ACYCLIC ORIENTATIONS AND CHROMATIC GENERATING FUNCTIONS

Ira M. Gessel[1]

Department of Mathematics
Brandeis University
Waltham, MA 02454-9110
gessel@brandeis.edu
www.cs.brandeis.edu/~ira

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Abstract. Let $P(k)$ be the chromatic polynomial of a graph with $n \geq 2$ vertices and no isolated vertices, and let $R(k) = P(k+1)/k(k+1)$. We show that the coefficients of the polynomial $(1-t)^{n-1}\sum_{k=1}^{\infty} R(k)t^k$ are nonnegative and we give a combinatorial interpretation to $R(k)$ when k is a nonpositive integer.

1. Introduction. The problem of characterizing chromatic polynomials was considered by Wilf [18], who gave several necessary conditions for a polynomial to be a chromatic polynomial. Linial [10] (see also Gansner and Vo [7], Tomescu [16], and Brenti [2]) gave another necessary condition: if $P_G(k)$ is the chromatic polynomial of a graph G with n vertices then the coefficients of the polynomial $A_G(t)$ defined by $A_G(t) = (1-t)^{n+1}\sum_{k=0}^{\infty} P_G(k)t^k$ are nonnegative. Linial's result is a consequence of Richard Stanley's theory of P-partitions ([12; 15, pp. 211–221]): Any proper coloring of G with colors $1, 2, \ldots, k$ yields an acyclic orientation of G in which each edge is directed from the lower color to the higher color. As noted by Stanley [13], the number of colorings corresponding to a given orientation $\mathcal{O}$ is a strict order polynomial $\overline{\Omega}_{\mathcal{O}}(k)$ and it is known [12, Prop 13.3; 15, Theorem 4.5.14] that the coefficients of $(1-t)^{n+1}\sum_{k=0}^{\infty} \overline{\Omega}_{\mathcal{O}}(k)t^k$ have nonnegative coefficients. Stanley observed that the same decomposition could be used to prove that the number of acyclic orientations of G is $(-1)^n P_G(-1)$ and to give an analogous combinatorial interpretation to $(-1)^n P_G(-k)$ for any positive integer k.

In this paper we prove a similar, but stronger result: Let G be graph with $n \geq 2$ vertices and no isolated vertices, and define the polynomial $R_G(k)$ of degree $n-2$ by

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$R_G(k) = P_G(k+1)/k(k+1)$. Then the coefficients of the polynomial $B_G(t)$, defined by

$$\frac{B_G(t)}{(1-t)^{n-1}} = \sum_{k=1}^{\infty} R_G(k)t^k, \tag{1}$$

are nonnegative. Our proof is similar to that of Linial's theorem, but requires a more careful analysis of the colorings corresponding to each acyclic orientation. We also obtain, in analogy with Stanley's theorem on acyclic orientations, a result of Greene and Zaslavsky [9, Theorem 7.2] on acyclic orientations with a single source and sink, and a combinatorial interpretation of $(-1)^n R_G(-k)$ when k is a nonnegative integer. We give two proofs of this combinatorial interpretation, one using P-partitions and one using the Cartier-Foata theory of free partially commutative monoids.

We note also that if G is connected, then the coefficients of $R_G(k)$ alternate in sign. Although this is a known result, it does not seem to be as well known as the weaker result that the coefficients of the chromatic polynomial alternate in sign.

2. Proof of the Main Theorem.

Theorem 2.1. *Let G be a graph with $n \geq 2$ vertices and at least one edge. Let $P_G(k)$ be the chromatic polynomial of G, and let $R_G(k) = P_G(k+1)/k(k+1)$.*

- (i) *If G is connected then $(-1)^n R_G(-k)$ is a monic polynomial in k of degree $n-2$ with nonnegative coefficients.*
- (ii) *If G has no isolated vertices then $B_G(t)$, as defined by (1), is a polynomial in t of degree at most $n-1$ with nonnegative coefficients, and $B_G(1) = (n-2)!$.*

Proof. (i) It is known [3, Proposition 6.3.1] that if G is a connected graph with n vertices then $P_G(k) = (-1)^{n-1} k\, t_G(1-k, 0)$, where $t_G(x,y)$ is the Tutte polynomial of G. Moreover, the Tutte polynomial is known to have nonnegative coefficients and no constant term [3, Theorem 6.2.13 (vii)]. It follows that $(-1)^n R_G(-k) = k^{-1} t_G(k,0)$ is a polynomial with nonnegative coefficients. Since $P_G(k)$ is monic of degree n, $R_G(k)$ must be monic of degree $n-2$.

