commuting indeterminates, the sum in the middle is over all integer partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $l(\lambda) = l \leqslant n$ parts, $m_i(\lambda) := \#\{j \in [l]\colon \lambda_j = i\}$, $\binom{n}{\bar{m}(\lambda)} := \binom{n}{m_0(\lambda), m_1(\lambda), \ldots, m_n(\lambda)}$, and for $\sigma = \sigma_1\cdots\sigma_n \in B_n$ and $i = 1, 2, \ldots, n$,

$$\begin{aligned} n_i(\sigma) &:= \#\{j\colon i \leqslant j \leqslant n, \sigma_j < 0\},\\ d_i(\sigma) &:= \#(\mathrm{Des}_B(\sigma)\cap\{i, i+1, \ldots, n\}),\\ \varepsilon_i(\sigma) &:= \chi(\sigma_i < 0),\\ f_i(\sigma) &:= 2d_i(\sigma) + \varepsilon_i(\sigma). \end{aligned}$$

By a specialization, (28) reduces to the generalized Carlitz identities of Adin *et al.* [2]:

$$\frac{\sum_{\sigma\in B_n} t^{\mathrm{fdes}(\sigma)}q^{\mathrm{fmaj}(\sigma)}}{(1-t)\prod_{i=1}^n(1-t^2q^{2i})} = \sum_{k\geqslant 0}[k+1]_q^n t^k = \frac{\sum_{\sigma\in B_n} t^{\mathrm{ndes}(\sigma)}q^{\mathrm{nmaj}(\sigma)}}{(1-t)\prod_{i=1}^n(1-t^2q^{2i})},$$

which in turn implies the equidistribution of (fdes, fmaj) and (ndes, nmaj). The multivariate identities (28) are considered algebraic interpretations of the generalized Carlitz identities for the hyperoctahedral group.

In view of the above algebraic interpretation of the generalized Carlitz identities of Adin *et al.*, it is natural to ask whether similar multivariate identity exists and whose specialization yields the type B Carlitz identity (17).

We shall show in this addendum that the answer to this existence question is positive. The multivariate identity being sought can be obtained by modifying the results given in [1].

6.1. The coinvariant algebra of B_n.

Let $x_0, x_1, \ldots, x_n$ be commuting indeterminates. The hyperoctahedral group B_n acts on the ring of polynomials $P_n = \mathbb{Q}[x_1, \ldots, x_n]$ by signed permutation of the variables, i.e., if $\sigma \in B_n$ and $P = P(x_1, \ldots, x_n) \in P_n$, then

$$\sigma P = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The ring of B_n-invariant polynomials is Λ_n^B, the ring of symmetric functions in $x_1^2, \ldots, x_n^2$. See, e.g., [30]. Let I_n^B be the ideal of P_n generated by elements of Λ_n^B without constant term. The quotient P_n/I_n^B is called the *coinvariant algebra* of B_n. The hyperoctahedral group B_n acts naturally on P_n/I_n^B, and the resulting representation is isomorphic to the regular representation. See, e.g., [21, 22] for details.

To any $\sigma \in B_n$ we associate the descent monomial

$$b_\sigma := \prod_{i=1}^n x_{|\sigma(i)|}^{f_i(\sigma)}.$$

The signed index permutation of a monomial $m = \prod_{i=1}^n x_i^{p_i} \in P_n$ is the unique signed permutation $\sigma = \sigma(m) \in B_n$ of m such that

$$
\begin{aligned}
&p_{|\sigma(i)|} \geqslant p_{|\sigma(i+1)|}, \quad i = 1, 2, \ldots, n-1,\\
&p_{|\sigma(i)|} = p_{|\sigma(i+1)|} \Rightarrow \sigma(i) < \sigma(i+1), \quad i = 1, 2, \ldots, n-1,\\
&p_{|\sigma(i)|} \equiv 0 \pmod 2 \iff \sigma(i) > 0, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

The next result [1, Claim 5.1] describes the exponents of m/b_σ.

Lemma 6.1. *The sequence $\{p_{|\sigma(i)|} - f_i(\sigma)\}_{i=1}^n$ of exponents in m/b_σ consists of weakly decreasing nonnegative even integers.*

Let $\mu_B(m)$ be the partition conjugate to $\{(p_{|\sigma(i)|} - f_i(\sigma))/2\}_{i=1}^n$. The following straightening law [1, Corollary 5.2] makes explicit that $\{b_\sigma + I_n^B : \sigma \in B_n\}$ spans P_n/I_n^B; a dimensional consideration then shows that the mentioned spanning set is linearly independent. Thus, $\{b_\sigma + I_n^B : \sigma \in B_n\}$ is a basis for P_n/I_n^B.

