LATTICE PATHS AND FABER POLYNOMIALS

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Abstract. The rth Faber polynomial of the Laurent series $f(t) = t + f_0 + f_1/t + f_2/t^2 + \cdots$ is the unique polynomial $F_r(u)$ of degree r in u such that $F_r(f) = t^r +$ negative powers of t. We apply Faber polynomials, which were originally used to study univalent functions, to lattice path enumeration.

1. Introduction. The classical ballot problem (see, e.g., Mohanty [4]) asks for the number $B(m, n)$ of paths from $(1, 0)$ to (m, n) (where $m > n$), with unit steps up and to the right, that never touch the line $x = y$. The number $B(m, n)$ can easily be computed by the recurrence

$$B(m, n) = B(m-1, n) + B(m, n-1) \quad \text{for } m > n \geq 0,\ (m, n) \neq (1, 0),$$

with the initial condition $B(1, 0) = 1$ and the boundary conditions $B(m, -1) = 0$ and $B(m, m) = 0$ for all $m \geq 0$. Displaying these values on the corresponding lattice points, we have the following array, showing $B(m, n)$ for $m \geq n \geq 0$:

5						0
4					0	14
3				0	5	14
2			0	2	5	9
1		0	1	2	3	4
0	0	1	1	1	1	1
n/m	0	1	2	3	4	5

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Let us now extend the values of $B(m,n)$ to the region in which $n > m \geq 0$ so that the same recurrence is satisfied; this can be done in only one way, since we may write the recurrence as $B(m,n-1) = B(m,n) - B(m-1,n)$. We obtain the following array:

$$\begin{array}{rrrrrr}
-1 & -4 & -9 & -14 & -14 & 0 \\
-1 & -3 & -5 & -5 & 0 & 14 \\
-1 & -2 & -2 & 0 & 5 & 14 \\
-1 & -1 & 0 & 2 & 5 & 9 \\
-1 & 0 & 1 & 2 & 3 & 4 \\
0 & 1 & 1 & 1 & 1 & 1
\end{array}$$

We observe that the recurrence $B(m,n) - B(m-1,n) - B(m,n-1) = 0$ is now satisfied for all $m, n \geq 0$ except $(m,n) = (1,0)$ and $(m,n) = (0,1)$, as long as we take $B(m,n)$ to be 0 for $m < 0$ or $n < 0$. In terms of generating functions, the recurrence and initial conditions are equivalent to the formula

$$(1 - x - y) \sum_{m,n=0}^{\infty} B(m,n) x^m y^n = x - y,$$

which gives

$$\sum_{m,n=0}^{\infty} B(m,n) x^m y^n = \frac{x-y}{1-x-y}. \tag{1}$$

Following MacMahon, we may call (1) a "redundant generating function," since it contains some terms which are not part of the solution of the original problem.

From (1) we may derive the well-known formula for the ballot numbers,

$$B(m,n) = \binom{m+n-1}{m-1} - \binom{m+n-1}{m} = \frac{m-n}{m+n}\binom{m+n}{m}. \tag{2}$$

There is a gap in our derivation of (1). It is clear that the numbers $B(m,n)$ defined by (1) do indeed have the property that for $m > n \geq 0$,

$$B(m,n) - B(m-1,n) - B(m,n-1) = \begin{cases} 1 & \text{if } (m,n) = (1,0) \\ 0 & \text{otherwise} \end{cases}$$

However, we have not yet proved that the boundary condition $B(m,m) = 0$ is satisfied. This follows easily from the explicit formula (2), or from the fact that the generating function (1) is anti-symmetric. The proof that the coefficients of (1) are indeed the solution to our problem is now complete.

By exactly the same reasoning, we find that for any positive integer r and any nonnegative integers $m > n \geq 0$, the number of paths from $(r,0)$ to (m,n) that never touch the line $x = y$ is the coefficient of $x^m y^n$ in $(x^r - y^r)/(1 - x - y)$.