(ii) Pick two adjacent vertices, a and z, in G. Then for $k \geq 1$, $P_G(k+1)/k(k+1) = R_G(k)$ is the number of proper colorings of G in $k+1$ colors, numbered 0 to k, in which a has color 0 and z has color k.

To any proper coloring of G there corresponds an acyclic orientation of G, in which each edge is directed from the lower-colored vertex to the higher-colored vertex. For each acyclic orientation $\mathcal{O}$, let $R_{\mathcal{O}}(k)$ be the number of corresponding colorings in which vertex a has color 0 and vertex z has color k. Then $R_G(k) = \sum_{\mathcal{O}} R_{\mathcal{O}}(k)$, where the sum is over all acyclic orientations $\mathcal{O}$ of G for which a is a source and z is a sink. So since

$$\sum_{k=1}^{\infty} \binom{k+n-2-i}{n-2} t^k = \frac{t^i}{(1-t)^{n-1}}, \quad i \geq 1,$$

it is sufficient to prove that $R_{\mathcal{O}}(k)$ is a nonnegative linear combination of the polynomials $\binom{k+n-2-i}{n-2}$, $i = 1, \ldots, n-1$.

The acyclic orientation $\mathcal{O}$ induces a partial order $\preceq$ on the vertices of G in which $x \preceq y$ if there is a directed path from x to y. Then a coloring c corresponds to the acyclic orientation $\mathcal{O}$ if and only if c has the property that $x \prec y$ implies $c(x) < c(y)$.

We now fix and acyclic orientation $\mathcal{O}$ and consider $R_{\mathcal{O}}(x)$ in more detail. First we make some assumptions about the labeling of the vertices of G that are helpful in describing a useful decomposition for the colorings counted by $R_{\mathcal{O}}(k)$. We assume that the vertex set of G is $[n] = \{1, 2, \ldots, n\}$. Without loss of generality, we may assume that the vertices of G have the property that if $i \prec j$ then $i < j$. Since G has no isolated vertices, no vertex is both a source and a sink of $\mathcal{O}$, and we may therefore assume that a is greater (in the usual total ordering of $[n]$) than every other source of $\mathcal{O}$, and that z is less than every other sink of $\mathcal{O}$.

For each permutation π of $[n]$, let $\mathcal{C}(\pi)$ be the set of colorings c of G with colors $0, 1, \ldots, k$ satisfying

(a) $c(\pi(i)) \le c(\pi(i+1))$ for $i = 1, \ldots, n-1$.
(b) If $i < j$ and $\pi(i) < \pi(j)$ then $c(\pi(i)) < c(\pi(j))$.

(Note that (b) is implied by (a) together with the special cases of (b) for which $j = i+1$, but the more general form of (b) will be useful.) Now let $\mathcal{L}(\mathcal{O})$ be the set of permutations of $\{1, 2, \ldots, n\}$ such that the sequence $\pi(1), \ldots, \pi(n)$ extends $\preceq$ to a total order; i.e., if $i \prec j$ then $\pi^{-1}(i) < \pi^{-1}(j)$. Then by Stanley's fundamental theorem on P-partitions [12, Theorem 6.2; 15, Lemma 4.5.3] the set of colorings corresponding to an acyclic orientation $\mathcal{O}$ is the disjoint union of $\mathcal{C}(\pi)$ over all permutations π in $\mathcal{L}(\mathcal{O})$.

We wish to count colorings in $\mathcal{C}(\pi)$ in which vertex a has color 0 and vertex z has color k. We claim that if c is a coloring in $\mathcal{C}(\pi)$ with $c(a) = 0$ then $\pi(1) = a$. To see this, suppose that $\pi(1) \neq a$ and let c be a coloring in $\mathcal{C}(\pi)$. Then $\pi(1)$ must be a source of $\mathcal{O}$ other than a, so $\pi(1) < a$, and thus by (b) we have $c(\pi(1)) < c(a)$, so $c(a) \neq 0$. This proves the claim. Similarly, if c is a coloring in $\mathcal{C}(\pi)$ with $c(z) = k$ then $\pi(n) = z$.

