LAGRANGE INVERSION FOR SPECIES

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1. Introduction. The Lagrange inversion formula is one of the fundamental results of enumerative combinatorics. It expresses the coefficients of powers of the compositional inverse of a power series in terms of the coefficients of powers of the original power series. G. Labelle [10] extended Lagrange inversion to cycle index series, which are equivalent to symmetric functions. Although motivated by Joyal's theory of species of structures [7], Labelle's proof was algebraic, and was based on the ordinary multivariable Lagrange inversion formula. We give here a new proof of this formula in the context of the theory of species. In contrast with the proof given in [10], the bijections involved are all natural in the categorical sense of the word. Our approach involves several new or little-known operations on species, some of which were studied earlier by Joyal [9], and which have other enumerative applications.

A different approach to Lagrange inversion for species has been taken by Ehrenborg and Mendez [5].

The basic combinatorial fact underlying the proof is very simple. Let U be a finite set and W a subset of U. Then a functional digraph from $U-W$ to U has weakly connected components of two types: trees with roots in W and other vertices in $U-W$, and cycles of trees with every vertex in $U-W$. (See Figure 1.)

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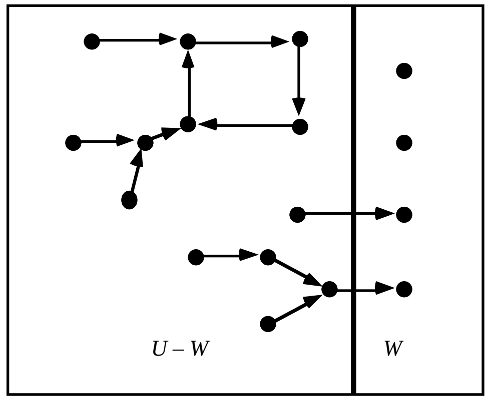

Figure 1

In essence, the same combinatorial fact underlies Gessel's combinatorial proof of the multivariable Lagrange inversion formula [6]. The approach we take here can also be used to derive a multivariable Lagrange inversion formula for species.

To get our result we need to construct these digraphs in a way that allows us to keep track of the preimage of each vertex. First we construct functions from $U - W$ to an arbitrary set V by partitioning $U - W$ and assign each block to an element of V. This construction is easily described by standard operations on "two-sorted species." Next we apply a "diagonal operator" which identifies V with U. This is a simple operation on species, but seems not to have been studied before.

Each operation on species that we use corresponds in a simple way to an operation on cycle index series, and this correspondence enables us to derive Labelle's formulas for cycle indexes from our formulas for species.

2. Lagrange inversion. Suppose that $r(x)$ is a formal power series in the variable x. Then the Lagrange inversion formula asserts that there is a unique formal power series $a(x)$ satisfying $a(x) = xr\big(a(x)\big)$, and that for any formal power series $f(x)$ and $g(x)$,

$$\left[\frac{x^n}{n!}\right] \frac{f\big(a(x)\big)}{1 - xr'\big(a(x)\big)} = \left[\frac{y^n}{n!}\right] f(y)r(y)^n. \tag{1}$$

and

$$\left[\frac{x^n}{n!}\right] g\big(a(x)\big) = \left[\frac{y^n}{n!}\right] g(y) \left(1 - \frac{yr'(y)}{r(y)}\right) r(y)^n. \tag{2}$$

Here $[z^n/n!]\, w(z)$ denotes the coefficient of $z^n/n!$ in $w(z)$. We can obtain (2) from (1) by setting $f(y) = g(y)\big(1 - yr'(y)/r(y)\big)$, since $a(x)/r\big(a(x)\big) = x$, and similarly (1) can be obtained from (2). Thus to prove both (1) and (2) it is sufficient to prove either one of them. We shall prove a species generalization of (1) and derive from it a virtual species generalization of (2).

Let us restate (1) in what may be appear to be a more complicated form, but which suggests the combinatorial proof we shall give. We define the *diagonal operator* ∇ from formal power series in x and y to formal power series in x by

$$\nabla\left(\sum_{m,n=0}^{\infty} a_{m,n} \frac{x^m}{m!}\frac{y^n}{n!}\right) = \sum_{n=0}^{\infty} a_{n,n}\frac{x^n}{n!}.$$

Then we may write (1) as

$$\frac{f\big(a(x)\big)}{1 - xr'\big(a(x)\big)} = \nabla\left(f(y)e^{xr(y)}\right). \tag{3}$$

This formulation of Lagrange inversion was given by Strehl [17].

It may be helpful to place the diagonal operator ∇ in an algebraic context. Let us consider the multiplication $\times$ on formal power series given by

$$\sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \times \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} b_n c_n \frac{x^n}{n!}. \tag{4}$$

Then the vector space of formal power series in x with the multiplication $\times$ is an algebra. An algebra on a vector space P is often described as a linear map $P \otimes P \to P$, where $\otimes$ denotes the tensor product. If we take P to be the vector space of formal power series in x, then we may identify $P \otimes P$ with the vector space of formal power series in x and y. Then ∇ is the map $P \otimes P \to P$ corresponding to $\times$.

3. Species. We refer the reader to [7] for the basic definitions of the theory of species. We recall here the following standard notations: If F is a species and U is a finite set then $F[U]$ (with square brackets) denotes the set of all F-structures on the (underlying) set U, while $F = F(X)$ (with parentheses) means that the points of the underlying sets of F-structures are declared to be of "sort" X. The corresponding notations $F[U, V, \dots]$ and $F = F(X, Y, \dots)$ are reserved for species whose structures live on underlying sets having several sorts X, Y, … of points. The combinatorial operations of sum (+), product ($\cdot$ or juxtaposition), differentiation ($'$), and substitution ($\circ$) can be used to define new species in terms of those already defined.

With this in mind, we now give a species interpretation to (3). Let R be any species and let X be the species of singletons ($X[U] = \{U\}$ if $|U| = 1$ and $X[U] = \emptyset$ otherwise). Then there is a unique species A satisfying $A = XR(A)$; A is the species of R-enriched rooted trees [7, 11]. An R-enriched rooted tree on a set U consists of

(1) a root $z \in U$
(2) a function $\phi : U - \{z\} \to U$ such that for each $u \in U - \{z\}$, $\phi^k(u) = z$ for some positive integer k
(3) an R-structure on each preimage $\phi^{-1}(u)$ for $u \in U$.

An R-enriched rooted tree is shown in Figure 2.

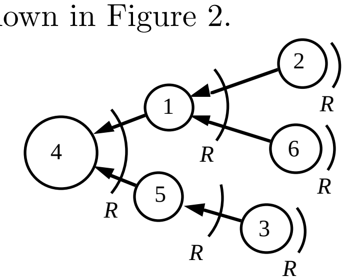

Figure 2

(In subsequent figures we omit the arrows since the figures are unambiguous without them.)