We can try a similar approach to paths that begin at $(1,0)$ and stay below the line $y = x/2$. Here the recurrence is again $C(m,n) = C(m-1,n) + C(m,n-1)$, but the boundary condition is $C(2n, n) = 0$. Extending the recurrence to the region $m < 2n$, we obtain the following array:

$$\begin{array}{ccccccc}
-2 & -5 & -8 & -10 & -10 & -7 & 0 \\
-2 & -3 & -3 & -2 & 0 & 3 & 7 \\
-2 & -1 & 0 & 1 & 2 & 3 & 4 \\
0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

As before, we find that the extended function $C(m, n)$, with $C(m, n) = 0$ for $m < 0$ or $n < 0$, satisfies the recurrence $C(m, n) = C(m-1, n) + C(m, n-1)$ everywhere except when (m, n) is $(1, 0)$ or $(0, 1)$, and thus the generating for the extended function is apparently

$$\frac{x - 2y}{1 - x - y}, \tag{3}$$

from which we may derive the formula

$$C(m, n) = \binom{m+n-1}{m-1} - 2\binom{m+n-1}{m} = \frac{m-2n}{m+n}\binom{m+n}{n}.$$

To complete the proof we must show that the coefficient of $x^{2n}y^n$ in (3) is indeed zero. Although this may be seen from the explicit formula for the coefficients, we use a different method that we will need later on. Let us substitute xt for x and y/t^2 for y in (3). Then it suffices to show that the constant term in t in

$$\frac{xt - 2y/t^2}{1 - xt - y/t^2},$$

when expanded as a power series in x and y, is zero. But

$$\frac{xt - 2y/t^2}{1 - xt - y/t^2} = t\frac{d}{dt}\log\frac{1}{1 - xt - y/t^2},$$

and since the coefficient of $1/t$ in the derivative with respect to t of a Laurent series in t is 0, the conclusion follows.

Note that this approach cannot easily be applied to paths that are required to stay below the line $y = 2x$: here we would require the boundary conditions $C(m, 2m) = 0$ and $C(m, 2m+1) = 0$, and this is not so easily achieved. However, there is no problem with paths starting at $(1, 0)$ that stay below the line $x = py$, where p is a positive integer, and we find in this case the generating function $(x - py)/(1 - x - y)$.

We now consider one final example before embarking on the general case. Suppose we want to count paths from $(3, 0)$ to (m, n) that stay below the line $y = x/2$, where $m > 2n$. The same recurrence is satisfied, and as before, we may extend its solution into the region where $m < 2n$, obtaining the following array:

$$\begin{array}{ccccccc}
-2 & -7 & -15 & -25 & -35 & -42 & -42 \\
-2 & -5 & -8 & -10 & -10 & -7 & 0 \\
0 & -3 & -3 & -2 & 0 & 3 & 7 \\
0 & 0 & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

The recurrence is satisfied except at the points $(3,0)$, $(1,2)$, and $(0,3)$, so the generating function is apparently

$$\frac{x^3-3xy^2-2y^3}{1-x-y}. \tag{4}$$

To prove this we must show that the coefficient of $x^{2n}y^n$ in (4) is zero, and we can do it exactly as in the previous example: we replace x with xt and y with y/t^2. Then we have

$$\frac{x^3t^3-3xy^2/t^3-2y^3/t^6}{1-xt-y/t^2}=t\frac{d}{dt}\left(\log\frac{1}{1-S(t)}-S(t)-S(t)^2/2\right), \tag{5}$$

where $S(t)=xt+y/t^2$, so the constant term in t in (5) is zero.

In the remainder of this paper, we shall explain the general theory of which (5) is a special case. It will turn out that the numerator polynomials are closely related to certain polynomials called *Faber polynomials* which have been studied in connection with univalent functions [6]; see also [1, 3, 7]. Faber polynomials were first applied to lattice path enumeration, in the special case we consider in Section 5, in [5].

2. Faber polynomials.

Let

$$f(t)=t+f_0+\frac{f_1}{t}+\frac{f_2}{t^2}+\cdots.$$

In the literature on Faber polynomials, the f_i are complex numbers, and the series converges in some neighborhood of infinity. However, for our applications we take t and the f_i to be indeterminates; i.e., we work in the ring of formal Laurent series $\mathbf{C}[[t,f_0,f_1/t,f_2/t^2,\ldots]]$.