If $\pi(1) = a$ and $\pi(n) = z$ then the number of colorings in $\mathcal{C}(\pi)$ satisfying (a) and (b) with colors from 0 to k, and with $c(a) = 0$ and $c(z) = k$, is easily seen to be $\binom{k+n-2-\alpha(\pi)}{n-2}$, where $\alpha(\pi)$ is the number of ascents of π, that is, the number of values of i from 1 to $n-1$ such that $\pi(i) < \pi(i+1)$. Since $a = \pi(1) < \pi(n) = z$, there must be at least one ascent, and there can be at most $n-1$ of them. It follows that

$$R_{\mathcal{O}}(k) = \sum_{i=1}^{n-1} u_i \binom{k+n-2-i}{n-2}, \tag{2}$$

where u_i is the number of permutations π in $\mathcal{L}(\mathcal{O})$ with i ascents satisfying $\pi(1) = a$ and $\pi(n) = z$.

Finally, $B_G(1) = (n-2)!$ since each permutation π of the vertex set of G with $\pi(1) = a$ and $\pi(n) = z$ contributes (after appropriate relabeling) one term t^i to $B_G(t)$. (Alternatively, this is a consequence of the fact that $R_G(k)$ is monic of degree $n-2$.) □

The prohibition of isolated vertices cannot be removed in Theorem 2.1, since if G is the graph on three vertices with one edge, then $P_G(k) = k^2(k-1)$, so $R_G(k) = k+1$, and

$$\sum_{k=1}^{\infty}(k+1)t^k = \frac{2t-t^2}{(1-t)^2}.$$

It remains to show that our result implies Linial's. Since $P_G(k) = k(k-1)R(k-1)$ it suffices (at least for graphs without isolated vertices) to show that for $j = 0, \dots, n-2$, $(1-t)^{n+1}\sum_{k=1}^{\infty} k(k-1)\binom{k+n-2-j}{n-2}t^k$ has nonnegative coefficients. We have

$$\sum_{k=1}^{\infty} k(k-1)\binom{k+n-2-j}{n-2}t^k = t^2\frac{d^2}{dt^2}\frac{t^j}{(1-t)^{n-1}}$$
$$= \frac{j(j-1)t^j + 2j(n-j)t^{j+1} + (n-j)(n-j-1)t^{j+2}}{(1-t)^{n+1}},$$

and the coefficients in the numerator are clearly nonnegative.

By the similar reasoning, one can show that Linial's theorem also hold for graphs with isolated vertices, since adding an isolated vertex to a graph multiples its chromatic polynomial by k.

3. Interpretation of $R_G(k)$ for $k \le 0$. Since $R_G(k)$ is a polynomial in k, it is well defined for all values of k. Our next theorem, due to Greene and Zaslavsky [9, Theorem 7.2] gives a combinatorial interpretation to $R_G(0)$, which is also equal to $P_G'(1)$. It is analogous to Stanley's theorem [13] that $(-1)^n P_G(-1)$ is the number of acyclic orientations of G.

Theorem 3.1. *Let G be graph with n vertices and with no isolated vertices and let a and z be adjacent vertices in G. Then $(-1)^n R_G(0)$ is the number of acyclic orientations of G in which a is the only source and z is the only sink.*

Proof. If $k = 0$ then $\binom{k+n-2-i}{n-2}$ is 0 for $i = 1, \dots, n-2$, and is $(-1)^n$ for $i = n-1$. Then by (2), $(-1)^n R_{\mathcal{O}}(0)$ is the number of permutations π in $\mathcal{L}(\mathcal{O})$ with $n-1$ ascents satisfying $\pi(a) = 1$ and $\pi(n) = z$. The only permutation of $[n]$ with $n-1$ ascents is the identity permutation, which is always in $\mathcal{L}(\mathcal{O})$, so the identity permutation will contribute to $(-1)^n R_{\mathcal{O}}(0)$ if and only if $a = 1$ and $z = n$, and this holds if and only if a is the only source and z the only sink of $\mathcal{O}$. $\square$

More generally, Stanley [13] gave the following combinatorial interpretation to the chromatic polynomial evaluated at a negative intger:

Theorem 3.2. *Let k be a nonnegative integer. Then $(-1)^n P_G(-k)$ is the number of pairs $(\mathcal{O}, f)$ in which $\mathcal{O}$ is an acyclic orientation of G and f is a function from the vertex set of G to $\{1, 2, \dots, k\}$ with the property that for all vertices x and y, if $\mathcal{O}$ has a directed edge from x to y then $f(x) \le f(y)$.*

We give a similar interpretation for $(-1)^n R(-k)$ that generalizes Theorem 3.1. As before, we fix two adjacent vertices a and z of G. Let $\mathcal{O}$ be an acyclic orientation of G for which a is a source and z is a sink. Let V be the vertex set of G. We say that a function $f : V \to \{0, 1, \dots, k\}$ is *$\mathcal{O}$-compatible* if the following conditions are satisfied:

(i) If (i, j) is an edge of $\mathcal{O}$ then $f(i) \le f(j)$.

(ii) $f(a) = 0$ and $f(z) = k$.
(iii) If s is a source of $\mathcal{O}$ other than a then $f(s) > 0$.
(iv) If t is a sink of $\mathcal{O}$ other than z then $f(t) < k$.

Theorem 3.3. *Let k be a nonnegative integer. Then under the hypotheses of Theorem 3.1, $(-1)^n R_G(-k)$ is the number of pairs $(\mathcal{O}, f)$ such that $\mathcal{O}$ is an acyclic orientation of G in which a is a source and z is a sink, and f is an $\mathcal{O}$-compatible function $V \to \{0, 1, \ldots, k\}$.*

Proof. With the notation used in the proof of Theorem 2.1, it is sufficient to prove that if $\mathcal{O}$ is an acyclic orientation of G in which a is a source and z is a sink, then $(-1)^n R_{\mathcal{O}}(-k)$ is the number of $\mathcal{O}$-compatible functions $V \to \{0, 1, \ldots, k\}$. Without loss of generality we assume that $V = [n]$ and that the vertices of G are labeled as in Theorem 1: if $i \prec j$ in $\mathcal{O}$ then $i < j$, a is greater than every other source of $\mathcal{O}$, and z is less than every other sink of $\mathcal{O}$. For each permutation π of $[n]$, let $\mathcal{F}(\pi)$ be the set of functions $[n] \to \{0, 1, \ldots, k\}$ satisfying

(a) $f(\pi(i)) \le f(\pi(i+1))$ for $i = 1, \ldots, n-1$.
(b) If $i < j$ and $\pi(i) > \pi(j)$ then $f(\pi(i)) < f(\pi(j))$.

By Stanley's fundamental theorem on P-partitions, the disjoint union of $\mathcal{F}(\pi)$ over all π in $\mathcal{L}(\mathcal{O})$ is the set of all functions $f : [n] \to \{0, 1, \ldots, k\}$ with the property that $i \prec j$ in $\mathcal{O}$ implies $f(i) \le f(j)$.

As in the proof of Theorem 1, we wish to count functions in $\mathcal{F}(\pi)$ which are $\mathcal{O}$-compatible. We first claim that if f is in $\mathcal{F}(\pi)$ and is $\mathcal{O}$-compatible then $\pi(1) = a$. To see this, suppose that $\pi(1) \ne a$ and let f be in $\mathcal{F}(\pi)$. Then $\pi(1)$ must be a source of $\mathcal{O}$ other than a, and so by (iii), we have $f(\pi(1)) > 0$. Thus $0 < f(\pi(1)) \le f(a)$, so $f(a) \ne 0$, and thus f is not $\mathcal{O}$-compatible. Similarly, if f is in $\mathcal{F}(\pi)$ and is $\mathcal{O}$-compatible then $f(z) = k$.

Next we show that if $\pi \in \mathcal{L}(\mathcal{O})$ with $\pi(1) = a$ and $\pi(n) = z$, and $f \in \mathcal{F}(\pi)$ with $f(a) = 0$ and $f(z) = k$, then f is $\mathcal{O}$-compatible. We need only show that properties (iii) and (iv) hold. To prove (iii), suppose that s is a source of $\mathcal{O}$ other than a. By our assumption on the labeling of the vertices of G, we have $a > s$, but $1 = \pi^{-1}(a) < \pi^{-1}(s)$. Then by (b), we have $0 = f(a) < f(s)$. The proof that (iv) holds is similar.