Now let E be the species of sets and C the species of cycles, and let $S = E \circ C$ be the species of permutations (sets of cycles). Then in the species analog of (3), $S(XR'(A))$ will take the place of $\bigl(1 - xr'(a(x))\bigr)^{-1}$.

Next we describe the species analog of ∇. First let us recall the operation of Cartesian product $\times$ of species: If F and G are two species then for any set U, $(F \times G)[U]$ is the Cartesian product of sets $F[U] \times G[U]$. In analogy with the relation between the operations $\times$ and ∇ on formal power series, we extend the Cartesian product of species to a diagonal operator, which we also denote by ∇, from two-sorted species (i.e., functors from ordered pairs of sets to sets) to (one-sorted) species: Given a two-sorted species $H = H(X, Y)$, we define the species ∇H by

$$(\nabla H)[U] = H[U, U].$$

As one might expect, the diagonal operator generalizes the Cartesian product: Let F and G be ordinary species, and define their "tensor product" $F \otimes G$ to be the two-sorted species given by $(F \otimes G)[U, V] = F[U] \times G[V]$. Then $\nabla(F \otimes G) = F \times G$. (In contrast to the situation for formal power series, it is not true that every two-sorted species is a linear combination of tensor products of one-sorted species.)

Theorem 1. *Let R be any species. Then there is a unique species $A = A(X)$ satisfying $A = XR(A)$, and for any species F,*

$$F(A)S\bigl(XR'(A)\bigr) = \nabla\bigl(F(Y)E(XR(Y))\bigr). \tag{5}$$

Proof. First we show uniqueness. We call a species G *homogeneous of degree* n if $G[U]$ is empty for $|U| \neq n$. It is well known that every species can be uniquely decomposed into homogeneous components. If we equate homogeneous components of degree n in the equation $A = XR(A)$ we find that $A[\emptyset] = \emptyset$ and for $n > 0$ the homogeneous component of A of degree n is uniquely determined by the homogeneous components of lower degree.

To prove (5) we examine the species $H = \nabla\bigl(F(Y)E(XR(Y))\bigr)$. First we note that $\bigl(XR(Y)\bigr)[U, V]$ is empty unless $|U| = 1$. If $U = \{u\}$ then an element of $\bigl(XR(Y)\bigr)[U, V]$ consists of u together with an R-structure on V. We can think of this as the (unique) function from V to U together with an R-structure on the (unique) preimage. Similarly, we can represent $\bigl(E(XR(Y))\bigr)[U, V]$ as the set of all functions from V to U with an R-structure on each preimage, and an element of $\bigl(F(Y)E(XR(Y))\bigr)[U, V]$ consists of a F-structure on a subset W of V, together with a function from $V - W$ to U with an R-structure on each preimage.

Let us consider $H[U] = \bigl(F(Y)E(XR(Y))\bigr)[U, U]$. Each element of $H[U]$ will consist of an F-structure on some subset W of U, together with a function ϕ from $U - W$ to U with an R-structure on each preimage. Now let W^* be the subset of U consisting of all u such that for some nonnegative integer k, $\phi^k(u)$ is in W. (Thus $W \subseteq W^*$.) Then ϕ restricted to $W^* - W$ is an acyclic function from $W^* - W$ to W^*, with an R-structure on each preimage, and with an F-structure on W; in other words an element of $\bigl(F \circ A\bigr)[W^*]$.

The restriction of ϕ to $U - W^*$ is simply an R-enriched endofunction on $U - W^*$. The digraph of such an endofunction is a set of weakly connected components, each of which contains a single cycle. It is not difficult to see (cf. [7,11] that the species of R-enriched endofunctions with a single cycle is $C\big(XR'(A)\big)$ and thus the species of all R-enriched endofunctions is $E(C(XR'(A)) = S\big(XR'(A)\big)$. ◻

Theorem 1 is illustrated by Figures 3 and 4. We can represent an element of

$$\nabla\big(F(Y)E(XR(Y))\big)[U] = \big(F(Y)E(XR(Y))\big)[U,U],$$

where $U = \{1, 2, \dots, 9\}$, as in Figure 3.

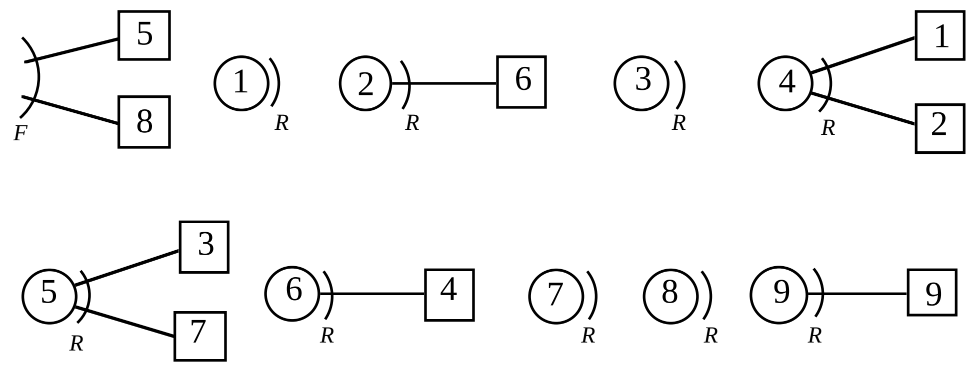

Figure 3

Here each label appears twice: of sort X in a circle and of sort Y in a square. But if we identify both occurrences of each label we obtain Figure 4.

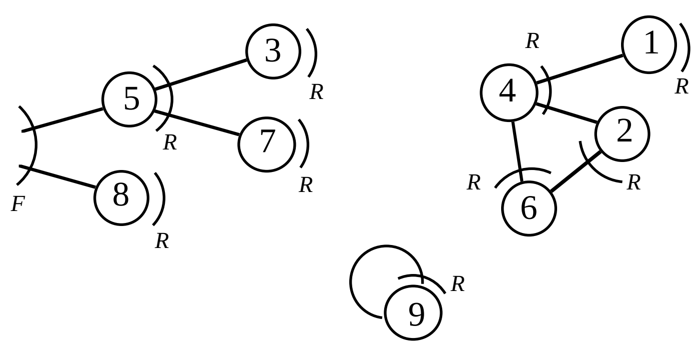

Figure 4

It is important to see that Figures 3 and 4 represent the same object; thus Figure 3 can be recovered from Figure 4.

We would now like to find a species generalization of (2). Unfortunately our result is more complicated than (2) might suggest. We can apply the same argument we used to derive (2) from (1), but to do this we will need to use *virtual species* [8, 9, 18], which are formal differences of species; thus they complete the semiring of species to a ring.