Let $F(u)$ be a polynomial in u of degree r such that

$$F(f)=t^r+\text{negative powers of } t.$$

We say that $F(u)$ is a *Faber polynomial* of f. It is easy to prove by induction that there is exactly one Faber polynomial $F_r(u)$ of degree r, which we call the *rth Faber polynomial of f*

For example, we have $F_1(u)=u-f_0$ and $F_2(f)=u^2-2f_0u+(f_0^2-2f_1)$.

M. Schiffer [6] gave the generating function

$$\log\frac{f(v)-u}{v}=-\sum_{r=1}^{\infty}F_r(u)\frac{v^{-r}}{r}. \tag{6}$$

If we set $f(v)=vh(1/v)$, so that $h(w)=1+\sum_{i=0}^{\infty}f_iw^{i+1}$ is a power series in w, then (6) may be rewritten in terms of formal power series as

$$\log\bigl(h(w)-uw\bigr)=-\sum_{r=1}^{\infty}F_r(u)\frac{w^r}{r}. \tag{7}$$

Expanding (6) or (7) gives the explicit formula

$$F_r(u)=\sum_{i=0}^{r}u^i\sum_{i+j_0+2j_1+3j_2\cdots=r}(-1)^{j_0+j_1+\cdots}r\frac{(i-1+j_0+j_1+\cdots)!}{i!\,j_0!\,j_1!\cdots}f_0^{j_0}f_1^{j_1}\cdots.$$

3. Counting paths.

Suppose we are given nonnegative integers r, k, and n and a subset S of the set $\{1, 0, -1, -2, \cdots\}$. We call the elements of S *steps*. We want to count sequences $(s_1, s_2, \ldots, s_n)$ of elements of S such that every partial sum $r + s_1 + s_2 + \cdots + s_i$ is positive and $r + s_1 + s_2 + \cdots + s_n = k$. We call such a sequence of steps a *good path of length n from r to k*. The ballot problem is equivalent to the case $S = \{1, -1\}$, with $r = 1$, and the other problems discussed in Section 1 are all equivalent to specializations of the case $S = \{1, -p\}$ for various values of p, r, and k.

It is convenient to consider a somewhat more general problem: We take as our set of steps the entire set $\{1, 0, -1, -2, \cdots\}$, but we assign to each path $(s_1, s_2, \ldots, s_n)$ the weight $f_{-s_1} f_{-s_2} \cdots f_{-s_n}$, where f_0, f_1, f_2, $\ldots$ are indeterminates and (to make all our formulas simpler) $f_{-1} = 1$. First we note that the weight of a path determines the number of steps, so taking $f_{-1} = 1$ does not lose any information.

Lemma 1. *A path from r to k with weight $f_0^{j_0} f_1^{j_1} \cdots$ has $k - r + j_1 + 2j_2 + \cdots$ steps equal to 1, and length $k - r + j_0 + 2j_1 + \cdots$.*

Proof. Let j_{-1} be the number of steps equal to 1. Since the path is from r to k, we have $r + j_{-1} - 0j_0 - 1j_1 - 2j_2 - \cdots = k$, and the first assertion follows. Then the length of the path is $j_{-1} + j_1 + j_2 + \cdots = k - r + j_0 + 2j_1 + \cdots$.

We now fix r throughout the following discussion. Let $G(n, k)$ be the sum of the weights of all good paths of length n from r to k. Thus the coefficient of $f_0^{j_0} f_1^{j_1} \cdots$ in $G(n, k)$ is the number of good paths of length n from r to k with j_0 steps equal to 0, j_1 steps equal to -1, and so on.

The following is clear:

Lemma 2.

(i) *$G(n, 0) = 0$ for all n.*
(ii) *$G(0, p) = 1$ and $G(0, k) = 0$ for $k \neq p$.*
(iii) *For $n > 0$, $G(n, k) = \sum_{i=-1}^{\infty} f_i G(n - 1, k + i)$.*

Moreover, $G(n, k)$ is uniquely determined by conditions (i)–(iii).