Now suppose that π is a permutation in $\mathcal{L}(\mathcal{O})$ with $\pi(1) = a$ and $\pi(n) = z$. Then the characterization of $\mathcal{O}$-compatible functions in $\mathcal{F}(\pi)$ just given implies that the number of these functions is $\binom{k+n-2-\delta(\pi)}{n-2}$, where $\delta(\pi)$ is the number of descents of π, that is, the number of values of i from 1 to $n-1$ such that $\pi(i) > \pi(i+1)$. Thus the number of $\mathcal{O}$-compatible functions is

$$\sum_{j=0}^{n-2} v_j \binom{k+n-2-j}{n-2}, \tag{3}$$

where v_j is the number of permutations in $\mathcal{L}(\mathcal{O})$ with j descents satisfying $\pi(1) = 1$ and $\pi(n) = z$. For any permutation π of $[n]$, $\alpha(\pi) + \delta(\pi) = n-1$, and we have $\binom{-k+n-2-i}{n-2} = (-1)^n \binom{k+n-2-j}{n-2}$, where $i + j = n-1$. Thus by comparing (2) and (3) we see that (3) is equal to $(-1)^n R_{\mathcal{O}}(-k)$. $\square$

It is easy to check that if we allow G to have isolated vertices, then the conclusion of Theorem 3.3 still holds for $k > 0$ (but not for $k = 0$).

Greene and Zaslavsky [9] also showed that $(-1)^{n-1}P'_G(0)$ is the number of acyclic orientations of G in which a specified vertex is the only source. By our methods we can obtain a generalization of this result analogous to Theorem 3.3.

We define the polynomial $S_G(k)$ to be $P_G(k)/k$, so that for $k > 0$, $S_G(k)$ is the number of colorings of G with k colors in which the color of a specified vertex is fixed. The proof of the next theorem is similar to that of Theorem 3.3, so we omit it here:

Theorem 3.4. *Let G be a graph with vertex set V, let a be a vertex of G, and let k be a nonnegative integer. Then $(-1)^{n-1}S_G(-k)$ is the number of pairs $(\mathcal{O}, f)$ such that $\mathcal{O}$ is an acyclic orientation of $\mathcal{O}$ in which a is a source, and f is a function $V \to \{0, 1, \ldots, k\}$ satisfying these conditions:*

(i) *If (i, j) is an edge of $\mathcal{O}$ then $f(i) \le f(j)$.*
(ii) *$f(a) = 0$.*
(iii) *If s is a source of $\mathcal{O}$ other than a then $f(s) > 0$.*

4. The Cartier-Foata theory of free partially commutative monoids. In this section, we give another approach to Theorems 3.3, 3.4, and 3.2, using the Cartier-Foata theory of free partially commutative monoids [4, 5, 17]. We note that it is also possible to prove these results using Stanley's generalized Ehrhart polyomial reciprocity theorem [14, Proposition 8.2]. To any graph G with vertex set V, we associate a monoid M_G whose elements are equivalence classes of words in the free monoid V^*: two words are equivalent if one can be obtained from the other by a sequence of steps each of which consists of switching two consecutive letters that are nonadjacent vertices in G. Thus if G is the path graph

$$a — b — c — d$$

then the two words *adcdb* and *dcabd* are equivalent via the seqence of switches

$$adcdb \to dacdb \to dacbd \to dcabd.$$

We shall follow the terminology of Mazurkiewicz [11] and call the equivalence classes with respect to this equivalence relation *traces*, and we call the monoid M_G the *trace monoid* of G. Of particular importance to us are equivalence classes of words in which every vertex of G appears exactly once. We call these equivalence classes *G-traces*.

Let w be a word in V^* in which no element of V appears more than once, and let H be the induced subgraph of G on the set of vertices that appear in w. Then we may associate to w the acyclic orientation of H in which each edge of G is oriented from the vertex appearing earlier in w to the vertex appearing later. It is clear that switching two consecutive letters that are nonadjacent vertices of G does not change the associated acyclic orientation, so equivalent words give the same acyclic orientation. One of the fundamental results of the theory [11, Theorem 1; 5, Proposition 1.2.4] is that the converse is true: if two words give the same acyclic orientation then they are equivalent. Thus traces in which no letter appears more than once may be identified with acyclic orientations of induced subgraphs of G. (A similar interpretation can be given for arbitrary traces.) In particular, G-traces correspond to acyclic orientations of G.