Lemma 2. *Theorem 1 is valid if F and R are virtual species.*

Proof. Since both sides of (5) are linear in F, we may replace F by a difference of two species. Now set $R = R_1 + nR_2$, where R_1 and R_2 are species and n is a nonnegative integer. Then the homogeneous components of both sides of (5) are polynomials in n, so we are justified in setting $n = -1$. $\square$

Now let G be an arbitrary virtual species. Applying Lemma 2 with

$$F(X) = G(X) \cdot S(XR'(X)R(X)^{-1})^{-1},$$

we obtain the virtual species analog of (2):

Theorem 3. *Let R be any virtual species. Then there is a unique virtual species $A = A(X)$ satisfying $A = XR(A)$, and for any virtual species G,*

$$G(A) = \nabla\big(G(Y) \cdot S(YR'(Y)R(Y)^{-1})^{-1} \cdot E(XR(Y))\big). \quad \square$$

4. Cycle indexes and symmetric functions.

Let $\lambda = (1^{m_1}2^{m_2}\dots)$ be a partition of the integer $n = \sum_i im_i$. We define the *power sum symmetric function* p_λ in the infinitely many variables x_1, x_2, ... to be $p_1^{m_1}p_2^{m_2}\cdots$, where $p_i = \sum_{j=1}^{\infty} x_j^i$. We denote by z_λ the integer $1^{m_1}m_1!\,2^{m_2}m_2!\cdots$. Note that $n!/z_\lambda$ is the number of permutations in the symmetric group S_n of cycle type λ. We denote by $\mathbf{n}$ the set $\{1, 2, \dots, n\}$, and similarly for other bold-face letters.

Now let F be a species. Any permutation $\pi : \mathbf{n} \to \mathbf{n}$ induces by functoriality a permutation $F[\pi] : F[\mathbf{n}] \to F[\mathbf{n}]$. Thus we have an action of the symmetric group S_n on $F[\mathbf{n}]$. We define $\operatorname{fix} F[\lambda]$ to be the number of elements of $F[\mathbf{n}]$ fixed by a permutation in S_n of cycle type λ. We define the *cycle index* of F to be the symmetric function

$$Z(F) = \sum_\lambda \operatorname{fix} F[\lambda] \frac{p_\lambda}{z_\lambda}. \tag{6}$$

If F is a molecular species of degree n, so that $F[\mathbf{n}]$ can be identified with the set of left cosets of some subgroup of S_n, then $Z(F)$ is just the classical cycle index polynomial of this subgroup. More generally, if F is homogenous of degree n, then $Z(F)$ is the “Frobenius characteristic” of the corresponding permutation representation of S_n.

We extend the definition of the cycle index to virtual species by linearity. It should be pointed out that in most of the literature on species and graphical enumeration our power sum p_i in the definition of a cycle index is replaced by a variable, usually x_i, z_i, or s_i, and the cycle index is considered to be a formal power series in these variables rather than a symmetric function (although the connection between cycle indexes and symmetric functions is well-known). However, we take the view (as did Redfield [15]) that the cycle index *is* a symmetric function. We use the notation of Macdonald [14] for symmetric functions.

It is well known that if F and G are species then $Z(FG) = Z(F)Z(G)$. Moreover, if the composition $F \circ G$ is defined then $Z(F \circ G) = Z(F) \circ Z(G)$, where $\circ$ on the right is the operation of *composition* (also called *plethysm*) for symmetric functions (see, e.g., Macdonald [14, pp. 65–68]). A slight generalization of this operation, which we call *substitution*, will be useful to us, especially when we work with symmetric functions in more than one set of variables: if $f = f(x_1, x_2, \dots)$ is a symmetric function and g is a sum of terms of the form

$u_1^{n_1}u_2^{n_2}\cdots$, where the u_j are variables (either the x_i or other indeterminates) then $f\circ g$, which we often write as $f(g)$, is obtained by substituting the terms of g for the x_i in f. (Each term in g must have coefficient 1, thus $2u^2v^3$ must be written as $u^2v^3+u^2v^3$.) In the general case (in which g may have negative or nonintegral coefficients), substitution may be reduced to the case of $p_m\circ g$ by the properties $(e+f)\circ g=e\circ g+f\circ g$ and $(ef)\circ g=(e\circ g)(f\circ g)$. To define $p_m\circ g$, if $g=\sum c_it_i$, where the t_i are terms and the c_i are their coefficients, we set $p_m\circ g=\sum c_it_i^m$. If g is a symmetric function in the x_i then we also have $p_m\circ g=g\circ p_m$, and $p_m\circ p_n=p_{mn}$. We note that for fixed g, the mapping $f\mapsto f\circ g$ is a homomorphism.

One further convention will be useful. In working with symmetric functions in the variables x_1, x_2, it is convenient to denote by x the sum $x_1+x_2+\cdots=p_1(x)$. Then $f(x)$, interpreted as a substitution, is exactly the same as $f(x_1,x_2,\dots)$. We make the same convention for other variables. Thus, for example, xy denotes $\sum_{i,j}x_iy_j$, and $f(xy)$ denotes $f(x_1y_1,\dots,x_iy_j,\dots)$.

There is another well-known operation on symmetric functions called the "inner," "internal," or "Kronecker" product, which is sometimes denoted by $*$; however for consistency we shall denote it by $\times$. It is defined by

$$\sum_\lambda a_\lambda\frac{p_\lambda}{z_\lambda}\times\sum_\mu b_\mu\frac{p_\mu}{z_\mu}=\sum_\lambda a_\lambda b_\lambda\frac{p_\lambda}{z_\lambda}.$$

Then $Z(F\times G)=Z(F)\times Z(G)$.

Corresponding to the derivative of a species is an operation on symmetric functions, which we also denote by a prime: $Z(F')=Z(F)'$. It is most easily defined by $f'=\partial f/\partial p_1$, where f is expressed in terms of the p_i. This operator is a special case of operators on symmetric functions called Hammond operators that we will discuss in the next section.

When we discuss symmetric functions in several sets of variables we shall write $f(x)$, $f(y)$, etc. to denote the symmetric function f in the variables x_i, y_i, etc. (When no variables are mentioned, we mean symmetric functions in the x_i.) If $H(X,Y)$ is a two-sorted species, then it has a cycle index which is a symmetric function in two sets of variables: we define the cycle index of H to be

$$Z(H)=\sum_{\lambda,\mu}\operatorname{fix}H[\lambda,\mu]\frac{p_\lambda(x)}{z_\lambda}\frac{p_\mu(y)}{z_\mu},$$

where for each pair of partitions λ of l and μ of m, $\operatorname{fix}H[\lambda,\mu]$ is the number of elements of $H[\mathbf{l},\mathbf{m}]$ fixed by an element (π,σ) of $S_l\times S_m$ in which π has cycle type λ and σ has cycle type μ. We may now define the mapping ∇ from symmetric functions in two sets of variables (the x's and y's) to symmetric functions in one set of variables (the x's) by

$$\nabla\Bigl(\sum_{\lambda,\mu}a_{\lambda,\mu}\frac{p_\lambda(x)}{z_\lambda}\frac{p_\mu(y)}{z_\mu}\Bigr)=\sum_\lambda a_{\lambda,\lambda}\frac{p_\lambda(x)}{z_\lambda}.$$

It is clear that if f and g are symmetric functions of one set of variables, then

$$\nabla\bigl(f(x)g(y)\bigr)=f(x)\times g(x),$$

and moreover, if H is a two-sorted species, then $Z(\nabla(H)) = \nabla(Z(H))$.