Now let us define

$$G_k = \sum_{n=0}^{\infty} G(n, k).$$

By Lemma 1, we can recover $G(n, k)$ from G_k as the sum of all terms in G_k in $f_0^{j_0} f_1^{j_1} \cdots$, where $k - r + j_0 + 2j_1 + \cdots = n$.

Now let $f(t)$, as in Section 2, be

$$f(t) = t + f_0 + \frac{f_1}{t} + \frac{f_2}{t^2} + \cdots .$$

Lemma 3. *Let $N(t)$ be a Laurent series in t such that*

(a) *$N(t) = t^r +$ negative powers of t*
(b) *$[t^0] N(t) / \big(1 - f(t)\big) = 0$.*

Then for $k > 0$, $G_k = [t^k]N(t)/\bigl(1 - f(t)\bigr)$.

Proof. Suppose that the hypotheses of the lemma are satisfied. For $k \geq 0$, let

$$g_k = [t^k]\frac{N(t)}{1 - f(t)}$$

and for each integer n, let $g(n,k)$ be the sum of all terms in g_k in $f_0^{j_0} f_1^{j_1} \cdots$, where

$$k - r + j_0 + 2j_1 + \cdots = n. \tag{8}$$

By Lemma 2, it suffices to show

(i) $g(n, 0) = 0$ for all n.
(ii) $g(0, r) = 1$ and $g(0, k) = 0$ for $k \neq r$.
(iii) For $n > 0$, $g(n, k) = \sum_{i=-1}^{\infty} f_i g(n - 1, k + i)$.

First note that (i) follows immediately from (b). By the definition of g_k, we have

$$\frac{N(t)}{1 - f(t)} = \sum_{k=1}^{\infty} g_k t^k + t^{-1} R(t),$$

where $R(t)$ is a power series in t^{-1}. Multiplying both sides by $1 - f(t)$ we get

$$N(t) = \bigl(1 - f(t)\bigr) \sum_{k=1}^{\infty} g_k t^k + S(t),$$

where $S(t) = \bigl(1 - f(t)\bigr)t^{-1}R(t)$ is a power series in t^{-1}. Equating coefficients of t^k for $k > 0$ on both sides and using (a), we obtain

$$g_k - \sum_{i=-1}^{\infty} f_i g_{k+i} = \begin{cases} 1, & \text{if } k = r \\ 0, & \text{if } k \neq r. \end{cases} \tag{9}$$

Extracting the terms in $f_0^{j_0} f_1^{j_1} \cdots$, where $k - r + j_0 + 2j_1 + \cdots = n$, we obtain

$$g(n,k) - \sum_{i=-1}^{\infty} f_i g(n - 1, k + i) = \begin{cases} 1, & \text{if } k = r \text{ and } n = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

since the nonzero case of (9) contributes to (10) only when $k = r$ and $j_0 = j_1 = \cdots = 0$. This proves (iii). Finally, (ii) will follow from the $n = 0$ case of (10) once we show that $g(-1, k) = 0$ for all k. We show in fact that $g(n, k) = 0$ for all $n < 0$: It is clear from (8) that $g(n, k) = 0$ for $n < -r$. It then follows from (10) by induction on n that $g(n, k) = 0$ for all negative n. Thus (ii) holds.

Theorem 1. *G_k is the coefficient of t^k in*

$$\frac{t}{r}\frac{d}{dt}F_r(f)/(1-f),$$

where $F_r(u)$ is the rth Faber polynomial of f.

Proof. It follows from the definition of Faber polynomials that

$$\frac{t}{r}\frac{d}{dt}F_r(f) = t^r + \text{negative powers of t}.$$

In view of the lemma, it is sufficient to show that

$$\frac{d}{dt}F_r(f)/(1-f),$$

is the derivative of some Laurent series in t, since this will imply that it has no term in t^{-1}.