Lemma 6.2 (Straightening Lemma). *Each monomial $m \in P_n$ has an expression*

$$m = e_{\mu_B(m)}(x_1^2, \ldots, x_n^2) b_{\sigma(m)} + \sum_{m' \prec_B m} n_{m',m} e_{\mu_B(m')}(x_1^2, \ldots, x_n^2) b_{\sigma(m')},$$

where $n_{m',m}$ are integers and $e_\mu(x_1, \ldots, x_n)$ denotes the elementary symmetric function in $x_1, \ldots, x_n$ indexed by the partition μ.

Here, $\prec_B$ is a partial ordering of monomials that need not concern us. With the above notations in place, we have the following bijective correspondence [1, Lemma 6.6].

Lemma 6.3. *The mapping $m \mapsto (\sigma(m), \mu_B(m)')$ is a bijection between the set of all monomials in P_n and the set of all pairs (σ, μ), where $\sigma \in B_n$ and μ is a partition with at most n parts.*

Given the basis $\{b_\sigma + I_n^B : \sigma \in B_n\}$ for P_n/I_n^B, the Hilbert series of the polynomial ring P_n by exponent partitions is precisely the first equality in (28). The second equality in (28) is obtained by considering the decomposition $B_n = \uplus_{u \in \mathfrak{S}_n} \{\sigma u : \sigma \in T\}$, where $T := \{\pi \in B_n : \operatorname{des}_B(\pi) = 0\}$, and $\uplus$ denotes disjoint union.

6.2. **The modified results.** We give in this section a modified version of (28) and a specialization of which yields the natural Carlitz identity of type B (Theorem 3.7).

Let $\sigma = \sigma_1 \cdots \sigma_n \in B_n$. Define the *modified* descent monomial

$$\tilde{b}_\sigma = \prod_{i=0}^n x_{|\sigma(i)|}^{f_i(\sigma)}.$$

Since $\sigma(0) := 0$, a comparison of the two descent monomials b_σ and $\tilde{b}_\sigma$ reveals that they differ only in the multiplicative factor $x_0^{f_0(\sigma)}$.

Lemma 6.4. *The number $f_0(\sigma)$ is the least even integer $\geqslant \max_{1 \leqslant i \leqslant n} f_i(\sigma)$.*

Proof. It is clear that $f_1(\sigma) \geqslant f_2(\sigma) \geqslant \cdots \geqslant f_n(\sigma)$. Since $\sigma_0 := 0 \not< 0$, $f_0(\sigma)$ being an even integer follows from $f_0(\sigma) := 2d_0(\sigma)$. To finish the proof, it suffices to show that $f_0(\sigma) \geqslant f_1(\sigma)$. But this follows from $0 \in \mathrm{Des}_B(\sigma) \iff d_0(\sigma) > d_1(\sigma)$ and $\varepsilon_1(\sigma) = 1$, and $0 \notin \mathrm{Des}_B(\sigma) \iff d_0(\sigma) = d_1(\sigma)$ and $\varepsilon_1(\sigma) = 0$. □

The above lemma expresses that $f_0(\sigma)$ is uniquely determined by $f_i(\sigma)$, $i = 1, 2, \ldots, n$, so that $\tilde{b}_\sigma$ carries no extra information about σ. We shall see later that the presence of $x_0^{f_0(\sigma)}$ in the descent monomial makes a difference in the multivariate identity.

Let $m = \prod_{i=1}^n x_i^{p_i} \in P_n$. Denote by $\tilde{m}$ the monomial $\prod_{i=0}^n x_i^{p_i} \in \mathbb{Q}[x_0, x_1, \ldots, x_n]$, where p_0 is the least even integer $\geqslant \max_{1 \leqslant i \leqslant n} p_i$. The linear extension of the map $m \mapsto \tilde{m}$ is an embedding of P_n into $\mathbb{Q}[x_0, x_1, \ldots, x_n]$. We denote by $\tilde{P}_n$ the image of P_n in $\mathbb{Q}[x_0, x_1, \ldots, x_n]$. Since $\sigma(0) := 0$, the B_n-action on P_n induces a B_n-action on $\tilde{P}_n$ defined by

$$\sigma P = P(x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

where $P \in \tilde{P}_n$, and $\tilde{P}_n$ and P_n are isomorphic B_n-modules.