We would like to take the cycle index of both sides of (5). First, let $s(x) = Z(S)$. It is well known that

$$s(x) = \prod_{i=1}^{\infty} \frac{1}{1 - p_i(x)}.$$

Now let $r(x) = Z(R)$, $a(x) = Z(A)$, and $f(x) = Z(F)$. We first determine $Z\big(E(XR(Y))\big)$. We have $Z(E) = \exp(\sum_{n=1}^{\infty} p_n/n)$, and $p_n\big(g(x)h(y)\big) = g\big(p_n(x)\big)h\big(p_n(y)\big)$, so

$$Z\big(E(XR(Y))\big) = \exp\bigg(\sum_{n=1}^{\infty} \frac{p_n(x)r\big(p_n(y)\big)}{n}\bigg).$$

Now let us set $r^{[n]} = r \circ p_n$, so $r^{[n]}$ is obtained from r by replacing each p_i with p_{ni} (or equivalently, replacing each x_j with x_j^n), and for any partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, let us set $r^{[\lambda]} = r^{[\lambda_1]} r^{[\lambda_2]} \cdots = p_\lambda \circ r$. Then

$$Z\big(E(XR(Y)\big) = \sum_{\lambda} \frac{p_\lambda(x) r^{[\lambda]}(y)}{z_\lambda}.$$

Thus

$$Z\big(F(Y)E(XR(Y))\big) = \sum_{\lambda} \frac{p_\lambda(x)}{z_\lambda} f(y) r^{[\lambda]}(y),$$

and so

$$Z\big(\nabla\big(F(Y)E(XR(Y))\big)\big) = \sum_{\lambda} c_{\lambda\lambda} \frac{p_\lambda(x)}{z_\lambda},$$

where $f(y)r^{[\lambda]}(y) = \sum_\mu c_{\lambda\mu} p_\mu(y)/z_\mu$.

We can now deduce the analog of (1) for symmetric functions, and deduce from it in the same way the analog of (2). We are using the fact that Theorem 1 is valid for virtual species and that any symmetric function is the cycle index of some virtual species. (Recall that x means $p_1(x) = x_1 + x_2 + \cdots$.)

Theorem 4. *(Labelle [10]) Let $r(x)$ be a symmetric function in $x_1, x_2, \ldots$. Then there is a unique symmetric function $a(x)$ satisfying $a = xr(a)$, and for any symmetric function $f(x)$,*

$$f(a)s\big(xr'(a)\big) = \sum_{\lambda} c_{\lambda\lambda} \frac{p_\lambda(x)}{z_\lambda},$$

where $s = \prod_{i=1}^{\infty}(1-p_i)^{-1}$ and $f(y)r^{[\lambda]}(y) = \sum_\mu c_{\lambda\mu} p_\mu(y)/z_\mu$. Moreover, for any symmetric function $g(x)$,

$$g(a) = \sum_{\lambda} b_{\lambda\lambda} \frac{p_\lambda(x)}{z_\lambda},$$

where

$$\frac{g(y)}{s\left(yr'(y)/r(y)\right)} r^{[\lambda]}(y) = \sum_{\mu} b_{\lambda\mu} p_\mu(y)/z_\mu. \qquad \square$$

We now give a simple example of Theorem 4. (See also [10] and [3].) Let R be the species E of sets. Then $r(x) = Z(R) = \sum_{n=0}^\infty h_n(x)$, which we write as $h(x)$, where $h_n(x)$ is the nth complete symmetric function. Let us first take $f(x) = 1$. Then $s\bigl(xr'(a)\bigr)$ is the cycle index series for endofunctions. We have $h^{[\lambda]} = h(p_{\lambda_1})h(p_{\lambda_2})\cdots = h(p_{\lambda_1} + p_{\lambda_2} + \cdots)$, since h has the property that $h(f)\cdot h(g) = h(f+g)$. Suppose that $\lambda = (1^{m_1}2^{m_2}\cdots k^{m_k})$. Then

$$\begin{aligned} r^{[\lambda]} &= h(m_1p_1 + m_2p_2 + \cdots) \\ &= \exp\Biggl(\sum_{j=1}^\infty (m_1p_j + m_2p_{2j} + m_3p_{3j} + \cdots)/j\Biggr) \\ &= \exp\Biggl(\sum_{n=1}^\infty \frac{p_n}{n}\sum_{d|n} dm_d\Biggr). \end{aligned}$$

Thus

$$c_{\lambda\lambda} = \prod_{i=1}^k \Biggl(\sum_{d|i} dm_d\Biggr)^{m_i}.$$

More generally, with $f(x) = \sum_{n=0}^\infty t^n h_n$, we have

$$s\bigl(xr'(a)\bigr)\cdot\sum_{n=0}^\infty t^n h_n(a) = \sum_\lambda \prod_{i=1}^k \Biggl(t^i + \sum_{d|i} dm_d\Biggr)^{m_i} \frac{p_\lambda(x)}{z_\lambda},$$

where $\lambda = (1^{m_1}2^{m_2}\cdots k^{m_k})$.

5. Other forms of Lagrange inversion. As noted by Labelle in [10], there are other forms of Lagrange inversion that have symmetric function analogs, and we can find species analogs of them also. We discuss here two that are closely related. First we recall a useful, but little-known, device that enables ordinary one-variable Lagrange inversion to solve equations such as $a(x) = xe^{xa(x)}$ in which x appears more than once, and equations such as $a = (1+xa)(1+ya)(1+za)$ involving more than one variable. (For applications of this idea to multivariable Lagrange inversion, see [6].) We write (1) as

$$\frac{f\bigl(a(x)\bigr)}{1 - xr'\bigl(a(x)\bigr)} = \sum_{n=0}^\infty x^n [y^n]\, f(y)r(y)^n. \tag{7}$$

Now suppose that the coefficients of $r(y)$ are indeterminates. Then the right side of (7) still makes sense if we set $x = 1$. It is easily verified that $\bar{a} = a(1)$ is a well-defined formal power series in the coefficients of r and that $\bar{a}$ is the unique formal power series solution to the equation $\bar{a} = r(\bar{a})$. Thus we may conclude that the unique formal power series solution of $\bar{a} = r(\bar{a})$ satisfies

$$\frac{f(\bar{a})}{1 - r'(\bar{a})} = \sum_{n=0}^{\infty} [y^n]\, f(y) r(y)^n. \tag{8}$$

Note that (7) can be recovered from (8) by replacing $r(y)$ with $xr(y)$.

