Note on enumeration of partitions contained in a given shape

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Abstract

Carlitz, Handa, and Mohanty proved determinantal formulas for counting partitions contained in a fixed bounding shape by area. Gessel and Viennot introduced a combinatorial method for proving such formulas by interpreting the determinants as counting suitable configurations of signed lattice paths. This note describes an alternative combinatorial approach that uses sign-reversing involutions to prove matrix inversion results. Combining these results with the classical adjoint formula for the inverse of a matrix, we obtain a new derivation of the Handa-Mohanty determinantal formula.

Let $M = (M_1 \leq M_2 \leq \cdots \leq M_s)$ and $N = (N_1 \leq N_2 \leq \cdots \leq N_s)$ be two fixed partitions such that $M \subseteq N$, i.e., $M_i \leq N_i$ for all i. Handa and Mohanty [3] proved the following determinantal formula that enumerates partitions lying between M and N by area:

$$\sum_{\lambda: M\subseteq\lambda\subseteq N} q^{|\lambda|} = \det\left(\begin{bmatrix} N_i - M_j + 1 \\ j-i+1 \end{bmatrix}_q q^{(j-i)(j-i+1)/2+(j-i+1)M_j}\right)_{1\leq i,j\leq s}. \tag{1}$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient, which enumerates partitions contained in a $k \times (n-k)$ box by area. An even more general determinant formula for enumerating plane partitions was proved by Carlitz [1]. Gessel and Viennot [5] introduced a powerful combinatorial technique for proving such determinantal identities. Their technique identifies terms in $\det(A) = \sum_{\tau\in S_s} \operatorname{sgn}(\tau) \prod_{i=1}^{s} A_{i,\tau(i)}$ as counting suitable signed, weighted s-tuples of lattice paths. Sign-reversing involutions are then defined that cancel out many of the objects, leaving only objects counted by the left side of the desired formula.

This note presents another combinatorial approach to proving identities like (1). Our viewpoint is that a determinantal formula such as (1) may be a consequence of the fact that two matrices with combinatorial entries are mutual inverses. This fact may be proved combinatorially using a sign-reversing involution. The determinantal formula can then be deduced by using the classical adjoint formula for the entries of the inverse of a matrix. One advantage of our approach, besides providing new information about these matrices, is that the objects involved in the involutions are a bit simpler than the objects counted in the Gessel-Viennot method. One minor drawback is that we must invoke the adjoint formula when passing to determinants, so that the proof of the final determinant formula is not wholly combinatorial.

Our approach may be viewed as an extension of that of Carlitz, Scoville, and Vaughan [2], who used a sign-reversing involution to give a combinatorial interpretation to reciprocal power series. This idea was applied to inverse matrices by Gessel [4]. A similar, but somewhat more complicated sign-reversing involution was given in the case discussed here by Loehr [6].

We illustrate our method by giving a new derivation of (1) based on the following theorem.

Theorem 1. *Let $M = (M_1, M_2, \dots, M_s)$ and $N = (N_1, N_2, \dots, N_s)$ be arbitrary sequences of integers. Call an indexed sequence of integers $(a_i, \dots, a_j)$* admissible *if $M_t \le a_t \le N_t$ for $i \le t \le j$. Let A be the lower-triangular matrix given by $A_{i,i} = 1$ for all i and*

$$A_{i,k} = (-1)^{i-k} \sum_{\substack{\text{strictly decreasing admissible} \\ \text{sequences } (a_{k+1},\dots,a_i)}} x_{a_{k+1}} \cdots x_{a_i} \qquad (0 \le k < i \le s).$$

Let B be the lower-triangular matrix given by $B_{k,k} = 1$ for all k and

$$B_{k,j} = \sum_{\substack{\text{weakly increasing admissible} \\ \text{sequences } (a_{j+1},\dots,a_k)}} x_{a_{j+1}} \cdots x_{a_k} \qquad (0 \le j < k \le s).$$

Then $AB = I$.

Proof. Since A and B are lower-unitriangular, it suffices to show that for $0 \le j < i \le s$, $\sum_{k=j}^{i} B_{k,j} A_{i,k} = 0$. The sum counts signed, weighted objects of the form

$$z = (a_{j+1}, \cdots, a_k;\ a_{k+1}, \cdots, a_i)$$

where the left-hand sequence is admissible and weakly increasing, the right-hand sequence is admissible and strictly decreasing, the weight of z is $\prod_{j<t\le i} x_{a_t}$, and the sign of z is $(-1)^{i-k}$. The following sign-reversing involution cancels all such objects in pairs, thus proving that the sum is zero. If $a_k \le a_{k+1}$ or $j = k$, map z to

$$z' = (a_{j+1}, \cdots, a_k, a_{k+1};\ a_{k+2}, \cdots, a_i).$$

If $a_k > a_{k+1}$ or $i = k$, map z to

$$z' = (a_{j+1}, \dots, a_{k-1};\ a_k, a_{k+1}, \dots, a_i).$$

It is immediate that z and z' have the same weight and opposite signs, and $z'' = z$. □

Keeping the notation of Theorem 1, assume now that M and N are weakly increasing sequences. In this case, a strictly decreasing sequence $(a_{k+1}, \dots, a_i)$ is admissible if and only if $N_{k+1} \ge a_{k+1}$ and $M_i \le a_i$, since these two inequalities imply $M_t \le M_i \le a_i \le a_t \le a_{k+1} \le N_{k+1} \le N_t$ for $k < t \le i$. (In fact, since consecutive a's differ by at least 1, this result would still hold provided that $M_{t+1} \ge M_t - 1$ and $N_{t+1} \ge N_t - 1$ for all $t < s$.) We now have

$$A_{i,k} = (-1)^{i-k} \sum_{N_{k+1} \ge a_{k+1} > \cdots > a_i \ge M_i} x_{a_{k+1}} \cdots x_{a_i} \quad (0 \le k < i \le s).$$

The map $(a_t : k < t \le i) \mapsto (a_t + t - i - M_i : k < t \le i)$ is a bijection from the set of sequences indexing the sum for $A_{i,k}$ onto the set of partitions μ contained in a rectangle with $i - k$ rows and $N_{k+1} - M_i + k + 1 - i$ columns, such that $\sum_{k<t\le i} a_t = |\mu| + (i-k)M_i + \binom{i-k}{2}$. Setting $x_m = q^m$ for all m, we obtain the following corollary.

Corollary 2. *Given $M_1 \leq M_2 \leq \cdots \leq M_s$ and $N_1 \leq N_2 \leq \cdots \leq N_s$, define lower-unitriangular matrices A and B by setting*

$$A_{i,k} = (-1)^{i-k} \begin{bmatrix} N_{k+1} - M_i + 1 \\ i-k \end{bmatrix}_q q^{(i-k)M_i + \binom{i-k}{2}} \qquad (0 \leq k \leq i \leq s);$$

$$B_{k,j} = \sum_{\substack{a_{j+1} \leq \cdots \leq a_k \\ M_t \leq a_t \leq N_t}} q^{a_{j+1} + \cdots + a_k} \qquad (0 \leq j \leq k \leq s).$$

Then $AB = I$.

We can now compute $B_{s,0} = (A^{-1})_{s,0}$ using the classical adjoint formula for A^{-1}, namely $(A^{-1})_{i,j} = \det(A(j|i))/\det(A)$. After some routine simplifications (left to the reader), the resulting identity is precisely the formula (1).

References

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