Suppose that $F_r(u) = \sum_{i=0}^{r} c_i u^i$. Then

$$\frac{d}{dt}F_r(f)/(1-f) = \sum_{i=1}^{r} ic_i f^{i-1} f' \sum_{j=0}^{\infty} f^j = \sum_{i=1}^{r}\sum_{j=0}^{\infty} ic_i f^{i+j-1} f'.$$

But $f^{i+j-1}f' = \frac{d}{dt}f^{i+j}/(i+j)$, so

$$\frac{d}{dt}F_r(f)/(1-f) = \frac{d}{dt}\sum_{i=1}^{r}\sum_{j=0}^{\infty}\frac{ic_i}{i+j}f^{i+j}.$$

4. A positivity result.

Let $N_r = \frac{t}{r}\frac{d}{dt}F_r(f)$ be the numerator in Theorem 1. We know that $N_r = t^r - M_r$, where M_r contains only negative powers of t.

Theorem 2. *The coefficients of M_r, as a power series in $t^{-1}, f_0, f_1, \ldots$ are non-negative integers.*

Proof. By setting $u = f(t)$ in Schiffer's formula (6), and then differentiating with respect to t, we obtain

$$\frac{tf'(t)}{f(v)-f(t)} = \sum_{r=1}^{\infty} N_r v^{-r}. \tag{11}$$

Thus

$$\begin{aligned}\sum_{r=1}^{\infty} M_r v^{-r} &= \sum_{r=1}^{\infty}\left[\left(\frac{t}{v}\right)^r - N_r v^{-r}\right]\\ &= \frac{t}{v-t} - \frac{tf'(t)}{f(v)-f(t)}\\ &= t\frac{v-t}{f(v)-f(t)}\left[\frac{f(v)-f(t)}{(v-t)^2} - \frac{f'(t)}{v-t}\right].\end{aligned} \tag{12}$$

We shall show that the last two factors in (12) have positive coefficients when expanded as power series in v^{-1} and t^{-1}. First, we have

$$\frac{f(v)-f(t)}{v-t}=\sum_{i=-1}^{\infty} f_i\left(\frac{v^{-i}-t^{-i}}{v-t}\right)=1-\sum_{i=1}^{\infty} f_i\left(\frac{1}{vt^i}+\frac{1}{v^2t^{i-1}}+\cdots+\frac{1}{v^it}\right).$$

Thus $(v-t)/\big(f(v)-f(t)\big)$ has nonnegative coefficients.

Next, we have

$$\frac{f(v)-f(t)}{(v-t)^2}-\frac{f'(t)}{v-t}=\sum_{i=-1}^{\infty} f_i\left(\frac{v^{-i}-t^{-i}}{(v-t)^2}+\frac{it^{-i-1}}{v-t}\right). \tag{13}$$

The coefficient of f_i in (13) is zero for $i=-1$ and $i=0$. It is easily verified (by multiplying both sides by $(v-t)^2$, for example) that for $i\geq 1$,

$$\frac{v^{-i}-t^{-i}}{(v-t)^2}+\frac{it^{-i-1}}{v-t}=\sum_{j=1}^{i}\frac{j}{v^{i-j+1}t^{j+1}},$$

and thus it follows that the coefficients of (13) are nonnegative. This completes the proof of Theorem 2.

5. Examples.

Let us now return to the problem discussed in the first section: given positive integers r and p, count paths in the plane with steps $(1,0)$ and $(0,1)$, from $(r,0)$ to (m,n), where $m>pn$, that never touch the line $x=py$. (Note that any starting point below the line $x=py$ would give an equivalent problem.) We can convert this problem to an instance of the problem introduced in section 3 by representing a horizontal step by a step equal to 1 and a vertical step by a step equal to $-p$. The transformed path will then go from r to k, where $k=m-pn$. The solution to the transformed problem is then obtained by setting all the f_i to 0 except for f_p in the general solution given in Theorem 1, where the weight of the transformed path is f_p^n. Explicitly, the required number is the coefficient of $t^{m-pn}f_p^n$ in

$$\frac{t}{r}\frac{d}{dt}F_r(t+f_p/t^p)/(1-t-f_p/t^p), \tag{14}$$

where the Faber polynomials $F_r(u)$ are given from (7) by

$$\begin{aligned}\sum_{r=1}^{\infty}F_r(u)\frac{w^r}{r}&=-\log(1+f_pw^{p+1}-uw)\\&=\sum_{j=1}^{\infty}(uw-f_pw^{p+1})^j/j\\&=\sum_{j=1}^{\infty}\sum_{i=0}^{j}\frac{(-1)^i}{j}\binom{j}{i}(f_pw^{p+1})^i(uw)^{j-i}\\&=\sum_{j=1}^{\infty}\sum_{i=0}^{j}\frac{(-1)^i}{j}\binom{j}{i}f_p^iw^{pi+j}u^{j-i}.\end{aligned}$$