If t is a trace than a *source* of t is a letter that occurs as a first element of some representative of t; thus if t is a G-trace then s is a source of t if and only if s is a source of the corresponding acyclic orientation of G. We define *sinks* of traces similarly. We call a letter in a trace t *isolated* if it occurs only once in t and commutes with every other letter in t. Clearly every isolated letter is both a source and sink; conversely, if a letter occurs only once in a trace and is both a source and a sink then it is isolated. We call a source or sink that is not isolated a *proper* source or sink.

It is not difficult to show that a letter v is a source of a trace t if and only if in any representative w of t, v commutes with every letter occurring before the first occurrence of v in w. A similar result holds for sinks.

For any commutative ring R, we define the ring $R\langle\langle M\rangle\rangle$ to be the ring of formal sums of elements of M, with coefficients in R, where addition and multiplication are defined in the obvious way. We may think of $R\langle\langle M\rangle\rangle$ as a ring of formal power series in variables of which only some pairs commute.

We shall derive Theorems 3.2, 3.3, and 3.4 from the following results on trace monoids, which we prove in Section 5.

We write $l(t)$ for the length of the trace t (which is the same as the length of any of its representatives). If W is an independent set of vertices in G (i.e., a set of mutually commuting letters) then the product of the elements of W is independent of their order and is thus well-defined.

Lemma 4.1. *Let M be the trace monoid of the graph G. Let C be the set of all products of independent sets in G, including 1 (the empty product) and each vertex of G. For each vertex v of G, let C_v be the set of products of all independent sets in G that contain v. Then the following identities hold in $\mathbf{Z}\langle\langle M\rangle\rangle$:*

(i) *The sum of all elements of M is*

$$\left(\sum_{x\in C}(-1)^{l(x)}x\right)^{-1}.$$

(ii) *Let u be a vertex of G. Then the sum of all words in M for which u is the only source is*

$$\left(\sum_{x\in C_u}(-1)^{l(x)-1}x\right)\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{-1}$$

and the sum of all words in M for which u is the only sink is

$$\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{-1}\left(\sum_{z\in C_u}(-1)^{l(z)-1}z\right).$$

(iii) *Let u and v be vertices of G. Then the sum of all words in M for which u is the only source and v is the only sink is*

$$\left(\sum_{x\in C_u}(-1)^{l(x)-1}x\right)\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{-1}\left(\sum_{z\in C_v}(-1)^{l(z)-1}z\right)+I,$$

where every trace occurring in I has at least one isolated letter.

Part (i) of Lemma 4.1 is due to Cartier and Foata [4, Théorème 2.4], part (ii) is due to Foata [6, Theorem 3.1] (see also Viennot [17, Proposition 5.3]), and part (iii) is closely related to a result of Bousquet-Mélou [1, Lemme 1.2].

Proof of Theorem 3.2. First recall that n denotes the number of vertices of G. If k is a positive integer, then $(-1)^n$ times the coefficient of a G-trace t in $\left(\sum_{x\in C}(-1)^{l(x)}x\right)^k$ is the number of k-tuples $(x_1, x_2, \dots, x_k)$ such that each x_i is in C and $x_1x_2\cdots x_k = t$; i.e., the number of ordered partitions $(A_1, \dots, A_k)$ of V into k independent sets such that the acyclic orientation of G corresponding to t is obtained if we direct each edge joining A_i with A_j from A_i to A_j, for all $i < j$. Thus $(-1)^n$ times the coefficient of t in $\left(\sum_{x\in C}(-1)^{l(x)}x\right)^k$ is the number of proper colorings of G in k colors whose associated acyclic orientation corresponds to t, and the sum of these numbers over all G-traces t is the chromatic polynomial $P_G(k)$. Since the coefficient of each trace is a polynomial in k, for k a nonnegative integer $(-1)^n P_G(-k)$ is equal to the sum of the coefficients of the G-traces in $\left(\sum_{x\in C}(-1)^{l(x)}x\right)^{-k}$, which by (i) of Lemma 4.1 is