To accommodate the extra variable x_0, the defining conditions of the signed index permutation $\sigma = \sigma(m)$ of a monomial $m \in P_n$ are relaxed so that they also hold when $i = 0$, and that the last condition becomes

$$p_{|\sigma(i)|} \equiv 0 \pmod 2 \iff \sigma(i) \geqslant 0, \quad i = 0, 1, \ldots, n.$$

For example, if $m = x_1^2 x_2^7 x_3^4 x_4^3 x_5^0 x_6^1$, then $\tilde{m} = x_0^8 x_2^7 x_3^4 x_4^3 x_1^2 x_6^1 x_5^0$, $\sigma(m) = \bar{2}3\bar{4}1\bar{6}5$, $\mathrm{Des}_B(\sigma) = \{0, 2, 4\}$, and $\tilde{b}_\sigma = x_0^6 x_2^5 x_3^4 x_4^3 x_1^2 x_6^1 x_5^0$.

Lemma 6.5. *The sequence $\{p_{|\sigma(i)|} - f_i(\sigma)\}_{i=0}^n$ of exponents in $\tilde{m}/\tilde{b}_\sigma$ consists of nonnegative even integers such that*

$$p_{|\sigma(i)|} - f_i(\sigma) \geqslant p_{|\sigma(i+1)|} - f_{i+1}(\sigma), \quad i = 1, 2, \ldots, n,$$

and $p_{|\sigma(0)|} - f_0(\sigma) = p_{|\sigma(1)|} - f_1(\sigma)$.

Proof. That $\{p_{|\sigma(i)|} - f_i(\sigma)\}_{i=1}^n$ being a sequence of weakly decreasing nonnegative even integers is the content of Lemma 6.1. We have $\sigma(1) < 0 \iff p_{|\sigma(0)|} = p_{|\sigma(1)|} + 1$ and $f_0(\sigma) = f_1(\sigma) + 1$, and $\sigma(1) > 0 \iff p_{|\sigma(0)|} = p_{|\sigma(1)|}$ and $f_0(\sigma) = f_1(\sigma)$, so that $p_{|\sigma(0)|} - f_0(\sigma) = p_{|\sigma(1)|} - f_1(\sigma)$ holds in both cases. □

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition with at most n parts, i.e., $l = l(\lambda) \leqslant n$. We shall regard any such λ as a sequence $(\lambda_1, \ldots, \lambda_l, 0, \ldots, 0)$ of length n with $n - l$ zeros appended. Define the *augmented* partition $\tilde{\lambda} := (\lambda_0, \lambda_1, \ldots, \lambda_l)$, where λ_0 is the least even integer $\geqslant \lambda_1$. It is clear that if λ, δ and μ are partitions such that at most one of δ and μ has its first part odd and $\lambda = \delta + \mu$ partwise, then $\tilde{\lambda} = \tilde{\delta} + \tilde{\mu}$.

Lemma 6.1 expresses that

$$(p_{|\sigma(1)|}, \ldots, p_{|\sigma(n)|}) = (f_1(\sigma), \ldots, f_n(\sigma)) + (\mu_1, \ldots, \mu_n),$$

where $\mu := (\mu_1, \ldots, \mu_n)$ is a partition with all its parts even and $l(\mu) \leqslant n$. The remark of the preceding paragraph and Lemma 6.4 then yield that

$$(p_{|\sigma(0)|}, p_{|\sigma(1)|}, \ldots, p_{|\sigma(n)|}) = (f_0(\sigma), f_1(\sigma), \ldots, f_n(\sigma)) + (\mu_1, \mu_1, \ldots, \mu_n).$$

Given a monomial $m = \prod_{i=1}^n x_i^{p_i} \in P_n$. Denote by $\lambda(m) = (p_{|\sigma(1)|}, p_{|\sigma(2)|}, \ldots, p_{|\sigma(n)|})$ its exponent partition, and $f(\sigma) = (f_0(\sigma), f_1(\sigma), \ldots, f_n(\sigma))$, where $\sigma = \sigma(m)$ is the index permutation of m. With these notations, Lemma 6.3 states that $\tilde{\lambda}(m) = f(\sigma) + \widetilde{2\mu}$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a partition with at most n positive parts.

Lemma 6.6. *We have*

$$\sum_{l(\lambda)\leqslant n} \bar{q}^{\widetilde{2\lambda}} = \frac{1}{\prod_{i=1}^n (1 - q_0^2 q_1^2 \cdots q_i^2)},$$

where the sum is over all partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $l(\lambda) = l \leqslant n$.