Setting $j = r - pi$ and equating coefficients of w^r, we obtain

$$\frac{F_r(u)}{r} = \sum_{i \le r/(p+1)} \frac{(-1)^i}{r-pi}\binom{r-pi}{i} f_p^i u^{r-(p+1)i},$$

and thus the numerator in (14) is

$$\begin{aligned}
&\frac{t}{r}F_r'(t+f_p/t^p)(1-pf_p/t^{p+1})\\
&\qquad = (t-pf_p/t^p)\sum_{i\le r/(p+1)}(-1)^i\frac{r-(p+1)i}{r-pi}\binom{r-pi}{i}f_p^i(t+f_p/t^p)^{r-(p+1)i-1}\\
&\qquad = (t-pf_p/t^p)\sum_{i< r/(p+1)}(-1)^i\binom{r-pi-1}{i}f_p^i(t+f_p/t^p)^{r-(p+1)i-1} \qquad (15)
\end{aligned}$$

To recover a generating function in x and y, as in section 1, we substitute x for t and $x^p y$ for f_p. Then (14) and (15) give as the redundant generating function for our problem

$$\frac{x-py}{1-x-y}\sum_{i<r/(p+1)}(-1)^i\binom{r-pi-1}{i}(x^py)^i(x+y)^{r-(p+1)i-1}. \qquad (16)$$

For example, if we take $p = 2$ and $r = 3$, (16) gives

$$\frac{x-2y}{1-x-y}(x+y)^2 = \frac{x^3-3xy^2-2y^3}{1-x-y},$$

as we obtained in (4).

Now let (16) equal $\widetilde{N}_r/(1-x-y)$ and let $\widetilde{N}_r = x^r - \widetilde{M}_r$. Then $\widetilde{N}_r$ and $\widetilde{M}_r$ can be obtained from N_r and M_r as defined in section 4 by the appropriate substitution. Since it is clear from (16) that $\widetilde{N}_r$ and $\widetilde{M}_r$ are homogeneous of degree r in x and y, they can be obtained from the generating functions $\sum_r \widetilde{N}_r$ and $\sum_r \widetilde{M}_r$. The formulas in the proof of Theorem 2 give

$$\sum_{r=1}^{\infty}\widetilde{N}_r = \frac{x-py}{1-x-y+x^py} \qquad (17)$$

and

$$\sum_{r=1}^{\infty}\widetilde{M}_r = y\frac{p+(p-1)x+(p-2)x^2+\cdots+x^{p-1}}{1-y(1+x+x^2+\cdots+x^{p-1})}. \qquad (18)$$

For $p = 1$, (18) gives $\widetilde{M}_r = y^r$, so that $\widetilde{N}_r = x^r - y^r$, as we observed in section 1. We can also obtain a simple explicit formula when $p = 2$. In this case, (18) gives

$$\sum_{r=1}^{\infty}\widetilde{M}_r = y\frac{2+x}{1-y(1+x)} = \sum_{i,j=0}^{\infty}x^iy^{j+1}\left[\binom{i}{j}+\binom{i+1}{j}\right].$$

Extracting the terms that are homogeneous of degree r and simplifying, we obtain

$$\widetilde{M}_r = \sum_{i=0}^{\lfloor r/2 \rfloor} \frac{2r-3i}{r-i} \binom{r-i}{i} x^i y^{r-i}.$$

The method we have described can be used for counting paths in the plane that stay below the line $x = py$, with arbitrary starting and ending points, and an arbitrary set of allowed steps subject only to the condition that every step (i, j) satisfies $i - pj \leq 1$. Another method, also using Laurent series, that is not subject to the restriction on steps, but does not allow an arbitrary starting point, is described in [2].

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