Finally we note that the coefficients of r need not all be indeterminates; as long as the sum in the right side of (8) converges as a sum of formal power series, (8) will hold. Thus $\bar{a} = 1 + z\bar{a}^2$ and $\bar{a} = z + \bar{a}^2$ are both admissible, but $\bar{a} = 1 - \bar{a} + z\bar{a}^2$ is not. It should be noted that we may lose uniqueness; thus the equation $\bar{a} = z + \bar{a}^2$ has two formal power series solutions (but only one with constant term zero).

To find the species analog of (8), we replace r with a species R on any number of sorts of points. We consider the equation

$$A(X, T^{(1)}, T^{(2)}, \dots) = XR(A, T^{(1)}, T^{(2)}, \dots), \tag{9}$$

where $R = R(Y, T^{(1)}, T^{(2)}, \dots)$ is a species on arbitrarily many sorts Y, $T^{(1)}$, $T^{(2)}$, … of points. The proof of Theorem 1 works exactly as before in this situation; and we deduce that for any (one-sorted) species F,

$$F(A) S\big(XR'(A, T^{(1)}, T^{(2)}, \dots)\big) = \nabla_{XY}\big(F(Y) E(XR(Y, T^{(1)}, T^{(2)}, \dots))\big), \tag{10}$$

where $R'(Y, T^{(1)}, \dots) = \partial R/\partial Y$ and ∇_{XY} denotes the diagonal operation with respect to sorts X and Y, so that $\nabla_{XY} G(X, Y, T^{(1)}, \dots)$ is a species of sorts X, $T^{(1)}$, … . The species analog of setting $x = 1$ in (7) is an operation that we now describe. (See also Joyal [9] for many applications of this operation.)

For a single-sorted species $F(X)$, we define $F(1)$ or $F(X)|_{X=1}$ to be the set of orbits of F. More precisely, suppose first that F is homogeneous of degree n. Then $F(1)$ is the set of orbits of $F[\mathbf{n}]$ under the action of the symmetric group S_n. We extend the definition to general species by linearity, assuming the necessary finiteness conditions.

For a many-sorted species $F(X, T^{(1)}, T^{(2)}, \dots)$ the definition is essentially the same, but we must check that we obtain a species in the remaining sorts. By linearity, we may assume that $F(X, T^{(1)}, T^{(2)}, \dots)$ is homogeneous in X of degree n, i.e., if $F[V, U_1, \dots]$ is nonempty then $|V| = n$. We define the species $F(1, T^{(1)}, T^{(2)}, \dots)$ or $F(X, T^{(1)}, T^{(2)}, \dots)|_{X=1}$ by specifying that for any sets $U_1, U_2, \dots,$ $F(1, T^{(1)}, T^{(2)}, \dots)[U_1, U_2, \dots]$ is the set of orbits of $F[\mathbf{n}, U_1, U_2, \dots]$ under the action of the symmetric group S_n on $\mathbf{n}$. Given bijections $\theta_i : U_i \to V_i$, the bijection

$$F(\mathrm{id}, \theta_1, \theta_2, \dots) : F[\mathbf{n}, U_1, U_2, \dots] \to F[\mathbf{n}, V_1, V_2, \dots]$$

takes orbits to orbits, and thus gives a functorial bijection

$$F(1, T^{(1)}, T^{(2)}, \dots)[U_1, U_2, \dots] \to F(1, T^{(1)}, T^{(2)}, \dots)[V_1, V_2, \dots],$$

so that $F(1,T^{(1)},T^{(2)},\dots)$ is in fact a species in the sorts $T^{(1)}$, $T^{(2)}$, $\dots$.

As observed by Joyal [9], we can use the operation $F(X)\mapsto F(1)$ to generalize the operation of substitution of species to substitutions $F(G)$ where $G[\emptyset]$ need not be $\emptyset$ (though a finiteness restriction is necessary): we define $F(G(X))$ to be $F(YG(X))|_{Y=1}$, and this definition applies more generally if G involves other sorts of points. In particular, with this notion of substitution, $F(1)$ becomes the substitution of the species 1 for X in $F(X)$, where the species 1 is defined by

$$1[U]=\begin{cases} U & \text{if } U=\emptyset\\ \emptyset & \text{if } U\neq\emptyset,\end{cases}$$

and this agrees with our earlier definition of $F(1)$.

By substituting 1 for X in (10), we conclude that the solution of

$$A(T^{(1)},T^{(2)},\dots)=R(A,T^{(1)},T^{(2)},\dots)$$

satisfies

$$F(A)S\big(R'(A,T^{(1)},T^{(2)},\dots)\big)=\nabla_{XY}\big(F(Y)E(XR(Y,T^{(1)},T^{(2)},\dots))\big)\big|_{X=1}, \tag{11}$$

where F is an arbitrary species (that may involve other sorts of points), and

$$R'(Y,T^{(1)},T^{(2)},\dots)=\frac{\partial}{\partial Y}R(Y,T^{(1)},T^{(2)},\dots).$$

The symmetric function analog of (10) is just like Theorem 4: The operation ∇_{xy} on symmetric functions is defined, as one would expect, by

$$\nabla_{xy}\Bigg(\sum_{\lambda,\mu,\nu,\dots} a_{\lambda,\mu,\nu,\dots}\frac{p_\lambda(x)}{z_\lambda}\frac{p_\mu(y)}{z_\mu}\frac{p_\nu(t^{(1)})}{z_\nu}\cdots\Bigg)=\sum_{\lambda,\nu,\dots} a_{\lambda,\lambda,\nu,\dots}\frac{p_\lambda(x)}{z_\lambda}\frac{p_\nu(t^{(1)})}{z_\nu}\cdots .$$

Here $t^{(1)}=t_1^{(1)}+t_2^{(1)}+\cdots$ as usual.

Next we show that the operation that takes a species $F(X,T^{(1)},\dots)$ to $F(1,T^{(1)},\dots)$ corresponds to setting each $p_\lambda(x)$ equal to 1.

To prove this we need the following lemma:

Lemma 5. *Suppose that G and H are finite groups such that their direct product $G\times H$ acts on a finite set S; G and H also act on S as subgroups of $G\times H$. Then H acts on the set of G-orbits of S and for any $h\in H$, the number of G-orbits fixed by h is*

$$\frac{1}{|G|}\sum_{g\in G}\operatorname{fix}(g,h),$$

where $\operatorname{fix}(g,h)$ is the number of elements of S fixed by $(g,h)\in G\times H$. □

This lemma is easily proved similarly to Burnside's lemma, and it reduces to Burnside's lemma when H is a trivial group. For the proof of a slightly more general result, with applications and further references, see Robinson [16]. Another application is given in [4].