Proof. Observe that the first two parts of $\widetilde{2\lambda}$ are equal. The lemma then follows from

$$\begin{aligned}\sum_{l(\lambda)\leqslant n} \bar{q}^{\widetilde{2\lambda}} &= \sum_{\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n\geqslant 0} q_0^{2\lambda_1} q_1^{2\lambda_1} q_2^{2\lambda_2} \cdots q_n^{2\lambda_n} = \prod_{i=1}^n \sum_{\lambda_i-\lambda_{i+1}\geqslant 0} (q_0^2 q_1^2 \cdots q_i^2)^{\lambda_i-\lambda_{i+1}} \\ &= \frac{1}{\prod_{i=1}^n (1 - q_0^2 q_1^2 \cdots q_i^2)},\end{aligned}$$

where $\lambda_{n+1} := 0$. □

Since $\tilde{\lambda}(m)$ and $\widetilde{2\mu}$ are uniquely determined by $\lambda(m)$ and 2μ, respectively, the bijection in Lemma 6.3 gives rise to a bijection between monomials $\tilde{m} \in \tilde{P}_n$ and $(\sigma, \widetilde{2\mu})$, where $\sigma \in B_n$ and μ a partition with $l(\mu) \leqslant n$.

Theorem 6.7. *We have*

$$\frac{\sum_{\sigma\in B_n} \prod_{i=0}^n q_i^{f_i(\sigma)}}{\prod_{i=1}^n (1 - q_0^2 q_1^2 \cdots q_i^2)} = \sum_{l(\lambda)\leqslant n} \binom{n}{\bar{m}(\lambda)} \bar{q}^{\tilde{\lambda}} \tag{29}$$

where the sum on the right ranges over all partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $l(\lambda) = l \leqslant n$, λ_0 is the least even integer $\geqslant \lambda_1$, $m_i(\lambda) := \#\{j \in [l] \colon \lambda_j = i\}$, and $\binom{n}{\bar{m}(\lambda)} := \binom{n}{m_0(\lambda), m_1(\lambda), \ldots, m_n(\lambda)}$.

Proof. For any partition λ with at most n parts, $\binom{n}{\bar{m}(\lambda)}$ is the number of monomials in P_n with exponent partition equal to λ. Therefore the Hilbert series of the polynomial ring P_n by the augmented exponent partition $\tilde{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_l)$ is equal to the right hand side of (29). On the other hand, since $\tilde{\lambda}(m) = \tilde{\lambda}(b_{\sigma(m)}) + \widetilde{2\mu}$, where μ is a partition with $l(\mu) \leqslant n$, we get

$$\begin{aligned}\sum_{m\in P_n} \bar{q}^{\tilde{\lambda}(m)} &= \sum_{m\in P_n} \bar{q}^{\tilde{\lambda}(b_{\sigma(m)})+2\mu(m)'} \\ &= \sum_{\sigma\in B_n} \bar{q}^{\lambda(b_\sigma)} \sum_{l=l(\mu)\leqslant n} \bar{q}^{2\tilde{\mu}} \\ &= \frac{\sum_{\sigma\in B_n} \prod_{i=0}^n q_i^{f_i(\sigma)}}{\prod_{i=1}^n (1 - q_0^2 q_1^2 \cdots q_i^2)}.\end{aligned}$$

□

A specialization of (29) yields Theorem 3.7.

Corollary 6.8. *We have*

$$\frac{\sum_{\sigma\in B_n} t^{\mathrm{des}_B(\sigma)} q^{\mathrm{fmaj}(\sigma)}}{\prod_{i=0}^{n}(1-tq^{2i})} = \sum_{r\geqslant 0}[2r+1]_q^n t^r.$$

Proof. Setting in Theorem 6.7 $q_0 = t^{1/2}$, and $q_1 = \cdots = q_n = q$, followed by dividing by $1-t$, we obtain

$$\begin{aligned}\frac{\sum_{\sigma\in B_n} t^{\mathrm{des}_B(\sigma)} q^{\mathrm{fmaj}(\sigma)}}{\prod_{i=0}^{n}(1-tq^{2i})} &= \sum_{m\geqslant 0} t^m \sum_{l(\lambda)\leqslant n, \lambda_0 \text{ even}} \binom{n}{\bar{m}(\lambda)} t^{\lambda_0/2} q^{\lambda_1+\cdots+\lambda_l}\\ &= \sum_{r\geqslant 0} t^r \sum_{l(\lambda)\leqslant n, \lambda_1\leqslant 2r} \binom{n}{\bar{m}(\lambda)} q^{\lambda_1+\cdots+\lambda_l}\\ &= \sum_{r\geqslant 0}[2r+1]_q^n t^r,\end{aligned}$$

where we have set $r = m + \lambda_0/2$ in the second equality. □

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Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan

Department of Mathematics, Brandeis University, Waltham, MA 02454–9110, U.S.A.
E-mail address: gessel@brandeis.edu