$$\left(\sum_{w\in M} w\right)^k. \tag{4}$$

Now the coefficient of a G-trace t in (4) is the number of k-tuples $(w_1, w_2, \dots, w_k)$ of traces for which $w_1w_2\cdots w_k = t$. To each such factorization of a G-trace t we associate the pair $(\mathcal{O}, f)$ where $\mathcal{O}$ is the acyclic orientation of G corresponding to t, and the function f is determined by the property that for each vertex v, v occurs in $w_{f(v)}$. This gives a bijection from these factorizations to the pairs $(\mathcal{O}, f)$ in the statement of Theorem 3.2.

A sketch of the case $k = -1$ of this proof can be found in [17, Section 8b]. □

Proof of Theorem 3.4. We choose a vertex v of G and consider the coefficient of a G-trace t in

$$\left(\sum_{x\in C_v}(-1)^{l(x)-1}x\right)\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{k-1}, \tag{5}$$

where, for now, we take k to be a nonnegative integer. Here $(-1)^{n-1}$ times this coefficient is the number of ordered partitions $(A_0, \dots, A_{k-1})$ of V into k independent sets such that $v \in A_0$ and the acyclic orientation of G corresponding to t is obtained if we direct each edge joining A_i and A_j from A_i to A_j, for all $i < j$. As in the proof of Theorem 3.2 above, the sum of these numbers over all G-traces t is the same as the number of proper colorings of G with colors $0, 1 \dots, k-1$ in which vertex v is colored in color 0, and this is $P_G(k)/k = S_G(k)$. As before, since $S_G(k)$ is a polynomial in k, this interpretation for the coefficient of (5) is valid for all k.

Now let M_v be the set of G-traces in which v is the only source. Then by (i) and (ii) of Lemma 4.1, for k a nonnegative integer, $(-1)^{n-1}S_G(-k)$ is the sum of the coefficients of all G-traces in

$$\left(\sum_{w\in M_v} w\right)\left(\sum_{w\in M} w\right)^k,$$

and this is easily seen to be the same interpretation as that given by Theorem 3.4. □

Proof of Theorem 3.3. Let u and v be two adjacent vertices of G and consider

$$\left(\sum_{x\in C_u}(-1)^{l(x)-1}x\right)\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{k-1}\left(\sum_{z\in C_v}(-1)^{l(z)-1}z\right). \tag{6}$$

As in the previous two proofs, if k is a positive integer, the sum of the coefficients of all G-traces in (6) is $(-1)^n$ times the number of proper colorings of G in colors $0,1,\dots,k$ in which u is colored 0 and v is colored k, and this is $(-1)^n R_G(k)$. As before, this interpretation holds for all k.

Next, we consider $R_G(0)$ and $R_G(-k)$ for k positive separately. First, $(-1)^n R_G(0)$ is the sum of the coefficients of the G-traces in

$$\left(\sum_{x\in C_u}(-1)^{l(x)-1}x\right)\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{-1}\left(\sum_{z\in C_v}(-1)^{l(z)-1}z\right),$$

which by (iii) of Lemma 4.1 (if G has no isolated vertices) is the same as the interpretation given of in Theorem 3.1 (which is the case $k=0$ of Theorem 3.3).

For k positive, $(-1)^n R_G(-k)$ is the sum of the coefficients of all the G-traces in

$$\left(\sum_{x\in C_u}(-1)^{l(x)-1}x\right)\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{-1} \times\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{-k+1}\times\left(\sum_{y\in C}(-1)^{l(y)}y\right)^{-1}\left(\sum_{z\in C_v}(-1)^{l(z)-1}z\right).$$

Then by (i) and (ii) of Lemma 4.1, we see that this gives the same result as Theorem 3.3 (with no restriction on isolated vertices). □

5. Counting traces by sources and sinks. We now prove Lemma 4.1. In fact, we prove a more general result, since it is just as easy to prove and is of independent interest. A similar result was used in [8] to count acyclic digraphs by sources and sinks.