Theorem 6. *$Z\big(F(1,T^{(1)},T^{(2)},\cdots)\big)$ is obtained from $Z\big(F(X,T^{(1)},T^{(2)},\cdots)\big)$ by replacing each $p_\lambda(x)$ with 1.*

Proof. By linearity, we may assume that F is homogeneous in X of degree l. Suppose that

$$Z\big(F(X,T^{(1)},T^{(2)},\cdots)\big) = \sum_{\lambda,\mu,\nu\cdots} a_{\lambda,\mu,\nu\cdots} \frac{p_\lambda(x)}{z_\lambda}\frac{p_\mu(t^{(1)})}{z_\mu}\frac{p_\nu(t^{(2)})}{z_\nu}\cdots$$

and

$$Z\big(F(1,T^{(1)},T^{(2)},\cdots)\big) = \sum_{\lambda,\mu,\nu,\cdots} b_{\mu,\nu,\cdots} \frac{p_\mu(t)}{z_\mu}\frac{p_\nu(w)}{z_\nu}\cdots.$$

Now let μ be a partition of m, ν a partition of n, and so on. Then $b_{\mu,\nu,\ldots}$ is the number of orbits of $F[\mathbf{l},\mathbf{m},\mathbf{n},\ldots]$ under the action of S_l that are fixed by an element $(\sigma,\tau,\ldots)$ in $S_m\times S_n\times\cdots$ in which σ has cycle type μ, τ has cycle type ν, $\ldots$. By the lemma this is

$$\frac{1}{l!}\sum_{\pi\in S_l}\operatorname{fix}(\pi,\sigma,\tau,\ldots) = \frac{1}{l!}\sum_{\lambda}\frac{l!}{z_\lambda}\operatorname{fix}(\lambda,\mu,\nu\cdots) = \sum_{\lambda}\frac{a_{\lambda,\mu,\nu\ldots}}{z_\lambda},$$

where the sum is over all partitions λ of l, and the theorem follows. ☐

By applying Theorem 6 to (10) we can find the cycle index series for the solution of (9).

6. Differential operators. Returning to ordinary formal power series, if we apply (8) to the equation

$$a = t + q(a),$$

we obtain

$$\frac{f(a)}{1-q'(a)} = \sum_n [y^n]\, f(y)\big(t+q(y)\big)^n.$$

A straightforward calculation shows that this is the same as

$$\frac{f(a)}{1-q'(a)} = \sum_{m=0}^{\infty}\frac{d^m}{dt^m}\left(f(t)\frac{q(t)^m}{m!}\right), \tag{12}$$

and as usual, setting $f(y) = g(y)\big(1-q'(y)\big)$ yields the alternate form

$$g(a) = \sum_{m=0}^{\infty}\frac{d^m}{dt^m}\left(g(t)\big(1-q'(t)\big)\frac{q(t)^m}{m!}\right), \tag{13}$$

Labelle [10] found a cycle index version of (13). For our combinatorial approach we need to consider the equivalent formula analogous to (12), which may be expressed in our notation as follows. (Here $a(t)$ is a symmetric function in the variables t_1, t_2, $\ldots$.)

Theorem 7. *For any symmetric function q, there is a unique symmetric function $a(t)$ satisfying*

$$a = t + q(a),$$

and for any symmetric function f,

$$f(a)s\big(q'(a)\big) = \sum_{m_1,m_2,\dots} \frac{\partial^{m_1+m_2+\cdots}}{\partial p_1^{m_1}\partial p_2^{m_2}\cdots}\left(f(t)\frac{q(p_1)^{m_1}}{m_1!}\frac{q(p_2)^{m_2}}{m_2!}\cdots\right), \tag{14}$$

where $p_i = p_i(t_1, t_2, \dots)$. □

Note that the right side of (12) could have been obtained by expanding $e^{zq(t)}f(t)$ and then replacing z with $\frac{d}{dt}$, being careful of the order of the factors. In the remainder of this paper we shall prove a species version of Theorem 7. To do this we shall discuss species versions of the differential operators appearing in Theorem 7, and in particular, of the operation of substituting a derivative for a variable in a power series.

The species equation that we now study is

$$A = T + Q(A). \tag{15}$$

Applying (11) with $R(Y,T) = T + Q(Y)$, we obtain

$$F(A)S\big(Q'(A)\big) = \nabla_{XY}\big(F(Y)E(XT + XQ(Y))\big)\big|_{X=1}. \tag{16}$$

We shall explain how the right side of (16) corresponds to the right side of (14).

First let us recall the scalar product $\langle\ ,\ \rangle$ on symmetric functions defined by

$$\left\langle \sum_\lambda a_\lambda \frac{p_\lambda}{z_\lambda}, \sum_\mu b_\mu \frac{p_\mu}{z_\mu}\right\rangle = \sum_\lambda \frac{a_\lambda b_\lambda}{z_\lambda}.$$

In other words, $\langle f, g\rangle$ is obtained from $f \times g$ by setting each p_λ equal to 1. It is interesting to note that this scalar product was first introduced by Redfield [15] in essentially the same combinatorial context as here. We will occasionally need to refer to the scalar product in another set of variables which will be denoted by a subscript; thus, $\langle f, g\rangle_y$ denotes the scalar product of the symmetric functions f and g in the variables y_1, y_2, $\dots$. If no subscript is give, then the variables are the x_i.

If f is any symmetric function, we define the operator $f(D)$ on symmetric functions to be the adjoint of multiplication by f, so that for any symmetric functions u and v,

$$\langle f(D)u, v\rangle = \langle u, fv\rangle.$$

Macdonald [14] uses the notation $D(f)$ instead of $f(D)$. However, the notation $f(D)$, due to Joyal, will be more convenient to us since, as we shall see, the species analog of the operation $f \mapsto f(D)$ is combinatorially like a substitution. It is clear that $(f+g)(D) = f(D) + g(D)$ and $(fg)(D) = f(D)g(D)$, and it is not hard to show that $p_n(D) = n\partial/\partial p_n$, acting on symmetric functions expressed in terms of the p's (see Macdonald [14, p. 44]). If follows that if $\lambda = (1^{m_1}2^{m_2}\cdots)$ then

$$p_\lambda(D) = 1^{m_1}2^{m_2}\cdots\frac{\partial^{m_1+m_2\cdots}}{\partial p_1^{m_1}\partial p_2^{m_2}\cdots}. \tag{17}$$

We can give an explicit formula for $f(D)$ using symmetric functions in two sets of variables, x's and y's. It is well known that the symmetric function

$$h(xy) = \sum_{n=0}^{\infty} h_n(x_1y_1,\ldots,x_iy_j,\ldots) = \prod_{i,j}(1-x_iy_j)^{-1}$$

has the property that for any symmetric function f, $f(x) = \langle f(y), h(xy)\rangle_y$. Thus with $f(D)g = w$ we have

$$f(D)g(x) = \langle w(y), h(xy)\rangle_y = \langle g(y), f(y)h(xy)\rangle_y. \tag{18}$$

There is another formula for $f(D)g$, which also has a simple species interpretation, though we won't need it:

$$f(D)g(x) = \langle g(x+y), f(y)\rangle_y, \tag{19}$$

where $g(x+y)$ means $g(x_1,x_2,\ldots,y_1,y_2,\ldots)$.

We now discuss the species analogs of these operations on symmetric functions. Following Joyal [9], we define the scalar product $\langle F, G\rangle$ of two species F and G to be $(F\times G)|_{X=1}$, with the obvious generalization to species of several sorts. The scalar product of species is illustrated in Figure 5.

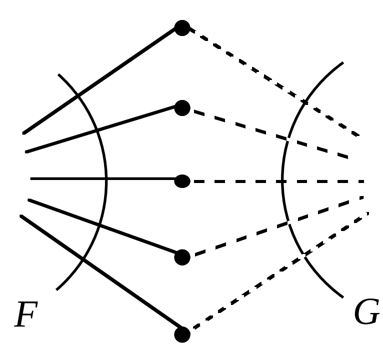

Figure 5: $\langle F, G\rangle$

Then we may define $F(D)G$ to be the species $\langle G(Y), F(Y)E(XY)\rangle_Y$, as shown in Figure 6.

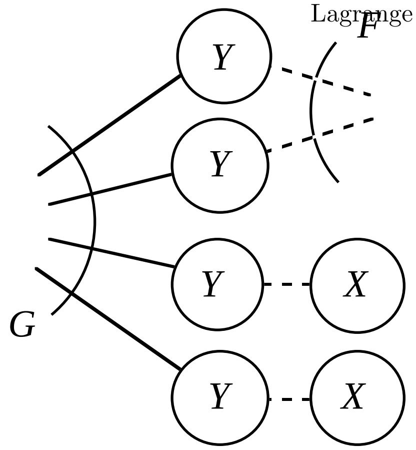

Figure 6a: $G(Y) \times_Y F(Y)E(XY)$

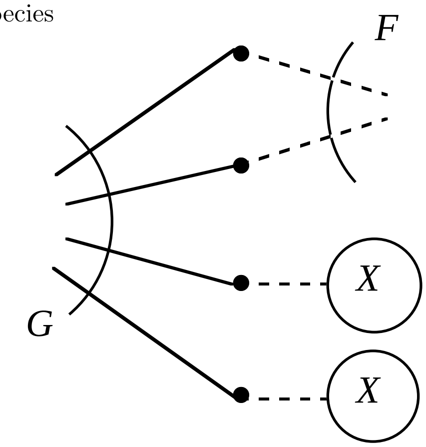

Figure 6b: $\langle G(Y), F(Y)E(XY)\rangle_Y = F(D)G$

It is easy to see that $F(D)G$ is also equal to $\langle G(X+Y), F(Y)\rangle_Y$, as shown in Figure 7,

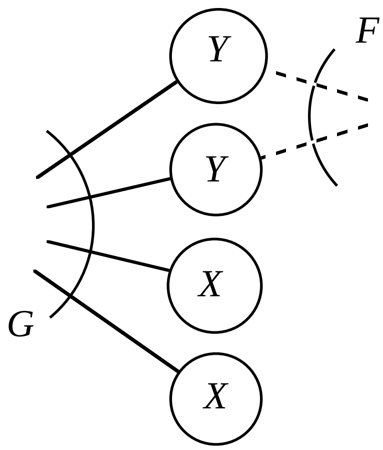

Figure 7a: $G(X+Y) \times_Y F(Y)$

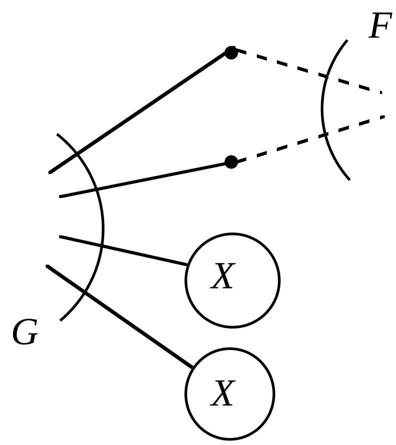

Figure 7b: $\langle G(X+Y), F(Y)\rangle_Y = F(D)G$

and that for any species H, $\langle F(D)G, H\rangle = \langle G, FH\rangle$, as shown in Figure 8.

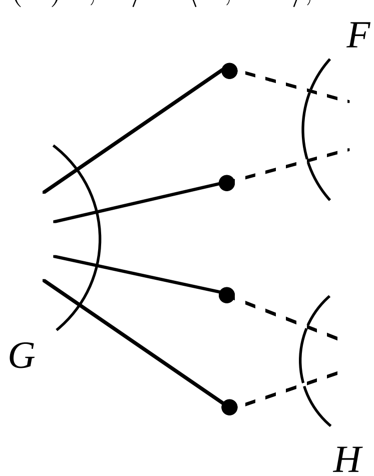

Figure 8: $\langle F(D)G, H\rangle = \langle G, FH\rangle$

7. de Bruijn's generalization of Pólya's theorem. Before we return to Lagrange inversion, we digress to explain how de Bruijn's generalizations of Pólya's theorem [2] can be easily derived from the operations we have introduced so far. (Thus they are useful for more than just Lagrange inversion!) de Bruijn considers the problem of counting orbits of functions from a finite set $\mathcal{D}$ to a finite set $\mathcal{R}$ under the action induced from finite groups acting on $\mathcal{D}$ and on $\mathcal{R}$. Restating de Bruijn's problem in terms of species, we take species F and G, which for simplicity of exposition we assume to be homogeneous of degrees m and n, and we want to count orbits of triples consisting of an F-structure on $\mathcal{D}$, a G-structure on $\mathcal{R}$, and a function $\mathcal{D} \to \mathcal{R}$ under the induced action of $S_m \times S_n$. de Bruijn also considers the analogous problem for injective functions. We may represent an orbit for such a triple with an injective function as in Figure 9,

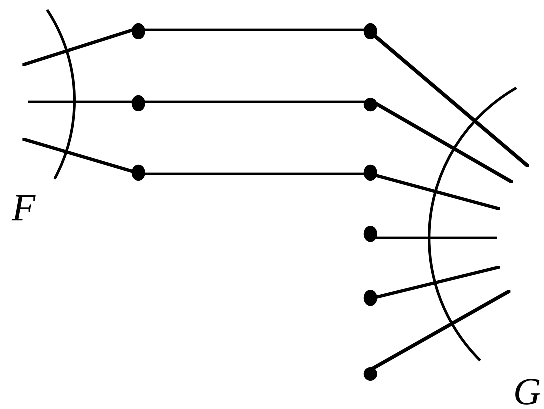

Figure 9

which clearly shows $\langle F(X), G(1+X)\rangle$. In the special case in which $m = n$, this reduces to $\langle F(X), G(X)\rangle$. The case of arbitrary functions is illustrated in Figure 10,

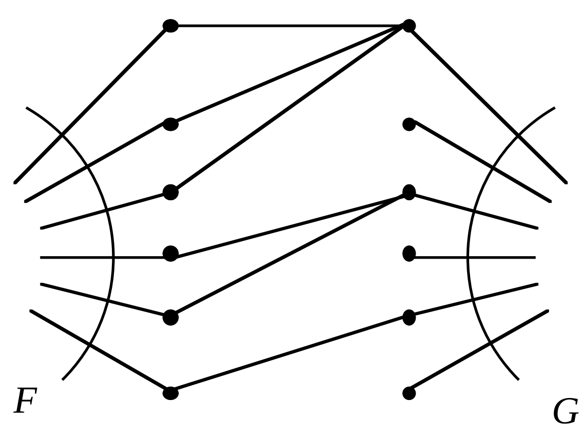

Figure 10

which is easily seen to be $\langle F(X), G\big(E(X)\big)\rangle$. To recover de Bruijn's formulas involving partial derivatives, we pass to cycle indexes, then we use the fact that $\langle f, h\rangle$ is the constant term in $f(D)h$, and we express $f(D)$ in terms of partial derivatives in the p_i using (17).