Lemma 5.1. *Let M be the trace monoid of a graph G. For any independent set S of vertices of G, let C_S to be the set of all products of independent sets containing S. Let A and B be independent sets of vertices in G (not necessarily disjoint) and let α and β be indeterminates. Then in the ring $\mathbf{Z}[\alpha,\beta]\langle\langle M\rangle\rangle$ we have the identity*

$$\sum_{x\in C_A}\alpha^{l(x)-|A|}x\cdot\sum_{y\in M}y\cdot\sum_{z\in C_B}\beta^{l(z)-|B|}z=\sum_t(1+\alpha)^{\mathrm{so}(t)}(1+\alpha+\beta)^{\mathrm{is}(t)}(1+\beta)^{\mathrm{si}(t)}t,$$

where the sum on the right is over all traces t for which the set of sources contains A, the set of sinks contains B, no element of $A\cap B$ is isolated, $\mathrm{so}(t)$ *is the number of proper* *sources of t not in A,* $\operatorname{is}(t)$ *is the number of isolated letters of t in neither A nor B, and* $\operatorname{si}(t)$ *is the number of proper sinks of t not in B.*

Proof. It is clear that in any trace appearing in the product, the set of sources contains A, the set of sinks contains B, and no element of $A \cap B$ is isolated. If t is such a trace, then every factorization $t = xyz$, where $x \in C_A$, $y \in M$, and $z \in C_B$, can be obtained by choosing for x all the letters in A plus some subset of the sources of t not in A and choosing for z all the letters in B plus some subset of the sinks of t not in B, with the restriction that an isolated letter of t cannot appear in both x and z. Thus each isolated letter of t not in A or B can appear in either x, y, or z, contributing a factor of $1+\alpha+\beta$, each proper source of t not in A can appear in either x or y, contributing a factor of $1+\alpha$, and each proper sink of t not in B can appear in either y or z, contributing a factor of $1+\beta$. □

Proof of Lemma 4.1. For (i), take $A = B = \varnothing$, $\alpha = -1$, and $\beta = 0$ in Lemma 5.1. This gives (with $C = C_\varnothing$)

$$\sum_{x \in C} (-1)^{l(x)} x \cdot \sum_{y \in M} y = \sum_t t,$$

where the sum on the right, is over all traces t in M with no proper sources and no isolated letters. The only such trace is the empty trace, so the sum on the right is equal to 1.

For (ii), take $A = u, B = \varnothing, \alpha = -1$, and $\beta = 0$. Then writing C_u for $C_{\{u\}}$, we have

$$\sum_{x \in C_u} (-1)^{l(x)-1} x \cdot \sum_{y \in M} y = \sum_t t,$$

where the sum on the right is over all traces t for which the set of sources (both proper and isolated) is $\{u\}$. Applying the formula of (i) gives the desired result.

For (iii), we take $A = \{u\}$, $B = \{v\}$, $\alpha = -1$ and $\beta = -1$. Then Lemma 5.1 yields

$$\sum_{x \in C_u} (-1)^{l(x)-1} x \cdot \sum_{y \in M} y \cdot \sum_{z \in C_v} (-1)^{l(z)-1} z = \sum_t (-1)^{\operatorname{is}(t)} t,$$

where the sum is over all traces t in which u is a source, v is a sink, if $u = v$ then u is not an isolated letter of t, every proper source is in $\{u\}$, every proper sink is in $\{v\}$, and $\operatorname{is}(t)$ is the number of isolated letters of t not equal to u or v. In particular, if t has no isolated letters then t appears with coefficient 1 if u is the only source of t and v is the only sink of t, and with coefficient 0 otherwise. □

6. Further comments. If G contains a triangle, then the chromatic polynomial $P_G(k)$ is divisible by $k(k-1)(k-2)$, so it is reasonable to ask whether the polynomial $T_G(k) = P_G(k)/k(k-1)(k-2)$ has the property that $(1-t)^{n-2} \sum_k T_G(k) t^k$ has nonnegative coefficients for some appropriate starting value of k. This is not always the case: if G is the graph on five vertices obtained from the complete graph K_4 by inserting a vertex of degree two in one edge then $P_G(k) = k(k-1)(k-2)(k^2-4k+5)$, so $T_G(k) = k^2-4k+5$, and $(1-t)^3 \sum_k (k^2-4k+5)t^k$ has some negative coefficients for every starting value of k.

Another natural question is whether the polynomials $B_G(t)$ are always unimodal. The answer is no, since if G is a 4-cycle, then $B_G(t) = t + t^3$.

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