8. Application to Lagrange inversion. Just as we found it necessary to extend the Cartesian product of two species to the operation ∇ from two-sorted species to one-sorted species, we now must extend the map $F \otimes G \mapsto F(D)G$ to an operation Ψ from two-sorted species to one-sorted species. For any two-sorted species $W(X,Y)$ (which may also involve other sorts of points) we define the one-sorted species $\Psi(W)(T)$ to be

$$\nabla_{XY}\big(W(X,Y)E(XT)\big)\big|_{X=1}.$$

If we represent the two-sorted species $W(X,Y)$ as in Figure 11

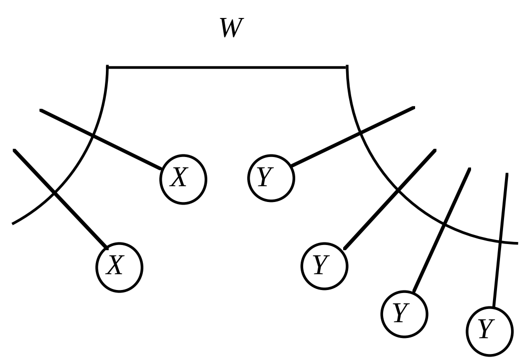

Figure 11

then $W(X,Y)E(XT)$ is as shown in Figure 12

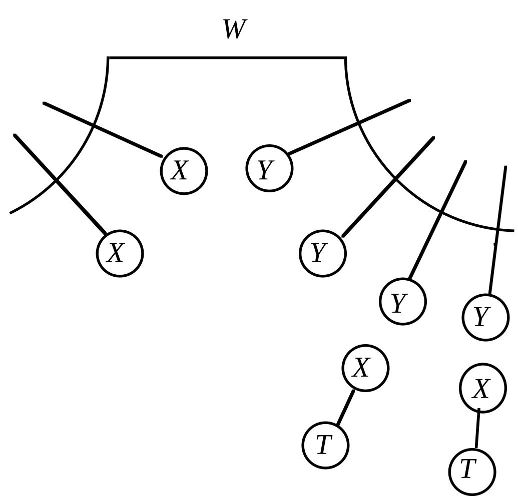

Figure 12

and $\Psi(W)(T) = \nabla_{XY}\big(W(X,Y)E(XT)\big)\big|_{X=1}$ is as shown in Figure 13.

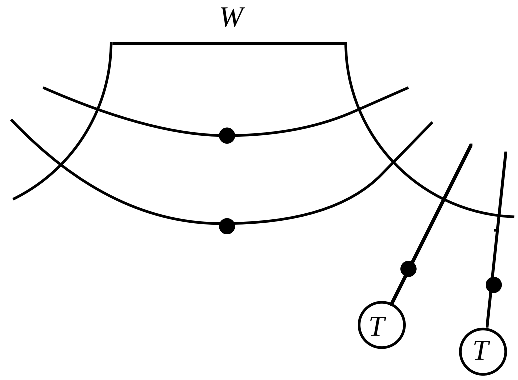

Figure 13

We note the slightly simpler formula, which we will not need,

$$\Psi(W)(T) = \nabla_{XY} W(X, T+Y)\big|_{X=1}.$$

It is easily verified that in the special case in which $W(X,Y) = F(X)G(Y)$, $\Psi(W)(X) = F(D)G(X)$.

We can now find the species analog of (14). In fact the right side of (16) reduces to

$$\nabla_{XY}\left(F(Y)E\bigl(XQ(Y)\bigr)E(XT)\right)\big|_{X=1} = \Psi(W)(T),$$

where

$$W(X,Y) = F(Y)E\bigl(XQ(Y)\bigr).$$

So our species version of Theorem 7 is:

Theorem 8. *For any species $Q(Y)$ there is a unique species $A(T)$ satisfying $A(T) = T + Q\bigl(A(T)\bigr)$, and for any species F,*

$$F(A)S\bigl(Q'(A)\bigr) = \Psi(W)(T),$$

where

$$W(X,Y) = F(Y)E\bigl(XQ(Y)\bigr). \quad \square$$

It remains to show that Theorem 7 is a consequence of Theorem 8. First we determine the effect of Ψ on the cycle index of a two-sorted species.

Lemma 9. *Let $W(X,Y)$ be a two-sorted species, and suppose that*

$$Z\bigl(W(X,Y)\bigr) = \sum_\lambda a_\lambda \frac{p_\lambda(x)}{z_\lambda} g_\lambda(y),$$

where for each λ, $g_\lambda(y)$ is a symmetric function in the y_i. Then

$$Z\bigl(\Psi\bigl(W(X,Y)\bigr)\bigr) = \sum_\lambda a_\lambda \frac{p_\lambda(D_t)}{z_\lambda} g_\lambda(t),$$

where $p_\lambda(D_t)$ denotes $p_\lambda(D)$ acting on symmetric functions in the t_i.

Proof. We have

$$Z\big(W(X,Y)E(XT)\big) = \sum_\lambda a_\lambda \frac{p_\lambda(x)}{z_\lambda} g_\lambda(y)h(xt),$$

so

$$Z\big(\Psi\big(W(X,Y)\big)\big) = \sum_\lambda a_\lambda \left\langle \frac{p_\lambda(x)}{z_\lambda} h(xt), g_\lambda(x) \right\rangle,$$

which by (18) is

$$\sum_\lambda a_\lambda \frac{p_\lambda(D_t)}{z_\lambda} g_\lambda(t). \quad \square$$

To derive Theorem 7 from Theorem 8, we take $W(X,Y) = F(Y)E(XQ(Y))$ in Lemma 9, and set $f(y) = Z\big(F(Y)\big)$ and $q(y) = Z\big(Q(Y)\big)$. Then

$$Z\big(W(X,Y)\big) = f(y)h\big(xq(y)\big) = f(y)\sum_\lambda \frac{p_\lambda(x)}{z_\lambda} q^{[\lambda]}(y),$$

so

$$Z\big(\Psi\big(W(X,Y)(T)\big)\big) = \sum_\lambda \frac{p_\lambda(D_t)}{z_\lambda} f(t)q^{[\lambda]}(t),$$

and by (17), this is easily seen to be the same as the right side of (14).

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